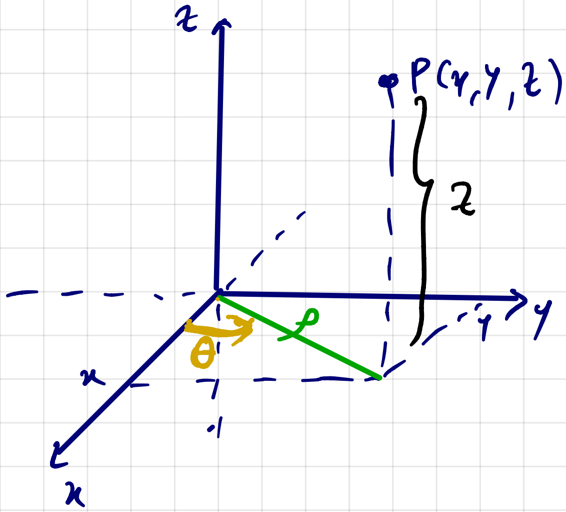
CÁLCULO IV

16/02/24 – AULA 15

No final da aula passada estudamos o sist. de coordenadas cilíndricas.



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\cdot\text{sen}\,\theta \\ z = z \end{cases}$$

Sendo $\Omega \subset \mathbb{R}^3$ e f integrável em Ω, então

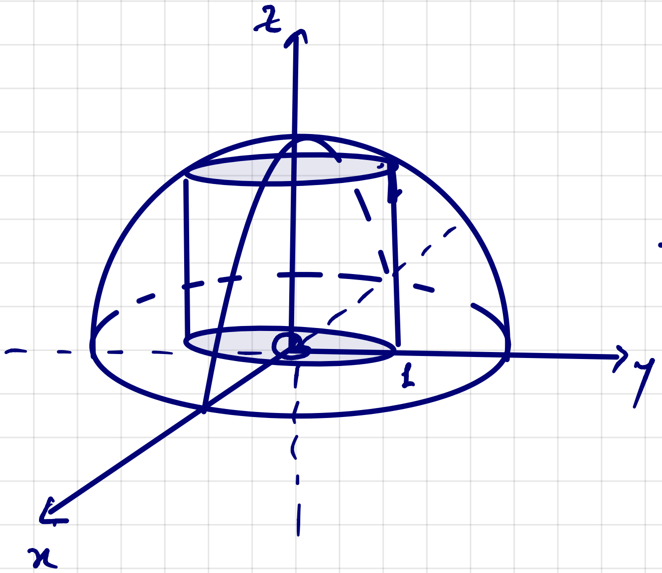
$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega'} f(\rho\cos\theta, \rho\,\text{sen}\,\theta, z)\,\rho\cdot d\rho\,d\theta\,dz$$

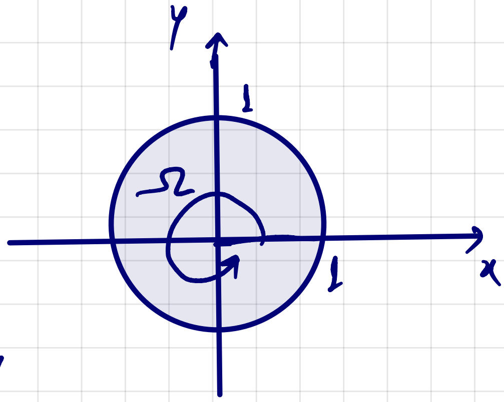
$\Omega' = \Omega$

Vejamos um exemplo.

EXEMPLO: Calcule o volume do sólido abaixo da semi-esfera $x^2+y^2+z^2=4$, $z \geq 0$, limitado pelo cilindro $x^2+y^2=1$ e pelo plano xy, usando integrais triplas.

