CÁLCULO 1.

LISTA 10:

$\boxed{\text{09/03/24 — AULA 24 (extra)}}$

3. Em cada item abaixo, mostre que a função dada satisfaz as hipóteses do Teorema de Lagrange (Teor. do Valor Médio) no intervalo $[a, b]$ dado e determine o valor de $c \in \mathbb{R}$ tal que

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(a) $f(x) = x^2 + 2x - 1$ em $[0, 1]$.
(b) $f(x) = x^{\frac{2}{3}}$ em $[0, 1]$.
(c) $f(x) = \operatorname{arcsen} x$ em $[-1, 1]$.
(d) $f(x) = \ln(x - 1)$ em $[2, 4]$.
(e) $f(x) = x^3 - 2x^2 - x$ em $[-2, 1]$.
(f) $f(x) = \sqrt{1 - \operatorname{sen} x}$ em $[0, \frac{\pi}{2}]$.

(b) $f(x) = x^{2/3}$ em $[0,1]$. Esta função é cont. em $[0,1]$

Além disso, é derivável em $(0,1)$. Estamos nas hipóteses do T.V.M. Então, $\exists\, c \in [0,1]$ tal que

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$f(1) = (1)^{2/3} = 1 \quad ; \quad f(0) = 0^{2/3} = 0.$$

$$f(x) = x^{2/3} \Rightarrow f'(x) = \frac{2}{3} \cdot x^{\frac{2}{3} - 1} \cdot 1$$

$$f'(x) = \frac{2}{3} \cdot x^{-1/3} \Rightarrow \boxed{f'(x) = \frac{2}{3\sqrt[3]{x}}}.$$

Precisamos achar $c \in [0,1]$ tal que

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\frac{2}{3\sqrt[3]{c}} = \frac{1 - 0}{1 - 0} \Leftrightarrow \frac{2}{3\sqrt[3]{c}} = 1$$

$$\Leftrightarrow \left(\frac{2}{3}\right)^3 = \left(\sqrt[3]{c}\right)^3 \Leftrightarrow c = \frac{8}{27} \in [0,1]$$

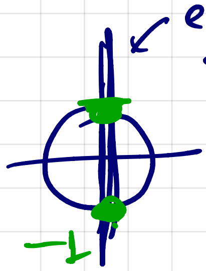
(c) $y = \operatorname{arcsen} x$ em $[-1,1]$

↳ esta função é cont. em $[-1,1]$ e derivável em $(-1,1)$.

$$(\operatorname{arcsen} x)' = \frac{x'}{\sqrt{1-x^2}}$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}.$$ Achar $c \in [-1,1]$

tal que $$f'(c) = \frac{f(1) - f(-1)}{1-(-1)}$$



$$f(1) = \operatorname{arcsen}(1) = \frac{\pi}{2}$$

$$f(-1) = \operatorname{arcsen}(-1) = -\frac{\pi}{2}$$

Disso, teremos:

$$\frac{1}{\sqrt{1-c^2}} = \frac{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}{1+1}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\pi}{2}$$

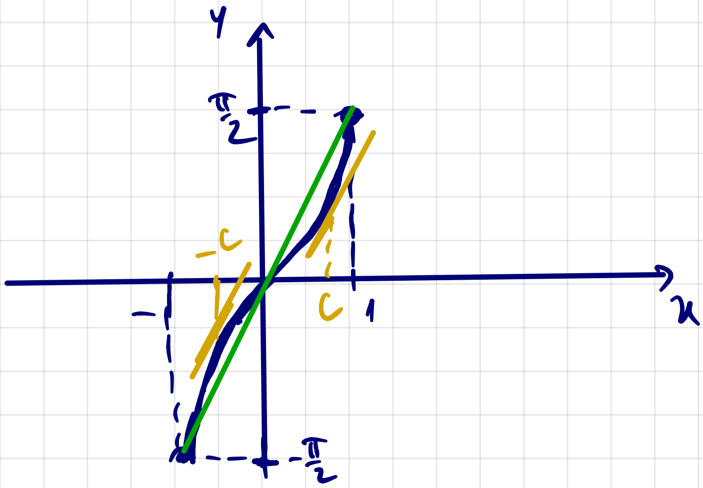
$$\left(\sqrt{1-c^2}\right)^{\cancel{2}} = \left(\frac{2}{\pi}\right)^2$$

