Fundação Universidade Federal de Pelotas
Departamento de Matemática e Estatística
Curso de Lic. em Matemática
Primeira Prova de Cálculo IV
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Nome: **Data:** 07/02/2024

Questão 01. [1,0 pt] Seja $X = \prod_{i=1}^{m} \mathbb{Q} \cap [0,1] \subset \mathbb{R}^m$ o conjunto de todas as m-uplas ordenadas com coordenadas racionais no interior do bloco m-dimensional do $\mathbb{R}^m$. Este conjunto é mensurável segundo Jordan? Justifique.

Questão 02. Seja $f : \mathbb{R}^2 \to \mathbb{R}$ a função dada por $f(x,y) = x^2 + xy$ e considere o bloco bidimensional $A = [0,2] \times [1,4]$.

(a) [0,5 pt] Sem calcular esta integral, encontre um intervalo fechado que contenha o valor da integral dupla de f.

(b) [1,5 pt] Use a definição de integral dupla como limite de somas de Riemann para calcular $\int_A f$.

(c) [0,5 pt] Calcule $\int_A f$ via integração iterada.

(d) [0,5 pt] Determine o valor médio de f em A.

Questão 03. [1,0 pt] Calcule $\displaystyle\iint_{\Omega} \arctan \frac{y}{x} dA$, onde Ω é a região dada por

$$\Omega = \{(x,y) \in \mathbb{R}^2 \,:\, 1 \leq x^2 + y^2 \leq 4,\ 0 \leq y \leq x\}.$$

Questão 04. [1,0 pt cada] Calcule as integrais abaixo:

(a) $\displaystyle\int_0^1 \int_y^1 \frac{\operatorname{sen} x}{x} dxdy$ (b) $\displaystyle\int_0^1 \int_0^z \int_0^y z e^{-y^2} dxdydz$ (c) $\displaystyle\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} dxdy$

Questão 05. [2,0 pt] Calcule

$$\iint_{\Omega} \frac{1}{x-y} \cos(2x+3y) dxdy,$$

onde Ω é a região do plano xy definida pelo quadrilátero $ABCD$ de vértices $A(\frac{3}{5}, -\frac{2}{5})$, $B(\frac{6}{5}, -\frac{4}{5})$, $C(\frac{9}{5}, -\frac{1}{5})$ e $D(\frac{6}{5}, \frac{1}{5})$.

Questão 06. [1,5 pt] Use integrais triplas para calcular o volume do sólido limitado pelo cilindro $x^2 + y^2 = 25$, pelo plano $x + y + z = 8$ e pelo plano xy.

01) $X = \prod_{i=1}^{m} \mathbb{Q} \cap [0,1] \subset \mathbb{R}^m = \{(x_1, x_2, \dots, x_m) : x_i \in \mathbb{Q}$ e $x_i \in [0,1]\}$

Seja $A = [0,1]^m$ um cubo m-dimensional.
Logo, $X \subset A$.

Para ver se X é J-mensurável, precisamos verificar se a função característica $\xi_X : A \to \mathbb{R}$, dada por

$$\xi_X(x) = \begin{cases} 1, & \text{se } x \in X \\ 0, & \text{se } x \notin X \end{cases},$$

é integrável, ou seja, se $\exists \int_A \xi_X$.

Seja P uma partição do bloco A determinando sub-blocos $B \in P$. Note que, $\forall x \in B$, temos

$$m_B = \inf_{x \in A} f(x) = 0 \; ; \; M_B = \sup_{x \in A} f(x) = 1;$$

devido à densidade de $\mathbb{Q}$ e do $\mathbb{I}$. Assim, teremos

$$\underline{\int_A} \xi_X = \sum_{B \in P} \underbrace{m_B}_{=0} \cdot Vol(B) = 0 \; ;$$

e

$$\overline{\int_A} \xi_X = \sum_{B \in P} \underbrace{M_B}_{=1} Vol(B) = \sum_{B \in P} Vol\, B = Vol(A) = 1^m = 1$$

Ou seja, $\underline{\int_A} \xi_X \neq \overline{\int_A} \xi_X$; i.e., ξ_X não é integrável.

