CÁLCULO 2 -

$\boxed{12/03/24 \text{ — AULA 24}}$

AULA DE EXERCÍCIOS:

1. Calcular o comprimento de arco de cada curva dada no intervalo considerado em cada caso:

(a) $f(x) = 2\sqrt{x}$, em $[0, 1]$.

SOLUÇÃO:

$$\ell = \int_0^1 \sqrt{1 + [f'(x)]^2} \cdot dx$$

$$f(x) = 2x^{\frac{1}{2}} \Rightarrow f'(x) = \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot x^{\frac{1}{2}-1} \cdot 1 = \frac{1}{\sqrt{x}}.$$

$-\frac{1}{2}$

$$\ell = \int_0^1 \sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2} \cdot dx = \int_0^1 \sqrt{1 + \frac{1}{x}}\, dx =$$

$$= \int_0^1 \sqrt{\frac{x+1}{x}}\, dx$$

- $\int \sqrt{\frac{x+1}{x}}\, dx = ?$

Escreva

$$\frac{x+1}{x} = p^2$$

$$x + 1 = x \cdot p^2$$

$$xp^2 - x = 1$$

$$x(p^2-1)=1$$

$$x=\frac{1}{p^2-1}=(p^2-1)^{-1}$$

Diferenciando em x, vem:

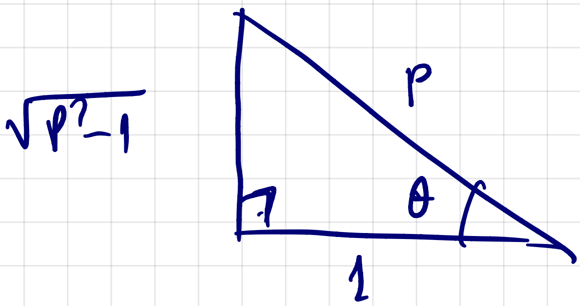
$$dx=-1.(p^2-1)^{-2}.2p.dp$$

$$dx=-\frac{2p}{(p^2-1)^2}dp$$

Assim, vamos obter:

$$\int\sqrt{\frac{x+1}{x}}\,dx=\int\sqrt{p^{\cancel{2}}}\cdot\left(\frac{-2p.\,dp}{(p^2-1)^2}\right)$$

$$=\int p.\frac{(-2p)dp}{(p^2-1)^2}=\int\frac{-2p^2dp}{(p^2-1)^2}$$



$$p^2=\left(\sqrt{p^{\cancel{2}}-1}\right)^{\cancel{2}}+1^2$$
$$=p^2-\cancel{1}+\cancel{1}$$

$$\tan\theta=\frac{\sqrt{p^2-1}}{1}\Rightarrow\boxed{p^2-1=\tan^2\theta}$$

$$\cos\theta=\frac{1}{p}\Rightarrow p=\sec\theta.$$

$$\Rightarrow d\rho = \sec\theta \tan\theta \, d\theta$$

Logo:

$$\int \frac{-2\rho^2 d\rho}{(\rho^2-1)^2} = \int \frac{-2 \cdot \sec^2\theta \cdot \sec\theta \cdot \tan\theta \, d\theta}{(\tan\theta)^2} =$$

$$= -2 \int \frac{\sec^3\theta \cdot \cancel{\tan\theta} \, d\theta}{\tan^{\cancel{4}}\theta} = -2 \int \frac{\sec^3\theta}{\tan^3\theta} \, d\theta .$$

$$= -2 \int \frac{1}{\cancel{\cos^3\theta}} \cdot \frac{\cancel{\cos^3\theta}}{\operatorname{sen}^3\theta} \, d\theta = -2 \cdot \int \csc^3\theta \, d\theta .$$