$$1 - c^2 = \frac{4}{\pi^2}$$

$$-c^2 = \frac{4}{\pi^2} - 1$$

$$c^2 = 1 - \frac{4}{\pi^2}$$

$$c = \pm\sqrt{1 - \frac{4}{\pi^2}}$$



LISTA 08

3. Calcule a derivada de cada função abaixo:

(a) $f(x) = \sqrt[5]{\dfrac{x-1}{x+2}}$ (b) $f(x) = \ln \dfrac{\cos x}{\sqrt{4-3x^2}}$ (c) $f(x) = \csc \dfrac{1-\sqrt{x}}{\ln(1-x)}$

(d) $f(x) = e^{\sqrt{\tan(2x^{-2}+x)}}$ (e) $f(x) = \sqrt{x \cdot \operatorname{sen}(1-x)}$ (f) $f(x) = \sqrt{x} \cdot \tan e^{\sqrt{x}}$

(c) $y = \csc \dfrac{1 - x^{\frac{1}{2}}}{\ln(1-x)}$

$(\csc v)' = -\csc v \cdot \cot v \cdot v'$

$v = \dfrac{1 - x^{\frac{1}{2}}}{\ln(1-x)} \Rightarrow v' = \left(\dfrac{u}{w}\right)' = \dfrac{w \cdot u' - u \cdot w'}{w^2}$

Neste caso, temos:

$$v' = \frac{\ln(1-x) \cdot \left(-\frac{1}{2} x^{-\frac{1}{2}}\right) - \left(1 - x^{\frac{1}{2}}\right) \cdot \frac{-1}{1-x}}{[\ln(1-x)]^2}$$

$$v' = \frac{\dfrac{\ln(1-x)}{2\sqrt{x}} + \dfrac{(1-\sqrt{x})}{1-x}}{\ln^2(1-x)}$$

Portanto, obtemos:

$$y' = -\csc \frac{1 - x^{\frac{1}{2}}}{\ln(1-x)} \cdot \cot \frac{1 - x^{\frac{1}{2}}}{\ln(1-x)} \cdot \frac{\dfrac{\ln(1-x)}{2\sqrt{x}} + \dfrac{(1-\sqrt{x})}{1-x}}{\ln^2(1-x)}$$

(e) $y = \sqrt{x \cdot \operatorname{sen}(1-x)}$. $\quad y' = ?$

$$y = \left[x \cdot \operatorname{sen}(1-x)\right]^{1/2}$$

$$(v^k)' = k \cdot v^{k-1} \cdot \underline{v'}$$

$$v = \underbrace{x}_{u} \cdot \underbrace{\operatorname{sen}(1-x)}_{w} = u \cdot w$$

$$\Rightarrow v' = u \cdot w' + u' \cdot w$$

$$\begin{cases} \underline{u = x} \Rightarrow u' = 1 \\ w = \operatorname{sen}(1-x) \Rightarrow w' = \underline{\cos(1-x) \cdot (-1)} \end{cases}$$

Assim, teremos:

$$v' = -x \cdot \cos(1-x) + 1 \cdot \operatorname{sen}(1-x)$$

Desta forma, obtemos:

$$\underset{\sim}{y'} = \left(\left[x \cdot \operatorname{sen}(1-x)\right]^{\frac{1}{2}}\right)' =$$

$$= \frac{1}{2} \cdot \left(x \cdot \operatorname{sen}(1-x)\right)^{\frac{1}{2}-1} \cdot \left[-x \cdot \cos(1-x) + \operatorname{sen}(1-x)\right] \qquad \frac{1}{2}-1 = -\frac{1}{2}$$

$$= \frac{\operatorname{sen}(1-x) - x \cdot \cos(1-x)}{2\sqrt{x \cdot \operatorname{sen}(1-x)}}$$