Portanto, o conj. X não é J-mensurável.

02) $f(x,y) = x^2 + xy$; $A = \overbrace{[0,2]}^{x} \times \overbrace{[1,4]}^{y}$.

(a) Como A é compacto do $\mathbb{R}^2$ (limitado e fechado), pelo T. de Weierstrass segue que f assume valores máximo e mínimo em A; e serão dados por:

- $\max_A f = (x^2 + xy)\Big|_{\substack{x=2\\y=4}} = (2)^2 + 2.4 = 12$

- $\min_A f = (x^2 + xy)\Big|_{\substack{x=0\\y=1}} = 0^2 + 0.1 = 0$

Assim, $\forall (x,y) \in A$, temos:

$$\min_A f \leq f(x,y) \leq \max_A f \text{ , i.e.;}$$

$$0 \leq f(x,y) \leq 12$$

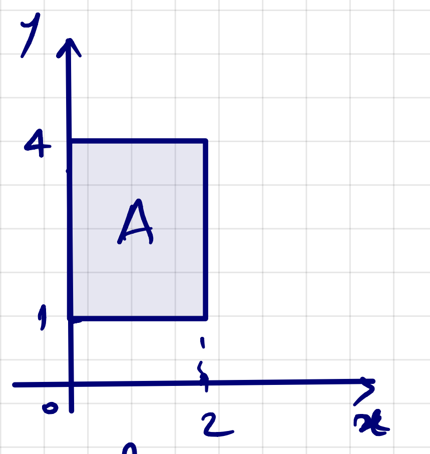
integrando em A; obtemos

$$\int_A 0 \leq \int_A f \leq \int_A 12$$

$$0 \leq \int_A f \leq 12 . \underbrace{\int_A 1}_{= \text{Vol } A}$$

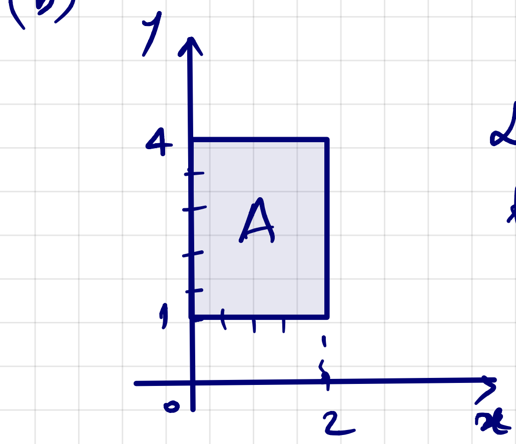
$$0 \leq \int_A f \leq 12 . 6$$

$$\Longrightarrow \boxed{0 \leq \int_A f \leq 72}$$



Vol(A) = 2.3 = 6

(b)

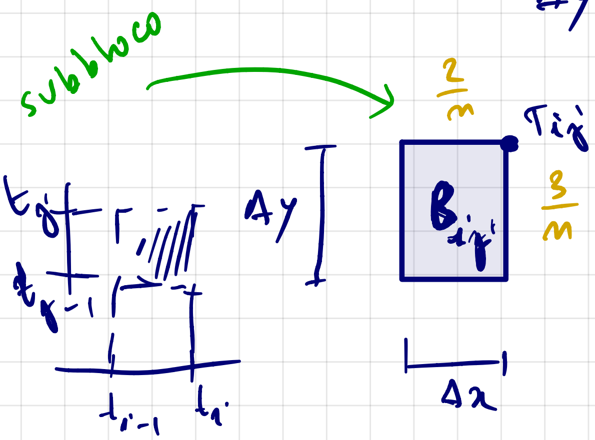


Seja $P = P_1 \times P_2$ partição regular que divide o bloco A em subblocos B, tal que divide P_1 em n subintervalos de comprimento