SOLUÇÃO:





$0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$x^2+y^2+z^2=4$

$z=\sqrt{4-(x^2+y^2)}$

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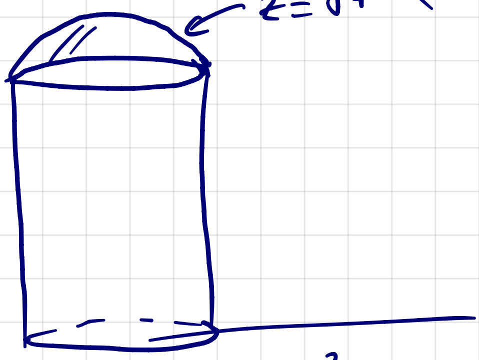


$z=0$

$0 \leq z \leq \sqrt{4-(\underbrace{x^2+y^2}_{=\rho^2})}$

$\boxed{\rho^2 = x^2+y^2}$

$V = \iiint_{\Omega} dV$. Usando coordenadas cilíndricas, temos:

$$V = \iiint_{\Omega} dx\,dy\,dz = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=0}^{z=\sqrt{4-\rho^2}} \rho \, dz\, d\rho\, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi}\left(\int_{\rho=0}^{\rho=1} \rho \,.\, z\Big|_{z=0}^{z=\sqrt{4-\rho^2}} . \, d\rho\right) d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\int_{\rho=0}^{\rho=1} \rho \sqrt{4-\rho^2}\, . \, d\rho\right) d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi}\left(-\frac{1}{2}\int_{\rho=0}^{\rho=1} (4-\rho^2)^{\frac{1}{2}} . (-2\rho\, d\rho)\right) d\theta = -\frac{1}{2}\int_{\theta=0}^{\theta=2\pi} \frac{(4-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{\rho=0}^{\rho=1} d\theta =$$

$\int r^k dr$

$r = 4-\rho^2 \rightarrow dr = -2\rho\, d\rho$

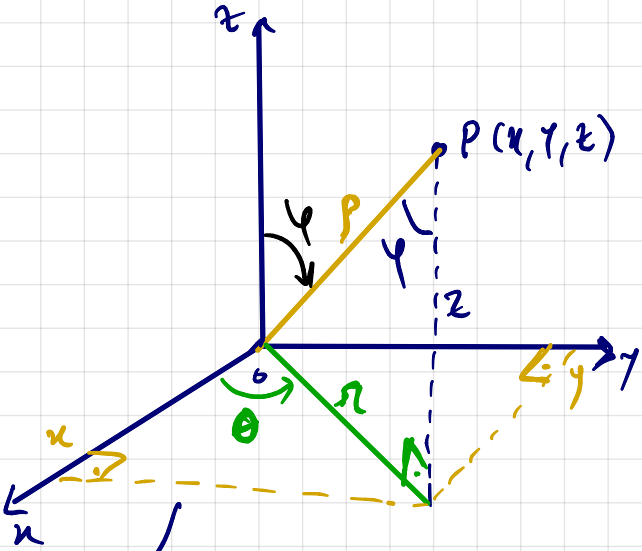
$$= -\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\int_{\theta=0}^{\theta=2\pi} \left[(4-1)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right] d\theta = -\frac{1}{3} . \left(3^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) . \int_0^{2\pi} d\theta$$

$$= -\frac{1}{3} . \left(\sqrt{3^3} - \sqrt{4^3}\right) . \theta\Big|_0^{2\pi} =$$

$$= -\frac{1}{3}\left(2\sqrt{3} - 4.2\right).\left(2\pi - 0\right) = \frac{2\pi.\left(8 - 2\sqrt{3}\right)}{3} \text{ u.v.}$$

SISTEMA DE COORDENADAS ESFÉRICAS:

Um ponto $P(x,y,z)$ no $\mathbb{R}^3$ fica identificado por um raio vetor ρ (distância do ponto P à origem) e dois ângulos: φ e θ, sendo φ partindo do eixo z até o raio vetor ρ, tal que $0 \leq \varphi \leq 180°$, e θ o ângulo da projeção do raio vetor no plano xy com o eixo x, $0° \leq \theta \leq 360°$.



