CÁLCULO IV

08/03/24 – AULA 21

Vimos no final da aula passada:

Se γ for um caminho parametrizado pelo comprim. de arco, $\vec{n}(s)$ o vetor normal unitário em γ, então,

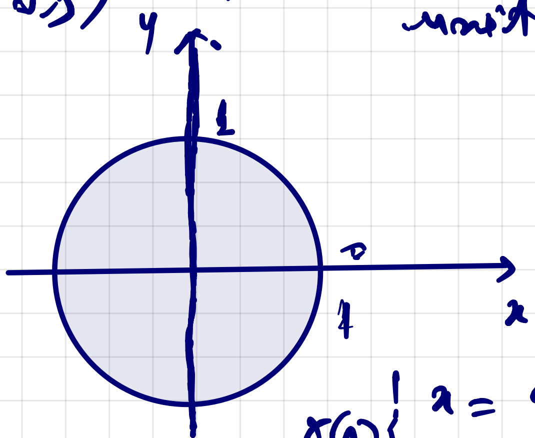
$$\oint_\gamma \vec{F}\,\vec{n}(s)\,ds = \iint_\Omega \operatorname{div} \vec{F}\,.\,dA,$$

chamado de TEOR DA DIVERGÊNCIA de GAUSS.

EX.: Verifique o teor. da divergência de Gauss para o campo $\vec{F}(x,y) = 2y\vec{i} + 5xy\vec{j}$ sendo γ a circunferência $x^2+y^2=1$.

SOLUÇÃO: Temos: $\vec{F}(x,y) = (2y, 5xy)$

$$\vec{n}(s) = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right),$$ e $\vec{n}(s)$ deve ser unitária.



Por um lado,

$$\oint_\gamma \vec{F}\,.\,\vec{n}(s)\,ds =$$

$\gamma(s) \begin{cases} x = \cos s \\ y = \operatorname{sen} s \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{dx}{ds} = -\operatorname{sen} s \\[2ex] \dfrac{dy}{ds} = \cos s \end{cases}$$

$$\|\gamma'(s)\| = \sqrt{(x'(s))^2 + (y'(s))^2}$$

$$= \sqrt{(-\operatorname{sen} s)^2 + (\cos s)^2}$$

$$= \sqrt{\operatorname{sen}^2 s + \cos^2 s} = \sqrt{1} = 1.$$

γ está parametrizada pelo compr. de arco.

Então, $\vec{n}(s) = \left(\dfrac{dy}{ds}, -\dfrac{dx}{ds}\right) = (\cos s, +\operatorname{sen} s)$

é um vetor normal (por construção) e unitário.

Disso:

$$\oint_\gamma \vec{F}\cdot n(s)\,ds = \int_0^{2\pi} (2y, 5xy)\cdot(\cos s, \operatorname{sen} s)\,ds$$

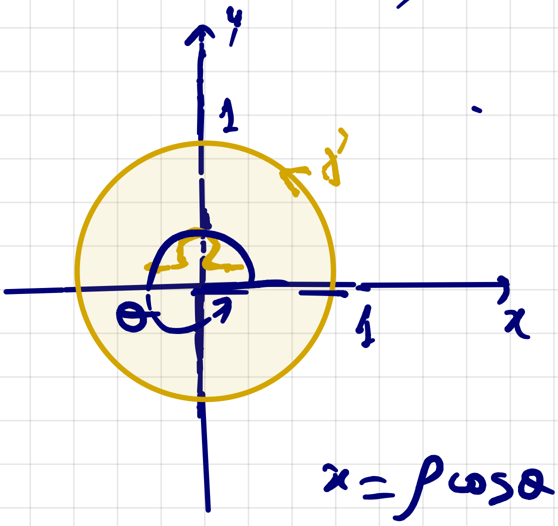
$$= \int_0^{2\pi} (2y\cos s + 5xy\operatorname{sen} s)\,ds \underset{\substack{x=\cos s\\ y=\operatorname{sen} s}}{=}$$