(f) $y = \sqrt{x} \cdot \tan e^{\sqrt{x}}$. $y' = ?$

$$y = x^{\frac{1}{2}} \tan e^{x^{\frac{1}{2}}} = u \cdot v \; ;$$

$(\tan w)' = \sec^2 w \cdot w'$

$$\left\{ \begin{array}{l} u = x^{\frac{1}{2}} \Rightarrow u' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} \cdot 1 = \frac{1}{2\sqrt{x}} \\ v = \tan e^{x^{\frac{1}{2}}} \Rightarrow v' = \sec^2 e^{x^{\frac{1}{2}}} \cdot \left(e^{x^{\frac{1}{2}}}\right)' \end{array} \right.$$

$\frac{1}{2}-1 = -\frac{1}{2}$

$$= \sec^2 e^{\sqrt{x}} \cdot e^{\sqrt{x}} \cdot (\sqrt{x})'$$

$(e^w)' = e^w \cdot w'$

$$= \sec^2 e^{\sqrt{x}} \cdot e^{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} \cdot 1$$

$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \cdot \sec^2 e^{\sqrt{x}} = v'$$

$$\boxed{y' = (u \cdot v)' = u \cdot v' + u' \cdot v}$$

Ansin, teremos:

$$y' = \sqrt{x} \cdot \frac{e^{\sqrt{x}}}{2\sqrt{x}} \cdot \sec^2 e^{\sqrt{x}} + \frac{1}{2\sqrt{x}} \cdot \tan e^{\sqrt{x}} .$$

$$y' = \frac{1}{2\sqrt{x}} \cdot \left(\sqrt{x} \cdot e^{\sqrt{x}} \cdot \sec^2 e^{\sqrt{x}} + \tan e^{\sqrt{x}} \right) ,$$

L8.

Calcule a derivada de cada função implícita abaixo.

(a) $y^3 - 3y + 2ax = 0$ (b) $\cos(xy) = x$ (c) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

(d) $y = \cos(x+y)$ (e) $\sqrt{xy} + 2y = \sqrt{x}$ (f) $\ln(x^2+y^2) - 3x^2y^3 = \sqrt{x+y}$

(b) $\cos(xy) = x$. $y' = ?$

$(\cos u)' = -\operatorname{sen} u \cdot u'$

Derivando toda a igualdade em x, vamos obter:

$$-\operatorname{sen}(xy) \cdot (xy)' = 1$$

$$-\operatorname{sen}(xy) \cdot [x \cdot y' + 1 \cdot y] = 1$$

$$-x \cdot \operatorname{sen}(xy) \cdot y' - y \cdot \operatorname{sen}(xy) = 1.$$

$$\boxed{y' = -\frac{1 + y \cdot \operatorname{sen}(xy)}{x \cdot \operatorname{sen}(xy)}}$$

(e) $\sqrt{xy} + 2y = \sqrt{x}$.

$$(xy)^{1/2} + 2y = x^{1/2}$$

$(u^k)' = k u^{k-1} \cdot u'$

Derivando em x, vamos obter:

$$\frac{1}{2}(xy)^{\frac{1}{2}-1} \cdot (x \cdot y)' + 2 \cdot y' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} \cdot 1.$$

$-\frac{1}{2}$

$$\frac{1}{2\sqrt{xy}} [x \cdot y' + y \cdot 1] + 2y' = \frac{1}{2\sqrt{x}}$$

$$\frac{x}{2\sqrt{xy}} \cdot y' + \frac{y}{2\sqrt{xy}} + 2y' = \frac{1}{2\sqrt{x}}$$

$$y' \cdot \left[\frac{x}{2\sqrt{xy}} + 2 \right] = \frac{1}{2\sqrt{x}} - \frac{y}{2\sqrt{xy}}$$

$$\boxed{y' = \frac{\frac{1}{2\sqrt{x}} - \frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}} + 2}}$$

(f) $y = \ln(x^2+y^2) - 3x^2y^3 = (x+y)^{\frac{1}{2}}$.