Deve-se agora integrar por partes. (AULA 15)

$$= -2 \cdot \left[\frac{1}{2} \ln|\csc\theta - \cot\theta| - \frac{1}{2} \csc\theta \cdot \cot\theta + C \right]$$

$$= -\ln|\csc\theta - \cot\theta| + \csc\theta \cdot \cot\theta + C =$$

$$\begin{cases} \csc\theta = \dfrac{1}{\operatorname{sen}\theta} = \quad ; \operatorname{sen}\theta = \dfrac{\sqrt{\rho^2-1}}{\rho} \\ \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\sqrt{\rho^2-1}} \end{cases}$$

$$= -\ln\left|\frac{p}{\sqrt{p^2-1}} - \frac{1}{\sqrt{p^2-1}}\right| + \frac{p}{\sqrt{p^2-1}}\cdot\frac{1}{\sqrt{p^2-1}} + c$$

$$= -\ln\left|\frac{p-1}{\sqrt{p^2-1}}\right| + \frac{p}{p^2-1} + c$$

$$= -\ln\left|\frac{\sqrt{\frac{x+1}{x}}-1}{\sqrt{\frac{x+1}{x}-1}}\right| + \frac{\frac{x+1}{x}}{\frac{x+1}{x}-1} + c = \ldots$$

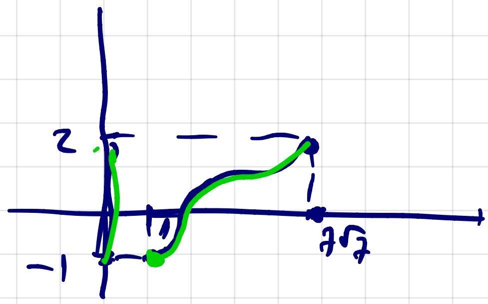
$\uparrow$

$p^2 = \frac{x+1}{x}$

L101

2. Ache o comprimento da curva $x^2 = (2y+3)^3$ de $P(1,-1)$ a $Q(7\sqrt{7},2)$.

solução:



$\Rightarrow$ [Ou escreva. $x = \sqrt{(2y+3)^3}$ e

calcule $\ell = \int_{-1}^{2} \sqrt{1+[x'(y)]^2}\,dy$]

Ou: $x^2 = (2y+3)^3$

$$\sqrt[3]{x^2} = \sqrt[3]{(2y+3)^3}$$

$$x^{\frac{2}{3}} = 2y+3 \Rightarrow 2y = x^{\frac{2}{3}} - 3$$

$$y = \frac{x^{2/3}-3}{2}$$

$$\Rightarrow l = \int_1^{7\sqrt{7}} \sqrt{1 + [y'(x)]^2}\, dx$$

Vamos considerar a primeira forma de resolver.

$$x = \sqrt{(2y+3)^3} = (2y+3)^{\frac{3}{2}}$$

$(u^k)' = k u^{k-1} \cdot u'$

$$x' = \frac{3}{\cancel{2}} (2y+3)^{\frac{1}{2}} \cdot \cancel{2} = 3\sqrt{2y+3}$$

Assim, a medida do comprimento l será:

$$l = \int_{-1}^{2} \sqrt{1 + [x'(y)]^2}\, dy$$

$$= \int_{-1}^{2} \sqrt{1 + [3 \cdot \sqrt{2y+3}]^2}\, dy =$$

$$= \int_{-1}^{2} \sqrt{1 + 9(2y+3)}\, dy = \int_{-1}^{2} \sqrt{1 + 18y + 27}\, dy$$

$$= \int_{-1}^{2} \sqrt{18y + 28}\, dy$$

- $\int \sqrt{18y+28}\, dy = \int (18y+28)^{\frac{1}{2}} dy = \int u^k du$

$u = 18y + 28 \Rightarrow du = 18 dy \Rightarrow dy = \frac{du}{18}$

$$= \int u^{\frac{1}{2}} \cdot \frac{du}{18} = \frac{1}{18} \int u^{\frac{1}{2}} du = \frac{1}{18} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{\cancel{18}_{9}} \cdot \frac{\cancel{2}}{3} \cdot (18y+28)^{\frac{3}{2}} + c$$

$$= \frac{1}{27} (18y+28)^{\frac{3}{2}} + c$$

$$\Rightarrow l = \int_{-1}^{2} \sqrt{18y+28}\, dy = \left(\frac{1}{27} (18y+28)^{\frac{3}{2}} \right) \Bigg|_{-1}^{2} =$$