$$= \int_0^{2\pi} (2\operatorname{sen} s\cdot\cos s + 5\cdot\cos s\cdot\operatorname{sen} s\cdot\operatorname{sen} s)\,ds$$

$$= 2\int_0^{2\pi} \underbrace{(\operatorname{sen}\theta)}_{u}^{1} \cdot \underbrace{(\cos\theta\, d\theta)}_{du} + 5\int_0^{2\pi} \underbrace{(\operatorname{sen}\theta)}_{u}^{2} \cdot \underbrace{(\cos\theta\, d\theta)}_{du} =$$

$$\left(\cancel{2} \cdot \frac{1}{\cancel{2}} \cdot \operatorname{sen}^2\theta + 5 \cdot \frac{1}{3} \cdot \operatorname{sen}^3\theta\right)\Bigg|_0^{2\pi} =$$

$$\operatorname{sen}^2 2\pi + \frac{5}{3}\operatorname{sen}^3 2\pi - \operatorname{sen}^2 0 + \frac{5}{3}\operatorname{sen}^3 0 = 0$$



Por outro lado,

$$\iint_\Omega \operatorname{div}\vec{F}\, dA = ?$$

$$\vec{F} = (\overbrace{2y}^{F_1}, \overbrace{5xy}^{F_2})$$

$x = \rho\cos\theta$
$y = \rho\operatorname{sen}\theta$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 5x = 5x.$$

Disso:

$$\iint_\Omega \operatorname{div}\vec{F} \cdot dA = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} 5 \cdot \underbrace{\rho\cos\theta}_{x} \cdot \underbrace{\rho\, d\rho\, d\theta}_{dA}$$

$$= \int_{\theta=0}^{\theta=2\pi} \cos\theta\, d\theta \cdot \int_{\rho=0}^{\rho=1} 5\rho^2\, d\rho = \operatorname{sen}\theta\Big|_0^{2\pi} \cdot \frac{5\rho^3}{3}\Big|_0^1 =$$

$$(\underbrace{\operatorname{sen} 2\pi}_{=0} - \underbrace{\operatorname{sen} 0}_{=0}) \cdot \left(\frac{5}{3} - 0\right) = 0$$

Então,
$$\oint_\gamma \vec{F} \cdot \vec{n}(s)\, ds = \iint_\Omega \operatorname{div} \vec{F} \, dA.$$

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segunda extensão vetorial do T. de Green:

Considere $\gamma(s) = (x(s), y(s))$ uma curva parametrizada pelo comprimento de arco; e seja $\vec{t}(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ o vetor tangente unitário em cada ponto da curva γ.

Vamos determinar

$$\oint_\gamma \vec{F} \cdot \vec{t}(s)\, ds. \quad ; \gamma : [a,b] \to \mathbb{R}^2 \text{ caminho fechado simples.}$$

e $\vec{F} = (P, Q)$ campo de $\mathbb{R}^2$.

Por um lado,

$$\oint_\gamma \vec{F} \cdot \vec{t}(s)\, ds = \int_a^b (P, Q) \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}\right) ds$$

PRODUTO ESCALAR

$$= \int_a^b P\,dx + Q\,dy = \underbrace{\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{}$$

T. de Green

Considere $\vec{F} = (P, Q, R)$, onde $R = 0$

Vamos determinar $\text{rot}\,\vec{F}$.