Derivando em x, vamos obter,

lembrete.

$$\boxed{\begin{aligned} (\ln u)' &= \frac{u'}{u} \\ (u^k)' &= k u^{k-1} \cdot u' \\ (u \cdot v)' &= u v' + u' v \end{aligned}}$$

$$\frac{(x^2+y^2)'}{x^2+y^2} - 3 \cdot \left[x^2 \cdot 3y^2 \cdot y' + 2x \cdot 1 \cdot y^3 \right] =$$

$$= \frac{1}{2}(x+y)^{\frac{1}{2}-1} \cdot (1+y')$$

$$\frac{2x + 2y \cdot y'}{x^2+y^2} - 9x^2y^2 \cdot y' - 6xy^3 = \frac{1}{2\sqrt{x+y}} \cdot (1+y')$$

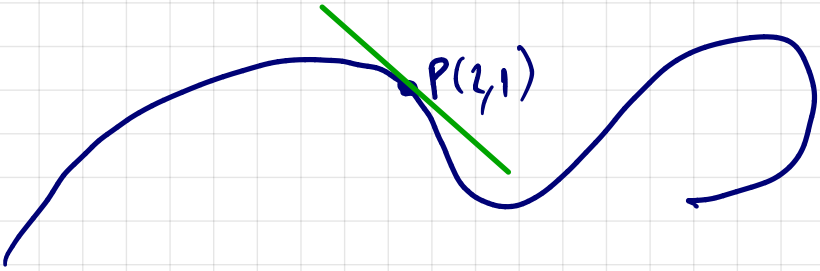
$$\frac{2x}{x^2+y^2} + \underline{\frac{2y}{x^2+y^2} \cdot y'} - \underline{9x^2y^2 \cdot y'} - 6xy^3 = \frac{1}{2\sqrt{x+y}} + \underline{\frac{1}{2\sqrt{x+y}} \cdot y'}$$

$$\left(\frac{2y}{x^2+y^2} - 9x^2y^2 - \frac{1}{2\sqrt{x+y}} \right) \cdot y' = \frac{1}{2\sqrt{x+y}} + 6xy^3 - \frac{2x}{x^2+y^2}$$

$$\boxed{y' = \frac{\frac{1}{2\sqrt{x+y}} + 6xy^3 - \frac{2x}{x^2+y^2}}{\frac{2y}{x^2+y^2} - 9x^2y^2 - \frac{1}{2\sqrt{x+y}}}}$$

L8.

6. Ache a equação da reta tangente à curva de equação $x^3 + y^3 = 2xy + 5$ em $P(2,1)$.



Eq. da reta tangente em $P(2,1)$:

$$y - y_p = y'_p \cdot (x - x_p) \qquad \begin{cases} x_p = 2 \\ y_p = 1 \end{cases}$$

$y'_p = ?$ inclinação da reta tangente.

$y' = ?$

$$x^3 + y^3 = 2xy + 5$$

Derivando em x, obtemos:

$$3x^2 + 3y^2 \cdot y' = \underbrace{2x \cdot y' + 2 \cdot y}_{u \cdot v' + u' \cdot v} + 0$$

$$3y^2 \cdot y' - 2x \cdot y' = 2y - 3x^2$$

$$y'(3y^2 - 2x) = 2y - 3x^2$$

$$\Rightarrow \boxed{y' = \frac{2y - 3x^2}{3y^2 - 2x}}$$

$$y'_p = y'(2,1) = \frac{2\cdot(1) - 3\cdot(2)^2}{3\cdot(1)^2 - 2\cdot(2)} = \frac{2-12}{3-4} = 10$$

Assim, a eq. da reta ficará:

$$y - y_p = y'_p \cdot (x - x_p)$$

$$y - 1 = 10 \cdot (x - 2)$$

$$y - 1 = 10x - 20$$

$$y = 10x - 20 + 1$$

$$\boxed{y = 10x - 19}$$