$$\Delta x = \frac{2-0}{n} = \frac{2}{n} \quad \text{e}$$

divide P_2 em n subintervalos de comprimento

$$\Delta y = \frac{4-1}{n} = \frac{3}{n}.$$



$$P_1 = \left\{0 < \frac{2}{n} < 2 \cdot \frac{2}{n} < \dots < n \cdot \frac{2}{n} = 2\right\}$$

$$P_2 = \left\{1 < 1 + \frac{3}{n} < 1 + 2 \cdot \frac{3}{n} < \dots < 1 + n \cdot \frac{3}{n} = 4\right\}$$

Como $f(x,y) = x^2 + xy$ é crescente em $A = [0,2] \times [1,4]$ então seu valor máximo em cada subbloco $B_{ij} \in P$ ocorre no vértice superior direito, i.e, no ponto T_{ij} destacado acima, onde $x = i \cdot \frac{2}{n}$ e $y = 1 + j \cdot \frac{3}{n}$.

Ou seja;

$$M_B = \max_{B_{ij}} f = f(x,y)\Big|_{\substack{x = \frac{2i}{n} \\ y = 1 + \frac{3j}{n}}} = \left(\frac{2i}{n}\right)^2 + \frac{2i}{n} \cdot \left(1 + \frac{3j}{n}\right)$$

$$= \frac{4i^2}{n^2} + \frac{2i}{n} + \frac{6ij}{n^2}$$

Além disso, o volume do bloco $B_{i'j'}$ será

$$\mathrm{vol}(B) = \Delta x \cdot \Delta y = \frac{2}{n} \cdot \frac{3}{n} = \frac{6}{n^2}.$$

Disso, temos a soma superior de f:

$$S(f;P) = \sum_{B \in P} M_B \cdot \mathrm{Vol}(B) = \sum_{j'=1}^{n} \sum_{i'=1}^{n} \left(\frac{4 i'^2}{n^2} + \frac{2i'}{n} + \frac{6 i' j'}{n^2} \right) \cdot \frac{6}{n^2} =$$

$$= \sum_{j'=1}^{n} \sum_{i'=1}^{n} \left(\frac{24 i'^2}{n^4} + \frac{12 i'}{n^3} + \frac{36 i' j'}{n^4} \right) =$$

$$= \frac{24}{n^4} \cdot \sum_{j'=1}^{n} 1 \cdot \sum_{i'=1}^{n} i'^2 + \frac{12}{n^3} \cdot \sum_{j'=1}^{n} 1 \cdot \sum_{i'=1}^{n} i' + \frac{36}{n^4} \cdot \sum_{j'=1}^{n} j' \cdot \sum_{i'=1}^{n} i' =$$

$$= \frac{\overset{4}{\cancel{24}}}{\cancel{n^4}_{2}} \cdot \cancel{n} \cdot \frac{\cancel{n}(n+1) \cdot (2n+1)}{\cancel{6}} + \frac{\overset{6}{\cancel{12}}}{\cancel{n^3}} \cdot \cancel{n} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{\overset{9}{\cancel{36}}}{\cancel{n^4}_{2}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}}$$

$$= 4 \cdot \left(\frac{n+1}{n} \right) \cdot \left(\frac{2n+1}{n} \right) + 6 \cdot \left(\frac{n+1}{n} \right) + 9 \left(\frac{n+1}{n} \right) \cdot \left(\frac{n+1}{n} \right)$$

$$= 4 \cdot \left(1 + \frac{1}{n} \right) \cdot \left(2 + \frac{1}{n} \right) + 6 \cdot \left(1 + \frac{1}{n} \right) + 9 \cdot \left(1 + \frac{1}{n} \right) \cdot \left(1 + \frac{1}{n} \right)$$