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{matrix}$$

$$= 0\vec{i} + \underbrace{\frac{\partial P}{\partial z}}_{\substack{\shortparallel \\ 0,\ \text{pois} \\ P = P(x,y)}}\vec{j} + \frac{\partial Q}{\partial x}\vec{k} - \frac{\partial P}{\partial y}\vec{k} - \underbrace{\frac{\partial Q}{\partial z}}_{\substack{= 0, \\ \text{pois } Q = Q(x,y)}}\vec{i} - 0\vec{j}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) . \vec{k}$$

Logo, $\text{rot}\,\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) . \vec{k}$

$$\text{rot}\,\vec{F} . \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) . \underbrace{\vec{k} . \vec{k}}_{= \|\vec{k}\|^2 = 1}$$

$$\Rightarrow \boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \text{rot}\,\vec{F} \cdot \vec{k}}$$

Ou seja, sendo γ curva seccionalmente suave, parametrizada pelo comprimento de arco, $\vec{F} = (P, Q)$ campo vetorial com derivadas parciais contínuas, então, sendo $\vec{t}(s)$ o vetor tangente unitário em γ,

$$\underbrace{\oint_\gamma \vec{F} \cdot \vec{t}(s)\, ds}_{} \underset{\text{T. GREEN}}{=} \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$= \underbrace{\iint_\Omega \text{rot}\,\vec{F} \cdot \vec{k} \cdot dA}_{},$$

conhecido como T. de STOKES.

<u>Ex.</u>: Verifique o T. de STOKES no plano para o campo $\vec{F}(x,y) = (xy^2, x^2y)$ na circunferência $x^2 + y^2 = 9$.

<u>SOLUÇÃO</u>: A parametrização para a circunf. γ será:

$$\gamma: \begin{cases} x = 3\cos s \\ y = 3 \cdot \text{sen}\, s. \end{cases}$$

O vetor $\vec{t}(s)$ será: $\vec{t}(s) = \dfrac{\gamma'(s)}{\|\gamma'(s)\|}$,

onde $\gamma'(s) = (-3\,\text{sen}\, s,\ 3\cos s)$

$$\Rightarrow \|\gamma'(s)\| = \sqrt{(-3\,\text{sen}\, s)^2 + (3\cos s)^2}$$

$$= \sqrt{9\,\text{sen}^2 s + 9\cos^2 s} = 3$$

$$\Rightarrow \underset{\sim}{\vec{t}(s)} = \frac{1}{\|\gamma'(s)\|}\cdot \gamma'(s) = \frac{1}{3}(-3\,\text{sen}\, s,\ 3\cos s)$$

$$= \underset{\sim}{(-\text{sen}\, s,\ \cos s)}$$

Por um lado, temos:

$$\oint_\gamma \vec{F}\cdot\vec{t}(s)\, ds = \int_0^{2\pi} (xy^2, x^2y)\cdot(-\text{sen}\, s, \cos s)\, ds$$

$$= \int_0^{2\pi} \left(\underline{3\cos s\cdot(3\,\text{sen}\, s)^2},\ (3\cos s)^2\cdot 3\,\text{sen}\, s\right)\cdot\left(\underline{-\text{sen}\, s},\ \cos s\right) ds$$

$$= \int_0^{2\pi} \left[\underline{27\,\text{sen}^2 s\,\cos s\cdot(-\text{sen}\, s)} + 27\cos^3 s\cdot\text{sen}\, s\right] ds$$

$$= -27\int_0^{2\pi} \underbrace{(\text{sen}\, s)^3}_{u}\cdot\underbrace{(\cos s\, ds)}_{du} + 27\int_0^{2\pi} \underbrace{(\cos s)^3}_{u}\underbrace{(-\text{sen}\, s\, ds)}_{du}$$

$$= -27\cdot\frac{\text{sen}^4 s}{4}\Bigg|_0^{2\pi} + 27\cdot\frac{\cos^4 s}{4}\Bigg|_0^{2\pi} = 0 \quad //$$

