CÁLCULO IV

09/02/2024 – AULA 14

MUDANÇA GERAL DE VARIÁVEIS NO $\mathbb{R}^3$.

Do mesmo modo que se fez no caso do $\mathbb{R}^2$, tem-se uma fórmula para mudar de variável no caso $\mathbb{R}^3$ (e que, no fim das contas, será a mesma). A sua construção será similar àquela feita no caso $\mathbb{R}^2$, com suas devidas adaptações.

Dada $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável, queremos efetuar uma mudança de variáveis adequada, de modo que o cálculo de

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz$$

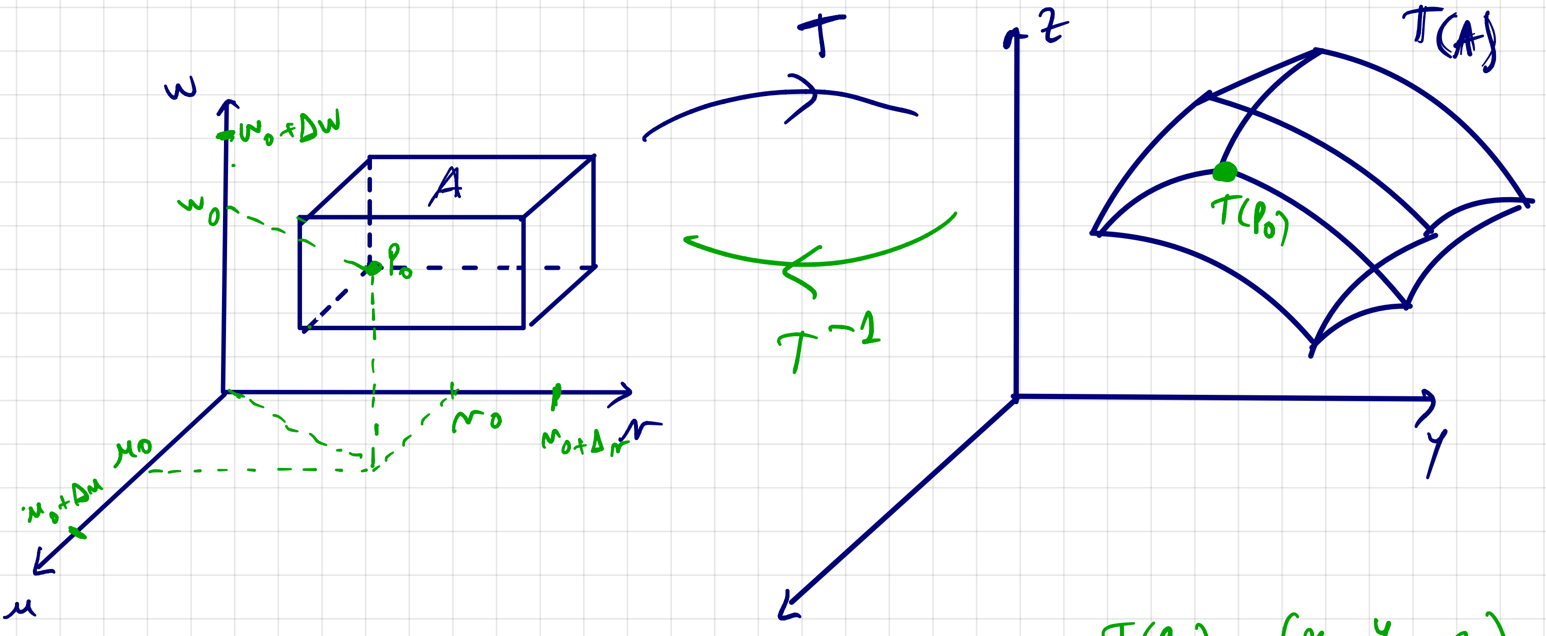
torne-se mais simples ou calculável.

Seja $T: \Omega \subset \mathbb{R}^3 \to T(\Omega) = \Omega' \subset \mathbb{R}^3$ uma transformação injetiva e de classe C^1. [i.e., com as suas derivadas parciais contínuas], dada por

$$T(u,v,w) = (x,y,z), \text{ onde}$$

$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w).$$

Seja A um paralelepípedo no sistema $u\,v\,w$, de dimensões Δu, Δv e Δw, e seja $P_0(u_0, v_0, w_0)$ no vértice inferior esquerdo, "ao fundo" do paralelepípedo, c.f. o esquema abaixo.