$$= \frac{1}{27} (18.(2)+28)^{\frac{3}{2}} - \frac{1}{27} (18.(-1)+28)^{\frac{3}{2}} =$$

$$= \frac{1}{27} (64)^{\frac{3}{2}} - \frac{1}{27} (10)^{\frac{3}{2}} = \frac{1}{27} \left[\sqrt{(64)^3} - \sqrt{10^3} \right]$$

$$= \frac{1}{27} \left[\sqrt{(64)^{\cancel{2}}. 8.8} - \sqrt{(10)^{\cancel{2}}.10} \right]$$

$$= \frac{1}{27} \left[64 \sqrt{2^{\cancel{2}}.2.\ 2^{\cancel{2}}.2} - 10\sqrt{10} \right]$$

$$= \frac{1}{27} \cdot \left[64.16 - 10\sqrt{10} \right] = \frac{1024 - 10\sqrt{10}}{27}$$

5. Determine o comprimento do arco da curva $y = \ln(1 - x^2)$ de $x = 0$ a $x = \frac{1}{2}$.

Solução:

$$l = \int_0^{\frac{1}{2}} \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = \ln(1 - x^2) \Rightarrow f'(x) = \frac{-2x}{1 - x^2}$$

$$(\ln u)' = \frac{u'}{u}$$

$$\Rightarrow l = \int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{-2x}{1-x^2}\right)^2}\, dx = \int_0^{\frac{1}{2}} \sqrt{1 + \frac{4x^2}{(1-x^2)^2}}\, dx$$

$$= \int_0^{\frac{1}{2}} \sqrt{\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^{\cancel{2}}}}\, dx = \int_0^{\frac{1}{2}} \frac{\sqrt{1 - 2x^2 + x^4 + 4x^2}}{(1-x^2)}\, dx$$

$$= \int_0^{\frac{1}{2}} \frac{\sqrt{x^4 + 2x^2 + 1}}{1 - x^2}\, dx = \int_0^{\frac{1}{2}} \frac{\sqrt{(x^2+1)^{\cancel{2}}}}{1 - x^2}\, dx =$$

$$= \int_0^{\frac{1}{2}} \frac{x^2 + 1}{1 - x^2}\, dx.$$

$$\bullet \int \frac{x^2+1}{1-x^2}\,dx = ?$$

$$\int \frac{x^2+1}{1-x^2}\,dx = -\int \frac{x^2+1}{x^2-1}\,dx \;(=)$$

$$\begin{array}{r|l} x^2+1 & x^2-1 \\ \hline -x^2+1 & 1 \\ \hline 2 & \end{array}$$

$$\frac{x^2+1}{x^2-1} = \frac{(x^2-1)\cdot 1 + 2}{x^2-1}$$

DIVIDENDO $\underline{\text{DIVISOR}}$ Q

R

$$= \frac{\cancel{x^2-1}}{\cancel{x^2-1}} + \frac{2}{x^2-1}$$

$$\Rightarrow \frac{\text{DIVIDENDO}}{\text{DIVISOR}} = \frac{Q \times \cancel{\text{DIVISOR}}}{\cancel{\text{DIVISOR}}} + \frac{R}{\text{DIVISOR}}.$$

$$= 1 + \frac{2}{x^2-1}$$

$$(=) \int \left(1 + \frac{2}{x^2-1}\right)dx = \int dx + 2\int \frac{dx}{x^2-1}$$

$$= x + 2\cdot\frac{1}{2}\cdot\ln\left|\frac{x-1}{x+1}\right| + C$$

$$\int \frac{dv}{v^2-a^2} = \frac{1}{2a}\cdot\ln\left|\frac{v-a}{v+a}\right| + c$$