NO SISTEMA ESFÉRICO $P(x,y,z)$ TORNA-SE $P(\rho,\varphi,\theta)$

$$\operatorname{sen}\varphi = \frac{r}{\rho}$$

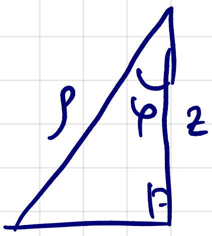
$$\Downarrow$$

$$\boxed{r = \rho.\operatorname{sen}\varphi}$$

$$\cos\theta = \frac{x}{r}$$

$$\Rightarrow x = r\cos\theta = \rho.\operatorname{sen}\varphi.\cos\theta.$$

$$\operatorname{sen}\theta = \frac{y}{r} \Rightarrow y = r.\operatorname{sen}\theta = \rho\operatorname{sen}\varphi.\operatorname{sen}\theta.$$



$$\cos\varphi = \frac{z}{\rho} \Rightarrow z = \rho.\cos\varphi$$

Ou seja, no sistema de coordenadas esféricas, um ponto $P(x,y,z)$ é tal que

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

Além disso, observamos que:

$$x^2+y^2+z^2 = (\rho \operatorname{sen}\varphi \cos\theta)^2 + (\rho \operatorname{sen}\varphi \operatorname{sen}\theta)^2 + (\rho\cos\varphi)^2$$

$$= \rho^2 \operatorname{sen}^2\varphi \cos^2\theta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\theta + \rho^2\cos^2\varphi$$

$$= \rho^2 \operatorname{sen}^2\varphi \, (\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1}) + \rho^2\cos^2\varphi$$

$$= \rho^2 (\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1}) = \rho^2$$

$$\Rightarrow \boxed{x^2+y^2+z^2=\rho^2}$$

(POR ISSO SISTEMA DE COORDENADAS "ESFÉRICAS".)

Vejamos como fica $\det J(T)(\rho,\varphi,\theta)$.

$$\det J(T)(\rho,\varphi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} =$$

$$= \left| \begin{array}{ccc|cc} \operatorname{sen}\varphi \cdot \cos\theta & \operatorname{sen}\varphi \cdot \operatorname{sen}\theta & \cos\varphi & \operatorname{sen}\varphi \cdot \cos\theta & \operatorname{sen}\varphi \cdot \operatorname{sen}\theta \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi & \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 & -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta \end{array} \right|$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$= 0 + \rho^2 \operatorname{sen}^3\varphi \operatorname{sen}^2\theta + \rho^2 \cos^2\varphi \cdot \operatorname{sen}\varphi \cos^2\theta +$$

$$+ \rho^2 \operatorname{sen}\varphi \cos^2\varphi \operatorname{sen}^2\theta + \rho^2 \operatorname{sen}^3\varphi \cos^2\theta - 0 =$$

$$= \rho^2 \operatorname{sen}^3\varphi \, (\underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{=1}) + \rho^2 \operatorname{sen}\varphi \cos^2\varphi \, (\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1})$$

$$= \rho^2 \operatorname{sen}^3\varphi + \rho^2 \operatorname{sen}\varphi \cos^2\varphi = \rho^2 \operatorname{sen}\varphi \, (\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1})$$

$$= \rho^2 \operatorname{sen}\varphi$$

$$\Rightarrow \det J(T)(\rho, \varphi, \theta) = \rho^2 \operatorname{sen}\varphi$$

$$\Rightarrow |\det J(T)(\rho, \varphi, \theta)| = |\rho^2 \cdot \operatorname{sen}\varphi| = \rho^2 \cdot \operatorname{sen}\varphi$$

$$0 \le \varphi \le 180^\circ$$

Assim, para o cálculo de uma integral tripla em coordenadas esféricas, escreveremos: (via mudança de variáveis)

Sendo $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável, então

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz =$$

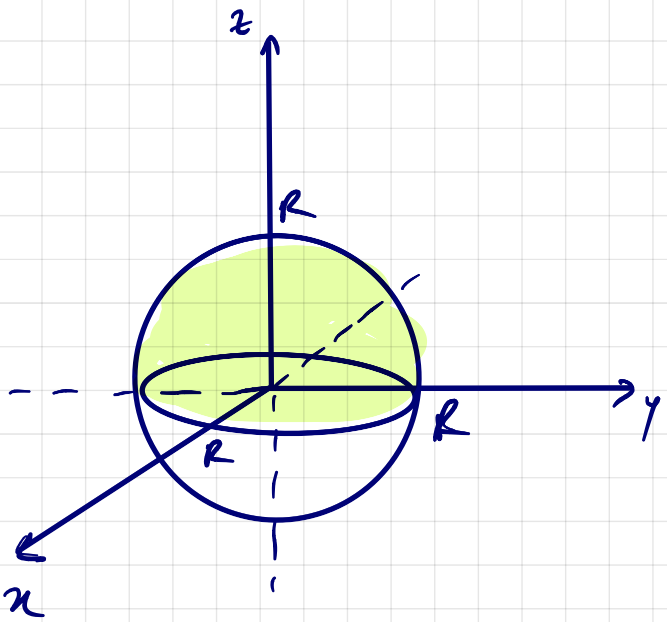
$$= \iiint_{\Omega'} f(\rho \operatorname{sen}\varphi \cos\theta, \rho \operatorname{sen}\varphi \operatorname{sen}\theta, \rho \cos\varphi) \cdot \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta.$$