$T(P_0) = (x_0, y_0, z_0)$

Sejam γ_1, γ_2 e γ_3, respectivamente, as curvas definidas por:

$$\gamma_1 : [u_0, u_0+\Delta u] \longrightarrow \mathbb{R}^3,$$

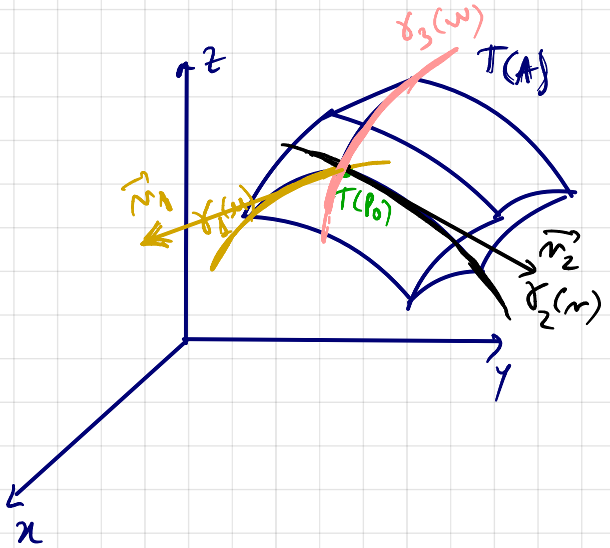
$$\gamma_1(u) = \big(x(u, v_0, w_0),\ y(u, v_0, w_0),\ z(u, v_0, w_0)\big)$$

$$\gamma_2 : [v_0, v_0+\Delta v] \longrightarrow \mathbb{R}^3,$$

$$\gamma_2(v) = \big(x(u_0, v, w_0),\ y(u_0, v, w_0),\ z(u_0, v, w_0)\big)$$

$$\gamma_3 : [w_0, w_0+\Delta w]$$

$$\gamma_3(w) = \big(x(u_0, v_0, w);\ y(u_0, v_0, w);\ z(u_0, v_0, w)\big).$$



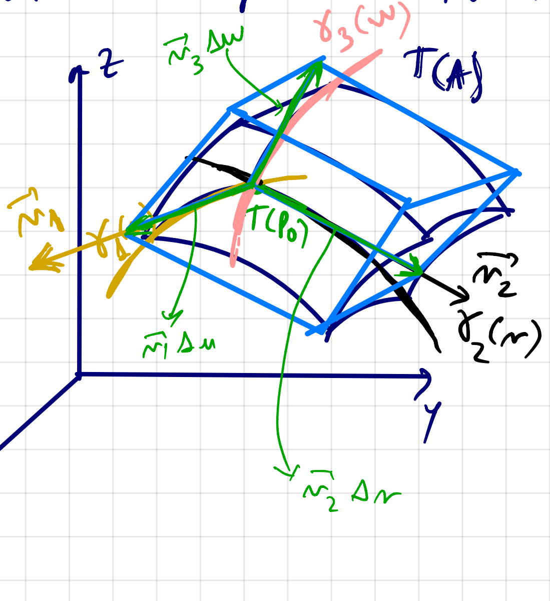
Sejam $\vec{v}_1, \vec{v}_2, \vec{v}_3$ os respectivos vetores tangentes às curvas γ_1, γ_2 e γ_3 no ponto $T(P_0) = (x_0, y_0, z_0)$, dados, por:

$$\vec{v}_1 = \gamma_1'(u_0) = \left(\frac{\partial x}{\partial u}(u_0, v_0, w_0), \frac{\partial y}{\partial u}(u_0, v_0, w_0), \frac{\partial z}{\partial u}(u_0, v_0, w_0)\right)$$

$$\vec{v}_2 = \gamma_2'(v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0, w_0), \frac{\partial y}{\partial v}(u_0, v_0, w_0), \frac{\partial z}{\partial v}(u_0, v_0, w_0)\right)$$

$$\vec{v}_3 = \gamma_3'(w_0) = \left(\frac{\partial x}{\partial w}(u_0, v_0, w_0), \frac{\partial y}{\partial w}(u_0, v_0, w_0), \frac{\partial z}{\partial w}(u_0, v_0, w_0)\right)$$

Considere, então, o paralelepípedo oblíquo no sistema xyz, determinado pelos vetores $\vec{v}_1 \Delta u$, $\vec{v}_2 \cdot \Delta v$ e $\vec{v}_3 \Delta w$