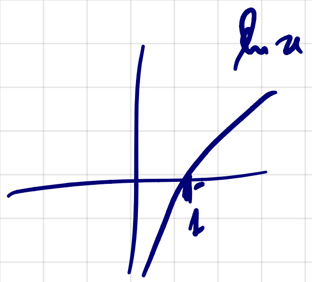
Assim, teremos:

$$I = -\int_0^{\frac{1}{2}} \frac{x^2+1}{x^2-1} = -\left(x + 2\ln\left|\frac{x-1}{x+1}\right|\right)\Bigg|_0^{\frac{1}{2}} =$$

$$= -\left[\frac{1}{2} + 2\ln\left|\frac{\frac{1}{2} - 1}{\frac{1}{2} + 1}\right| - 0 - 2\ln\left|\frac{-1}{1}\right|\right]$$

$$= -\left[\frac{1}{2} + 2\cdot\ln\left|\frac{-\frac{1}{\cancel{2}}}{\frac{3}{\cancel{2}}}\right| - 2\cdot\underbrace{\ln 1}_{=0}\right]$$

$$= -\left[\frac{1}{2} + 2\ln\left|-\frac{1}{3}\right|\right]$$



$$= -\frac{1}{2} - 2\cdot\ln\left(\frac{1}{3}\right) \approx \underline{\underline{1{,}7}}$$

LISTA 09

3. Esboçar a região entre as curvas e ache as áreas compreendidas:

(a) $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$.

(b) $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$.

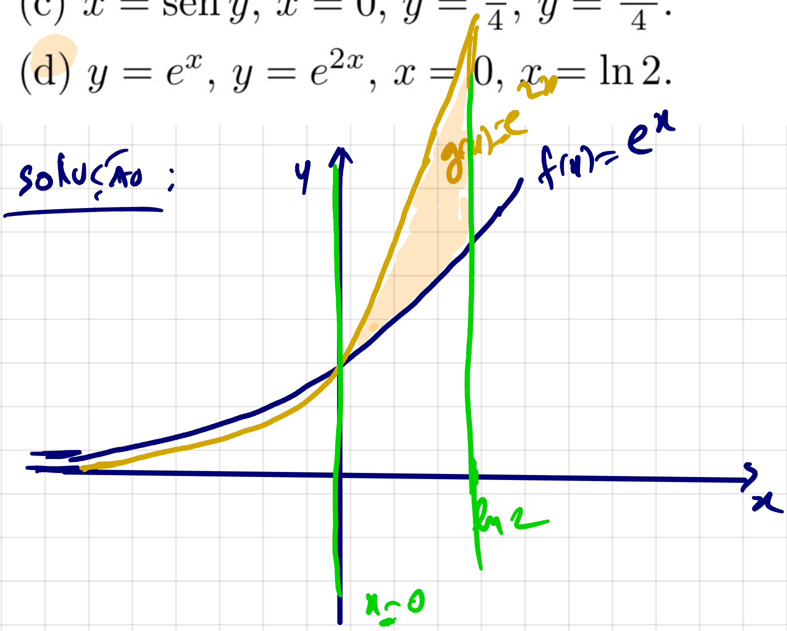
(c) $x = \operatorname{sen} y$, $x = 0$, $y = \frac{\pi}{4}$, $y = \frac{3\pi}{4}$.

(d) $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$.

$y = e^x$

x	$y = e^x$	$y = e^{2x}$
0	$e^0 = 1$	$e^0 = 1$
1	$e^1 = e$	$e^2 > e$

SOLUÇÃO:



$$A = \int_0^{\ln 2} (e^{2x} - e^x)\,dx = \frac{1}{2}\int_0^{\ln 2} e^{2x}(2dx) - \int_0^{\ln 2} e^x dx$$

$v = 2x \Rightarrow dv = 2dx$

$$= \frac{1}{2}\, e^{2x}\Big|_0^{\ln 2} - e^x\Big|_0^{\ln 2} =$$

$$= \frac{1}{2}\cdot\left(e^{2\ln 2} - e^0\right) - \left[e^{\ln 2} - e^0\right]$$

$$= \frac{1}{2}\left[e^{\ln 2^2} - 1\right] - \left[2 - 1\right] = \frac{1}{2}\left[e^{\ln 4} - 1\right] - 1$$

$$\begin{cases} a^{\log_a b} = b \\ e^{\ln a} = e^{\log_e a} = a \end{cases}$$

$$= \frac{1}{2}\left[4 - 1\right] - 1 = \frac{3}{2} - 1 = \frac{1}{2} \text{ u.a.}$$

LISTA Nº 1

2. Determine se a sequência (x_n) dada converge ou diverge. Se convergir, determine o seu limite.

(a) $x_n = \dfrac{3+5n^2}{n+n^2}$ (b) $x_n = \dfrac{\sqrt{n}}{1+\sqrt{n}}$ (c) $x_n = \dfrac{2^n}{3^{n+1}}$

Para mostrar se uma seq. converge, precisamos mostrar que ela é MONÓTONA e LIMITADA.