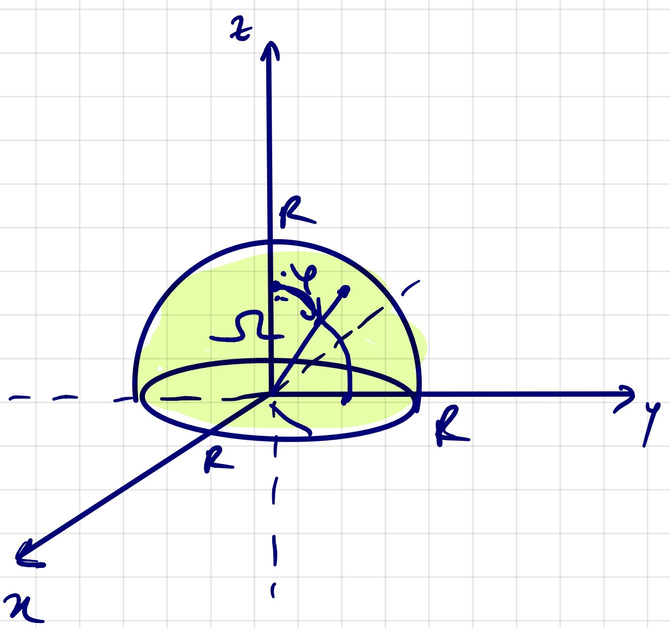
$\Omega' = \Omega$

POIS O SISTEMA ESFÉRICO NÃO DEFORMA A REGIÃO

EXEMPLO: Usando o sistema de coordenadas esféricas, deduza a fórmula do volume de uma esfera de raio R.

SOLUÇÃO: Por simetria, o volume pode ser calculado como sendo o "dobro da metade".





$$V = 2 \iiint_{\Omega} dV$$

$0 \leq \theta \leq 2\pi$ $\quad$ $0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq \rho \leq R$

$$V = 2 \cdot \iiint_{\Omega} dV = 2 \cdot \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} \rho^2 \operatorname{sen}\varphi \cdot d\rho \, d\varphi \, d\theta =$$

$$= 2 \cdot \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \operatorname{sen}\varphi \cdot \left. \frac{\rho^3}{3} \right|_{\rho=0}^{\rho=R} d\varphi \, d\theta$$

$$= 2 \cdot \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \operatorname{sen}\varphi \cdot \left(\frac{R^3}{3} - 0 \right) d\varphi \, d\theta =$$

$$= \frac{2R^3}{3} \cdot \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \operatorname{sen}\varphi \, d\varphi =$$

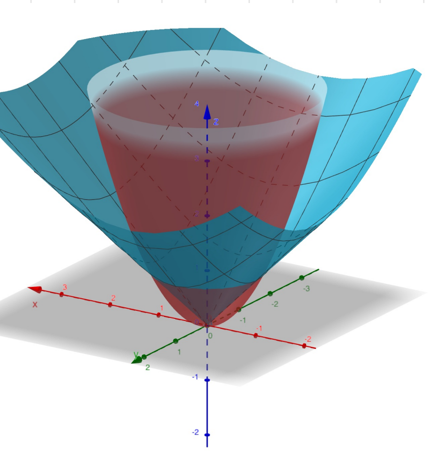
$$= \frac{2R^3}{3} \cdot \theta \Big|_0^{2\pi} \cdot (-\cos\varphi) \Big|_0^{\frac{\pi}{2}} = \frac{2R^3}{3} \cdot (2\pi) \cdot \Big(\underbrace{-\cos\frac{\pi}{2}}_{0} + \underbrace{\cos 0}_{1} \Big)$$