O volume V desse paralelepípedo oblíquo formado pelos vetores $\vec{n}_1 \Delta u$, $\vec{n}_2 \Delta v$ e $\vec{n}_3 \Delta w$ é dado pelo módulo do produto misto entre eles [apud geom. Analítica]

Ou seja;

$$V = \left| \left[\vec{n}_1 \Delta u, \vec{n}_2 \Delta v, \vec{n}_3 \Delta w \right] \right| \quad , \text{ onde}$$

$$\left[\vec{n}_1 \Delta u, \vec{n}_2 \Delta v, \vec{n}_3 \Delta w \right] =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0,w_0)\cdot\Delta u & \frac{\partial y}{\partial u}(u_0,v_0,w_0)\cdot\Delta u & \frac{\partial z}{\partial u}(u_0,v_0,w_0)\cdot\Delta u \\ \frac{\partial x}{\partial v}(u_0,v_0,w_0)\cdot\Delta v & \frac{\partial y}{\partial v}(u_0,v_0,w_0)\Delta v & \frac{\partial z}{\partial v}(u_0,v_0,w_0)\Delta v \\ \frac{\partial x}{\partial w}(u_0,v_0,w_0)\Delta w & \frac{\partial y}{\partial w}(u_0,v_0,w_0)\cdot\Delta w & \frac{\partial z}{\partial w}(u_0,v_0,w_0)\Delta w \end{vmatrix} .$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \cdot \Delta u \, \Delta v \cdot \Delta w$$

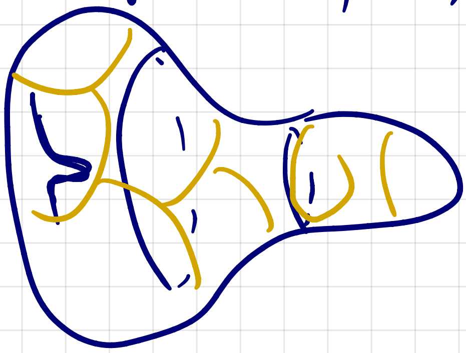
$$= \det J(T)(u_0,v_0,v_0) \cdot \Delta u \, \Delta v \, \Delta w$$

Ou seja; assumindo $\Delta u, \Delta v, \Delta w > 0$

$$V = \left| \det J(T)(u_0,v_0,w_0) \right| \Delta u \, \Delta v \, \Delta w$$

Seja $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ função integrável e considere $T: \Omega \subset \mathbb{R}^3 \to T(\Omega) = \Omega' \subset \mathbb{R}^3$ transformação injetiva e de classe C^1.

Seja D uma decomposição qualquer de Ω.



A soma de Riemann dessa decomposição será

$$\sum_{i=1}^{n} (f;D) := \sum_{i=1}^{n} f(x_i, y_i, z_i) \cdot V_i =$$

$$= \sum_{i=1}^{n} f(x(u_i,v_i,w_i), y(u_i,v_i,w_i), z(u_i,v_i,w_i)) \cdot |\det J(T)| \cdot \Delta u_i \Delta v_i \Delta w_i$$

Então,

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} (f;D)$$

$$= \lim_{\substack{n\to\infty \\ \Updownarrow \\ \|D\|\to 0}} \sum_{i=1}^{n} f(x(u_i,v_i,w_i), y(u_i,v_i,w_i), z(u_i,v_i,w_i)) \cdot |\det J(T)| \cdot \Delta u_i \Delta v_i \Delta w_i .$$

$$\iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot |\det J(T)(u,v,w)| \cdot du\,dv\,dw$$

EXEMPLOS:

01) Calcule $\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z}\, dx\, dy\, dz$, onde Ω é a região dada por: $\begin{cases} 0 \leq x-y+z \leq 1 \\ 1 \leq x+y-z \leq 2 \\ 0 \leq z \leq 1. \end{cases}$