(c) $x_n = \dfrac{2^n}{3^{n+1}}$.

AF-01 (x_n) é decrescente (monótona)

$$\frac{x_{n+1}}{x_n} = \frac{\dfrac{2^{n+1}}{3^{n+2}}}{\dfrac{2^n}{3^{n+1}}} = \frac{2^{n+1}}{3^{n+2}} \times \frac{3^{n+1}}{2^n}$$

$$= \frac{\cancel{2^n}\cdot 2 \cdot \cancel{3^n}\cdot \cancel{3}}{\cancel{3^n}\cdot 3^{\cancel{2}}\cdot \cancel{2^n}} = \frac{2}{3} < 1$$

$$\Rightarrow \frac{x_{n+1}}{x_n} < 1, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow x_{n+1} < x_n, \quad \forall n \in \mathbb{N}$$

Logo, (x_n) é decrescente, i.e., monótona.

AF·02 (a_n) é limitada (inferiormente)

(pois é decrescente)

De fato, $a_n = \frac{2^n}{3^{n+1}} > 0$, $\forall n$.

$\Rightarrow 0 < a_n$, $\forall n$.

Logo, vale a AF 02.

Portanto, por teorema segue que (a_n) é convergente.

L11

5. Analisar a convergência das séries a seguir e determinar a sua soma.

(a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ (b) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ (c) $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

(d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (e) $\sum_{n=1}^{\infty} \frac{2n-1}{3^n}$ [Sugestão: $2n-1 = 3n-(n+1)$]

(a) $\sum_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)}$.

A ideia é efetuar uma decomposição em frações parciais:

$$a_n = \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$= \frac{A\cdot(2n+1) + B\,(2n-1)}{(2n-1)(2n+1)}$$

$\Leftrightarrow \quad 1 \equiv 2An + A + 2Bn - B$

$$\Leftrightarrow \begin{cases} 2A + 2B = 0 \\ A - B = 1 \end{cases} \rightsquigarrow A = 1 + B$$

$$2(1+B) + 2B = 0$$

$$2 + 4B = 0 \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$\Rightarrow A = 1 + B = 1 - \frac{1}{2} \Rightarrow A = \frac{1}{2}$$

Logo, o termo geral a_n será:

$$a_n = \frac{\frac{1}{2}}{2n-1} + \frac{-\frac{1}{2}}{2n+1}$$

A sequência (s_n) das somas parciais da série é:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \dots + a_n$$

$$= \sum_{k=1}^{n} \left(\frac{\frac{1}{2}}{2k-1} - \frac{\frac{1}{2}}{2k+1} \right) =$$

$$= \sum_{k=1}^{n} \frac{1}{2(2k-1)} - \frac{1}{2 \cdot (2k+1)} =$$

$$= \sum_{k=1}^{n} \frac{1}{4k-2} - \frac{1}{4k+2} =$$

$$= \underbrace{\frac{1}{2} - \cancel{\frac{1}{6}}}_{k=1} + \underbrace{\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}}_{k=2} + \underbrace{\cancel{\frac{1}{10}} - \cancel{\frac{1}{14}}}_{k=3} + \cancel{} \dots \cancel{} + \underbrace{\cancel{\frac{1}{4n-2}} - \frac{1}{4n+2}}_{k=n}$$

$$= \frac{1}{2} - \frac{1}{4n+2}$$

$$\Rightarrow s_n = \frac{1}{2} - \frac{1}{4n+2}$$

A soma S da série será:

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{2} - \underbrace{\frac{1}{4n+2}}_{0} \right) = \frac{1}{2}$$