$$\Rightarrow \overline{\int_A} f = \lim_{n \to \infty} S(f;P) = 4 \cdot 1 \cdot 2 + 6 \cdot 1 + 9 \cdot 1 \cdot 1$$

$$= 8 + 6 + 9 = 23$$

Analogamente se mostra que $\int_{-A} f = 32.$

Portanto, $\int_A f = 32.$

(c) $A = [\underbrace{0,2}_{x}] \times [\underbrace{1,4}_{y}]$. Assim, temos:

$$\int_A f = \int_{x=0}^{x=2} \left(\int_{y=1}^{y=4} (x^2 + xy)\, dy \right) dx = \int_{x=0}^{x=2} \left(x^2 y + x\frac{y^2}{2} \right) \Bigg|_{y=1}^{y=4} =$$

$$= \int_{x=0}^{x=2} \left(4x^2 + 8x - x^2 - \frac{x}{2} \right) dx = \int_0^2 \left(3x^2 + \frac{15}{2} x \right) dx =$$

$$= \left(x^3 + \frac{15}{2} \frac{x^2}{2} \right) \Bigg|_0^2 = 8 + 15 - 0 - 0 = \underline{\underline{23}}$$

(d) Seja $f(\alpha, \beta)$ o valor médio de f em A.

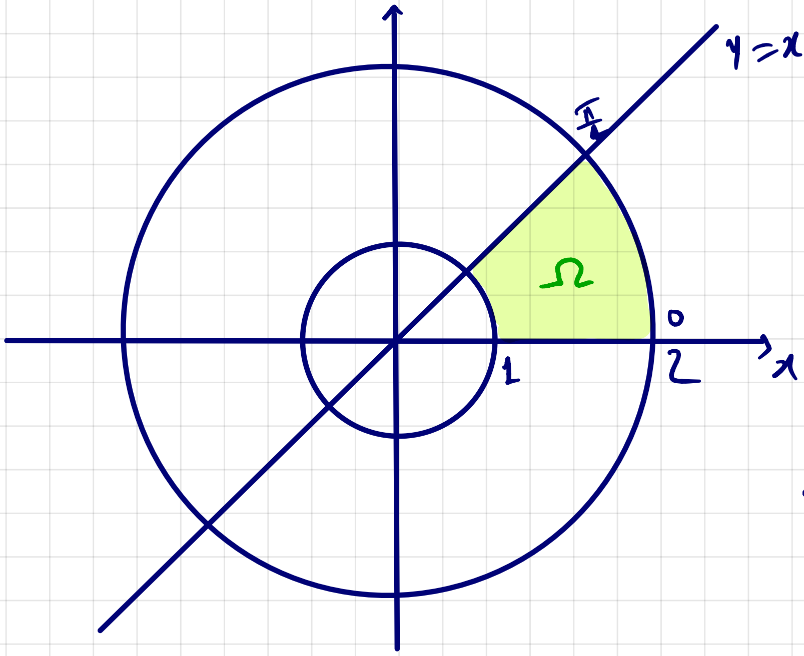
Então,

$$f(\alpha, \beta) = \frac{\int_A f}{\text{Vol}(A)};$$

e como $\text{Vol}(A) = 2.3 = 6$, segue que

$$f(\alpha, \beta) = \frac{23}{6} \quad //$$

03)



$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$

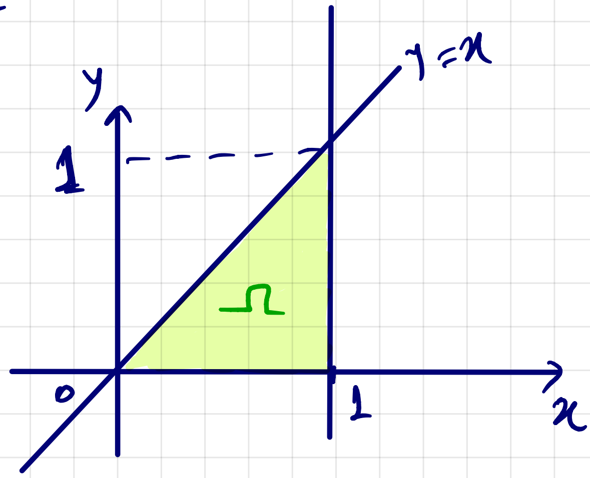
Assim:

$$\iint_{\Omega} \arctan\left(\frac{y}{x}\right) dA = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{\rho=1}^{\rho=2} \theta \cdot \rho \cdot d\rho \, d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \theta \, d\theta \cdot \int_{\rho=1}^{\rho=2} \rho \, d\rho = \left.\frac{\theta^2}{2}\right|_0^{\frac{\pi}{4}} \cdot \left.\frac{\rho^2}{2}\right|_1^2 =$$

$$= \left(\frac{\frac{\pi^2}{16}}{2} - 0\right) \cdot \left(\frac{4}{2} - \frac{1}{2}\right) = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3\pi^2}{64}$$

04) (a) $$\int_{y=0}^{y=1}\int_{x=y}^{x=1} \frac{\operatorname{sen} x}{x}\, dx\, dy = ?$$



$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} \frac{\operatorname{sen} x}{x}\, dx\, dy = \int_{x=0}^{x=1}\int_{y=0}^{y=x} \frac{\operatorname{sen} x}{x}\, dy\, dx = \int_{x=0}^{x=1} \frac{\operatorname{sen} x}{x}\left(\int_{y=0}^{y=x} dy\right)dx$$

TROCA DE ORDEM DE INTEGRAÇÃO

$$= \int_{x=0}^{x=1} \frac{\operatorname{sen} x}{x} \cdot y\Big|_{y=0}^{y=x} dx = \int_{x=0}^{x=1} \frac{\operatorname{sen} x}{\cancel{x}} \cdot \cancel{x}\, dx = \int_{x=0}^{x=1} \operatorname{sen} x\, dx$$

$$= -\cos x\Big|_0^1 = -\cos(1) - (-\cos(0)) = \underline{\underline{1 - \cos(1)}}$$

(b) $$\int_{z=0}^{z=1}\int_{y=0}^{y=z}\left(\int_{x=0}^{x=y} z\cdot e^{-y^2} dx\right)dy\, dz = \int_{z=0}^{z=1}\int_{y=0}^{y=z} z e^{-y^2} \int_{x=0}^{x=y} dx\, dy\, dz =$$

$$= \int_{z=0}^{z=1}\int_{y=0}^{y=z} z e^{-y^2} \cdot x\Big|_{x=0}^{x=y} dy\, dz = \int_{z=0}^{z=1} z \int_{y=0}^{y=z} e^{-y^2} \cdot y \cdot dy\, dz =$$

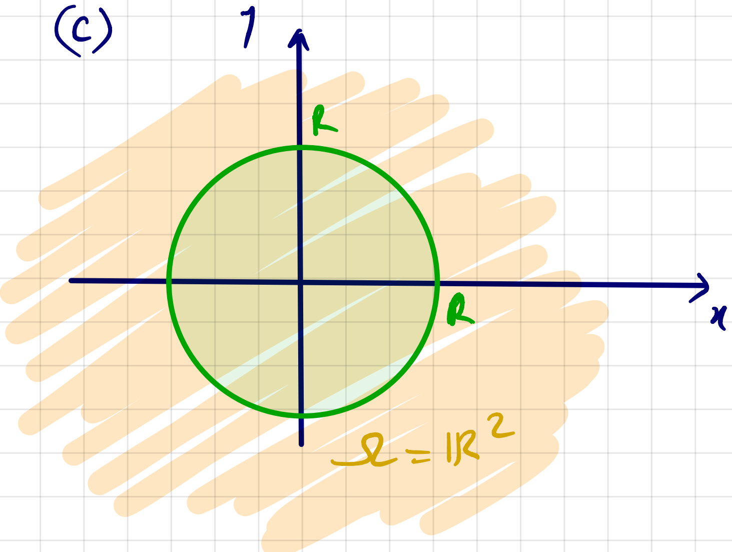
$$= \int_{z=0}^{z=1} z\cdot\left(-\frac{1}{2}\right)\int_{y=0}^{y=z} e^{-y^2}\cdot(-2y\,dy)\cdot dz = \int_{z=0}^{z=1} -\frac{z}{2}\cdot e^{-y^2}\Big|_{y=0}^{y=z} dz =$$