Por outro lado, sendo $\vec{F}(x,y) = (xy^2, x^2y)$, temos

$$\text{rot}\,\vec{F} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x^2y & 0 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy^2 & x^2y \end{matrix}$$

$$= 0\vec{i} + 0\vec{j} + \cancel{2xy}\,\vec{k} - \cancel{2xy}\,\vec{k} - 0\vec{i} - 0\vec{j}$$

$$= \vec{0}\,;$$ e disso:

$$\iint_{\Omega} \text{rot}\,\vec{F}\cdot\vec{k}\,dA = \iint_{\Omega} \vec{0}\cdot\vec{k}\,dA = \iint_{\Omega} 0\,dA = 0$$

Logo, se cumpre que

$$\oint_{\gamma} \vec{F}\cdot\vec{T}(s)\,ds = \iint_{\Omega} \text{rot}\,\vec{F}\cdot\vec{k}\,dA.$$

EXERCÍCIO:

LISTA 06, nº 8.

8. Use o Teorema da Divergência para mostrar que, dado um campo escalar $g : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ com derivadas parciais contínuas até a segunda ordem no aberto Ω e γ uma curva suave fechada simples em Ω, então

$$\oint_{\gamma} g\frac{\partial g}{\partial \vec{n}}ds = \iint_{\Omega}(g\Delta g + ||\nabla g||^2)dA.$$

Obs.: a quantidade $\nabla g \cdot \vec{n} := \dfrac{\partial g}{\partial \vec{n}}$ aparece na integral de linha. A derivada direcional de g na direção do vetor normal $\vec{n}$ é chamada de *derivada normal* de g.

Solução: Pelo T. da divergência:

$$\oint_{\gamma} \vec{F} \cdot \vec{n}(s)\, ds = \iint_{\Omega} \operatorname{div} \vec{F} \cdot dA.$$

No nosso caso:

$$\oint_{\gamma} g \cdot \frac{\partial g}{\partial \vec{n}}\, ds = \oint_{\gamma} \underbrace{g \cdot \nabla g}_{= \vec{F}} \cdot \vec{n}\, ds \; ; \quad \text{onde}$$

$$F(x,y) = g(x,y) \cdot \nabla g(x,y)$$

$$= g \cdot \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right)$$

$$= \left(\underbrace{g \cdot \frac{\partial g}{\partial x}}_{F_1}, \underbrace{g \cdot \frac{\partial g}{\partial y}}_{F_2} \right)$$

Então:

$$\operatorname{div} \vec{F} = \operatorname{div}(g \cdot \nabla g) = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 =$$

$$= \frac{\partial}{\partial x}\left(g \cdot \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y}\left(g \cdot \frac{\partial g}{\partial y} \right) =$$

$(u \cdot v)'$

$$= g \cdot \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x} \cdot \frac{\partial g}{\partial x} + g \cdot \frac{\partial^2 g}{\partial y^2} + \frac{\partial g}{\partial y} \cdot \frac{\partial g}{\partial y}$$

$$= g \cdot \underbrace{\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right)}_{\Delta g} + \frac{\partial g}{\partial x} \cdot \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \cdot \frac{\partial g}{\partial y}$$

$$= g \cdot \Delta g + \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \cdot \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$$

prod. escalar

$$= g \cdot \Delta g + \nabla g \cdot \nabla g = \underbrace{g \cdot \Delta g + \|\nabla g\|^2}$$

$\bar{u} \cdot \bar{u} = \|\bar{u}\|^2$

$$\Rightarrow \operatorname{div} \bar{F} = g \cdot \Delta g + \|\nabla g\|^2 .$$

Portanto;

$$\underbrace{\oint_\gamma g \cdot \frac{\partial g}{\partial \bar{n}} \cdot ds} = \oint_\gamma \underbrace{g \cdot \nabla g}_{= \vec{F}} \cdot \bar{n} \cdot ds = \oint_\gamma \bar{F} \cdot \bar{n} \, ds =$$

T. GAUSS

$$= \iint_\Omega \operatorname{div} \bar{F} \cdot d_A = \underbrace{\iint_\Omega (g \cdot \Delta g + \|\nabla g\|^2) \, dA}_{} ,$$

o que prova o desejado.

$\square$