SOLUÇÃO: Escreva

$$T: \begin{cases} u = x-y+z \\ v = x+y-z \\ w = z \end{cases}$$

Neste caso, teremos

$$\left.\begin{cases} 0 \leq u \leq 1 \\ 1 \leq v \leq 2 \\ 0 \leq w \leq 1 \end{cases}\right) \Omega'$$

$$J(T)(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

; onde:

$$\begin{cases} u = x-y+z \\ v = x+y-z \\ w = z \end{cases} \quad + \implies u+v = 2x \implies x = \frac{1}{2}u + \frac{1}{2}v$$

$$z = w$$

$$y = v - x + z$$

$$y = v - \left(\frac{1}{2}u + \frac{1}{2}v\right) + w$$

$$y = -\frac{1}{2}u + \frac{1}{2}v + w$$

Assim,

$$J(T)(u,v,w) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \underline{\det J(T)(u,v,w)} = \left|\begin{array}{ccc|cc} \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 0 & 1 \end{array}\right.$$

$$= \frac{1}{4} + 0 + 0 - 0 - 0 + \frac{1}{4} = \underline{\frac{1}{2}}$$

Disso, temos:

$$\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z}\,dx\,dy\,dz = \iiint_{\Omega'} \frac{e^{u}}{v} \cdot \underbrace{\left|\det J(T)(u,v,w)\right|}_{\frac{1}{2}} \cdot du\,dv\,dw$$

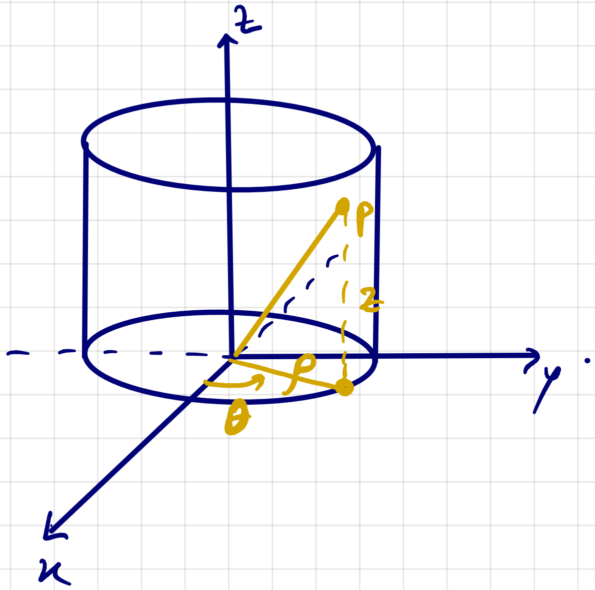
$$\int_{w=0}^{w=1} \int_{v=1}^{v=2} \int_{u=0}^{u=1} \frac{e^{u}}{v} \cdot \frac{1}{2} \cdot du\,dv\,dw =$$

$$\frac{1}{2} \cdot \int_{w=0}^{w=1} dw \cdot \int_{v=1}^{v=2} \frac{dv}{v} \cdot \int_{u=0}^{u=1} e^{u}\,du =$$

$$= \frac{1}{2} \cdot w\Big|_{0}^{1} \cdot \ln|v|\Big|_{1}^{2} \cdot e^{u}\Big|_{0}^{1} = \frac{1}{2} \cdot (1-0) \cdot (\ln 2 - \underbrace{\ln 1}_{0}) \cdot (e^{1} - e^{0})$$

$$= \underline{\frac{1}{2} \cdot \ln 2 \cdot (e-1)}$$

02) SISTEMA DE COORDENADAS CILÍNDRICAS: (é o equivalente ao sistema polar do $\mathbb{R}^2$, agora no $\mathbb{R}^3$)



$P(x, y, z)$; onde

$$\left\{\begin{array}{l} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{array}\right. \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Polar}$$

$$\Omega' = \Omega$$

POIS NÃO "DEFORMAMOS" O SÓLIDO, SÓ MUDAMOS DE COORDENADAS.

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = ?$$

$$\det(J)(T)(\rho,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} =$$

$$= \left|\begin{array}{ccc} \cos\theta & \text{sen}\,\theta & 0 \\ -\rho\,\text{sen}\,\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right| \begin{array}{cc} \cos\theta & \text{sen}\,\theta \\ -\rho\,\text{sen}\,\theta & \rho\cos\theta \\ 0 & 0 \end{array}$$

$$= \rho\cdot\cos^2\theta + 0 + 0 - 0 - 0 + \rho\cdot\text{sen}^2\theta = \rho\cdot(\underbrace{\cos^2\theta + \text{sen}^2\theta}_{=1}) = \rho$$

Assim, concluímos que

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega'} f(\rho\cos\theta, \rho\,\text{sen}\,\theta, z)\,\rho\,.d\rho\,d\theta\,dz$$

$\Omega = \Omega'$