$$= \frac{1}{2}\int_0^1 \left(e^{-z^2}-e^{0}\right)(-z\,dz) = \frac{1}{2}\cdot\frac{1}{2}\int_0^1 e^{-z^2}\cdot(-2z\,dz)+\frac{1}{2}\int_0^1 z\,dz$$

$$= \frac{1}{4}\cdot e^{-z^2}\Big|_0^1 + \frac{1}{2}\frac{z^2}{2}\Big|_0^1 =$$

$$= \frac{1}{4}\cdot\left(e^{-1}-e^{0}\right)+\frac{1}{4}\cdot(1-0) = \frac{1}{4}\cdot e^{-1}\cancel{-\frac{1}{4}}\cancel{+\frac{1}{4}} = \frac{1}{4e}$$

(c)



Seja $R>0$ e considere

$$\Omega_R = B_R(0).$$

Então,

$$\iint_{\Omega} f = \lim_{R\to+\infty}\iint_{\Omega_R} f.$$

Assim, usando coordenadas polares, vamos obter:

$$\iint_{\Omega} f = \lim_{R\to+\infty}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=R} e^{-\rho^2}\cdot\rho\,d\rho\,d\theta =$$

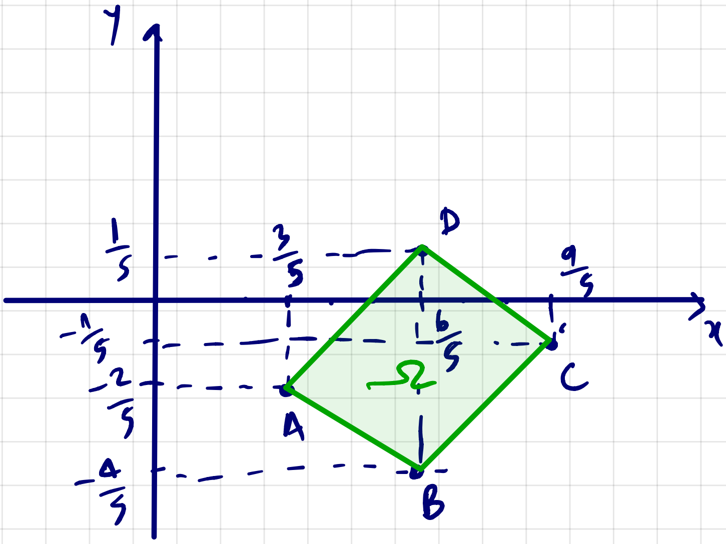
$$= \lim_{R\to+\infty}\int_{\theta=0}^{\theta=2\pi} d\theta\cdot\left(-\frac{1}{2}\right)\int_{\rho=0}^{\rho=R} e^{-\rho^2}\cdot(-2\rho)\,d\rho =$$

$$= -\frac{1}{2} \cdot \theta\Big|_0^{2\pi} \cdot \lim_{R\to+\infty} e^{-\rho^2}\Big|_0^{R} =$$

$$= -\frac{1}{2} \cdot 2\pi \cdot \lim_{R\to+\infty} \left(e^{-R^2} - 1\right) = -\pi \cdot \lim_{R\to+\infty} \left(\frac{1}{e^{R^2}} - 1\right)$$

$$= -\pi \cdot (0 - 1) = \underline{\underline{\pi}}$$

05) $\displaystyle\iint_\Omega \frac{1}{x-y} \cdot \cos(2x+3y)\,dx\,dy$



Efetuando a mudança de variáveis:

$$\begin{cases} u = x - y \rightsquigarrow x = u + y \\ v = 2x + 3y \end{cases}$$

$$v = 2(u+y) + 3y$$

$$\Rightarrow v = 2u + 5y$$

$$\Rightarrow \boxed{y = -\frac{2}{5}u + \frac{1}{5}v}$$

$$x = u + y = u - \frac{2}{5}u + \frac{1}{5}v$$

$$\Rightarrow \boxed{x = \frac{3}{5}u + \frac{1}{5}v}$$

A transformação será

$$T(u,v) = (x,y) = \left(\frac{3}{5}u + \frac{1}{5}v, -\frac{2}{5}u + \frac{1}{5}v\right)$$

O determinante da matriz jacobiana da T será:

$$\det(J(T)(u,v)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \partial y \end{vmatrix} = \begin{vmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{vmatrix} = \frac{3}{25} + \frac{2}{25} = \frac{1}{5}$$

Região Ω' no plano uv:

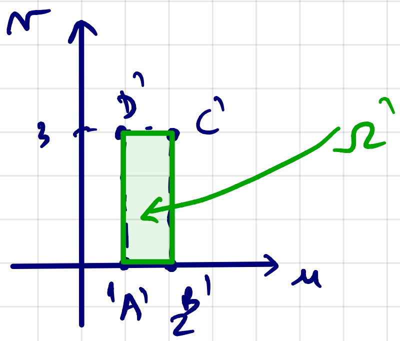
$$(x,y) \xrightarrow{T} (u,v) = (x-y,\ 2x+3y)$$

$$A\left(\tfrac{3}{5}, -\tfrac{2}{5}\right) \longrightarrow \left(\tfrac{3}{5}+\tfrac{2}{5},\ \tfrac{6}{5}-\tfrac{6}{5}\right) = (1,0)\ A'$$

$$B\left(\tfrac{6}{5}, -\tfrac{4}{5}\right) \longrightarrow \left(\tfrac{6}{5}+\tfrac{4}{5},\ \tfrac{12}{5}-\tfrac{12}{5}\right) = (2,0)\ B'$$

$$C\left(\tfrac{9}{5}, -\tfrac{1}{5}\right) \longrightarrow \left(\tfrac{9}{5}+\tfrac{1}{5},\ \tfrac{18}{5}-\tfrac{3}{5}\right) = (2,3)\ C'$$

$$D\left(\tfrac{6}{5}, \tfrac{1}{5}\right) \longrightarrow \left(\tfrac{6}{5}-\tfrac{1}{5},\ \tfrac{12}{5}+\tfrac{3}{5}\right) = (1,3)\ D'$$



Assim, temos:

$$\iint_{\Omega} \frac{1}{x-y} \cdot \cos(2x+3y)\,dx\,dy = \iint_{\Omega'} \frac{1}{u} \cdot \cos v\, \left|\det J(T)(u,v)\right| \cdot du\,dv$$

$$= \int_{v=0}^{v=3} \int_{u=1}^{u=2} \frac{1}{u} \cdot \cos v \cdot \left|\frac{1}{5}\right| \cdot du\,dv = \frac{1}{5} \cdot \int_{v=0}^{v=3} \cos v\,dv \cdot \int_{u=1}^{u=2} \frac{du}{u} =$$

$$= \frac{1}{5}\cdot(\operatorname{sen} r)\Big|_{r=0}^{r=3} \cdot \ln|u|\Big|_{u=1}^{u=2} = \frac{1}{5}\cdot(\operatorname{sen}3 - \underbrace{\operatorname{sen}0}_{=0})\cdot(\ln 2 - \underbrace{\ln 1}_{=0})$$