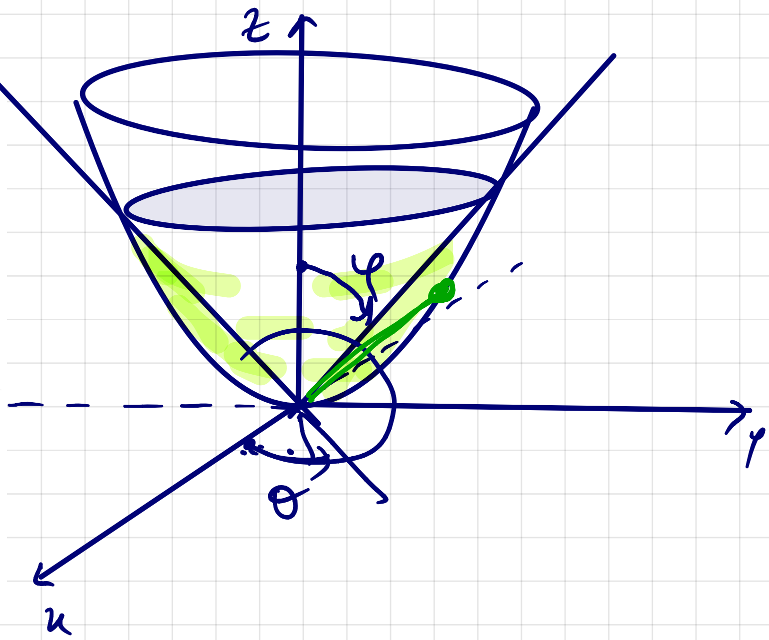
$$= \frac{4\pi R^3}{3} \cdot \Longrightarrow \boxed{V = \frac{4\pi R^3}{3}}$$

EXEMPLO 2: Calcule $\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz$, onde

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : \; x^2+y^2 \leq z \leq \sqrt{x^2+y^2}\}.$$

SOLUÇÃO: Ω é a região compreendida entre o paraboloide $z = x^2+y^2$ e o cone $z = \sqrt{x^2+y^2}$.





Usaremos o sist. de coordenadas esféricas para resolver. Note que devido ao cone "canônico" $z = \sqrt{x^2+y^2}$, teremos a variação seguinte para φ:

$$0 \leq \varphi \leq \frac{\pi}{4}$$

Obviamente, teremos $0 \leq \theta \leq 2\pi$.

A maior dificuldade será determinar uma variação para ρ. Pelo esquema desenhado ρ varia desde 0 até pontos sobre o paraboloide $z = x^2+y^2$. De posse do sistema esférico, teremos:

$$z = x^2 + y^2$$

$$\rho\cos\varphi = (\rho\,\mathrm{sen}\,\varphi\cos\theta)^2 + (\rho\cdot\mathrm{sen}\,\varphi\cdot\mathrm{sen}\,\theta)^2$$

$$\rho\cos\varphi = \rho^2\,\mathrm{sen}^2\varphi\cdot\cos^2\theta + \rho^2\,\mathrm{sen}^2\varphi\cdot\mathrm{sen}^2\theta.$$

$$\cancel{\rho}\cos\varphi = \rho^{\cancel{2}}\,\mathrm{sen}^2\varphi\cdot(\underbrace{\cos^2\theta + \mathrm{sen}^2\theta}_{=1})$$

$$\cos\varphi = \rho\,\mathrm{sen}^2\varphi \Rightarrow \rho = \frac{\cos\varphi}{\mathrm{sen}^2\varphi}$$

Portanto, teremos $0 < \rho \leq \dfrac{\cos\varphi}{\mathrm{sen}^2\varphi}$

Dessa forma, podemos escrever:

$$\iiint_\Omega \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{4}}\int_{\rho=0}^{\rho=\frac{\cos\varphi}{\mathrm{sen}^2\varphi}} \cancel{\sqrt{\rho^2}}\cdot\rho^2\,\mathrm{sen}\,\varphi\,d\rho\,d\varphi\,d\theta$$

ρ^2

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \mathrm{sen}\,\varphi\int_{\rho=0}^{\rho=\frac{\cos\varphi}{\mathrm{sen}^2\varphi}} \rho^3\,d\rho\,d\varphi\,d\theta = \underline{\underline{\text{et cetera} \ldots}}$$