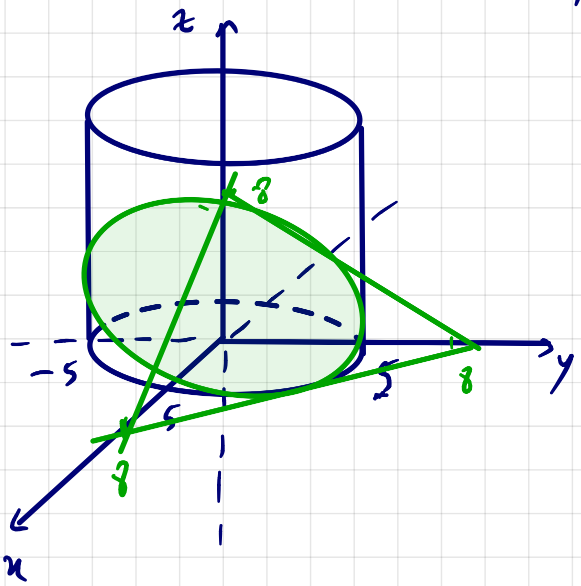
$$= \frac{1}{5}\cdot \operatorname{sen}3 \cdot \ln 2$$

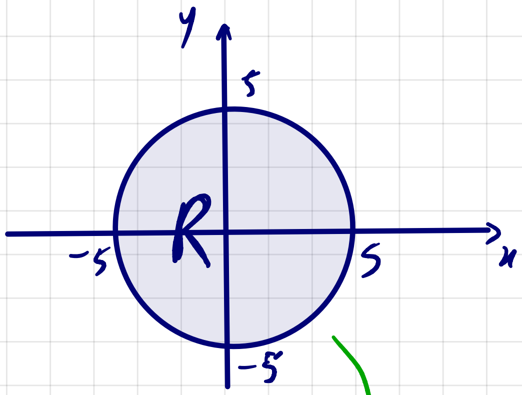
06) $x+y+z=8 \Rightarrow z = 8-x-y$

$x=0$ (plano yz): $y+z=8$

$y=0$ (plano xz): $x+z=8$

$z=0$ (plano xy): $x+y=8$





$$V = \iint_R \left(\int_{z=0}^{z=8-x-y} dz\right) dx\,dy =$$

$$= \iint_R z\Big|_{z=0}^{z=8-x-y} dx\,dy = \iint_R (8-x-y)\,dx\,dy =$$

COORDENADAS POLARES NO PLANO xy

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=5} (8-\rho\cos\theta-\rho\operatorname{sen}\theta)\cdot\rho\,d\rho\,d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\int_{\rho=0}^{\rho=5} 8\rho - \rho^2(\cos\theta + \operatorname{sen}\theta)\, d\rho \right) d\theta = \int_{\theta=0}^{\theta=2\pi} \left(4\rho^2 - \frac{\rho^3}{3}(\cos\theta + \operatorname{sen}\theta) \right) \Bigg|_{\rho=0}^{\rho=5} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[100 - \frac{125}{3}(\cos\theta + \operatorname{sen}\theta) - 0 \right] d\theta =$$

$$100 \int_0^{2\pi} d\theta - \frac{125}{3} \int_0^{2\pi} \cos\theta \, d\theta - \frac{125}{3} \int_0^{2\pi} \operatorname{sen}\theta \, d\theta =$$

$$100 \cdot \theta \Big|_0^{2\pi} - \frac{125}{3} \cdot \operatorname{sen}\theta \Big|_0^{2\pi} - \frac{125}{3} \cdot (-\cos\theta) \Big|_0^{2\pi} =$$

$$200\pi - \frac{125}{3} \cdot \Big(\underbrace{\operatorname{sen} 2\pi - \operatorname{sen} 0}_{=0} \Big) + \frac{125}{3} \cdot \Big(\underbrace{\cos 2\pi}_{1} - \underbrace{\cos 0}_{1} \Big)$$

$$= 200\pi - 0 + 0 = \underline{\underline{200\pi}}$$ unidades de volume.