CÁLCULO 2

06/08/24 – AULA 15

AULA DE EXERCÍCIOS:

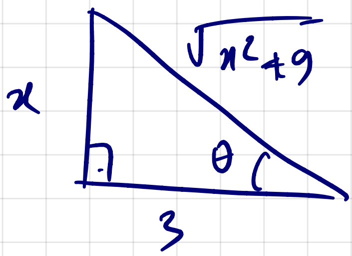
LISTA 04

8. Calcule as integrais a seguir, através de uma substituição trigonométrica:

(a) $\int \frac{dx}{9x^2+5}$ (b) $\int \frac{dx}{\sqrt{x^2-16}}$ (c) $\int \frac{dx}{4x^2+4x-6}$

(d) $\int \sqrt{4x^2+9}dx$ (e) $\int \frac{dx}{\sqrt{1+2x+3x^2}}$ (f) $\int \frac{\sqrt{x^2+9}}{x^3}dx$

(f) $\int \frac{\sqrt{x^2+9}}{x^3}\, dx.$




$$\tan\theta = \frac{x}{3} \Rightarrow x = 3.\tan\theta.$$

$$\Rightarrow dx = 3.\sec^2\theta\, d\theta.$$

$$\cos\theta = \frac{3}{\sqrt{x^2+9}} \Rightarrow \sqrt{x^2+9} = \frac{3}{\cos\theta}$$

$$\sqrt{x^2+9} = 3.\sec\theta.$$

Assim, teremos:

$$\int \frac{\sqrt{x^2+9}}{x^3}\, dx = \int \frac{\cancel{3}.\sec\theta.(\cancel{3}.\sec^2\theta\, d\theta)}{3^{\cancel{3}}\tan^3\theta} =$$

$$= \frac{1}{3}\int \frac{\sec^3\theta}{\tan^3\theta}\, d\theta = \frac{1}{3}.\int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^3\theta}{\cos^3\theta}}\, d\theta =$$

$$= \frac{1}{3}\int \frac{1}{\cancel{\cos^3\theta}}.\frac{\cancel{\cos^3\theta}}{\sin^3\theta}\, d\theta = \frac{1}{3}\int \csc^3\theta\, d\theta = $$

- $\int \csc^3\theta \, d\theta = \int \csc^2\theta \cdot \csc\theta \, d\theta = \int u \, dv = u \cdot v - \int v \cdot du$

$$\begin{cases} dv = \csc^2\theta \, d\theta \Rightarrow v = -\cot\theta \\ u = \csc\theta \Rightarrow du = -\csc\theta \cdot \cot\theta \, d\theta . \end{cases}$$

$$= \left[ -\csc\theta \cdot \cot\theta - \int (-\cot\theta) \cdot (-\csc\theta \cdot \cot\theta) \, d\theta \right]$$

$$= -\csc\theta \cdot \cot\theta - \int \cot^2\theta \cdot \csc\theta \, d\theta =$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\cot^2\theta = \csc^2\theta - 1$$

$$= -\csc\theta \cdot \cot\theta - \int (\csc^2\theta - 1) \cdot \csc\theta \, d\theta .$$

$$= -\csc\theta \cdot \cot\theta - \int \csc^3\theta \, d\theta + \int \csc\theta \, d\theta$$

Ou seja, obtemos:

$$\int \csc^3\theta \, d\theta = -\csc\theta \cot\theta - \int \csc^3\theta + \ln|\csc\theta - \cot\theta| + C$$

$$\Rightarrow 2 \cdot \int \csc^3\theta \, d\theta = \ln|\csc\theta - \cot\theta| - \csc\theta \cdot \cot\theta + C$$

$$\Rightarrow \int \csc^3\theta\, d\theta = \frac{1}{2}\ln|\csc\theta - \cot\theta| - \frac{1}{2}\csc\theta\cdot\cot\theta + C$$

Portanto:

$$= \frac{1}{3}\int \csc^3\theta\, d\theta = \frac{1}{3}\left(\frac{1}{2}\ln|\csc\theta - \cot\theta| - \frac{1}{2}\csc\theta\cdot\cot\theta\right) + C$$

$$= \frac{1}{6}\ln|\csc\theta - \cot\theta| - \frac{1}{6}\csc\theta\cdot\cot\theta + C;$$

como $\tan\theta = \frac{x}{3} \Rightarrow \cot\theta = \frac{3}{x}$;

$\sin\theta = \frac{x}{\sqrt{x^2+9}} \Rightarrow \csc\theta = \frac{\sqrt{x^2+9}}{x}$;

segue que:

$$\int \frac{\sqrt{x^2+9}}{x^3}\, dx = \frac{1}{6}\ln|\csc\theta - \cot\theta| - \frac{1}{6}\csc\theta\cdot\cot\theta + C$$

$$= \frac{1}{6}\ln\left|\frac{\sqrt{x^2+9}}{x} - \frac{3}{x}\right| - \frac{1}{6}\cdot\frac{\sqrt{x^2+9}}{x}\cdot\frac{3}{x} + C$$

$$= \frac{1}{6}\cdot\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| - \frac{1}{2}\,\frac{\sqrt{x^2+9}}{x^2} + C.$$

LISTA 04

01) (h) $\int \frac{\ln x \, dx}{(x+1)^2} = \int u \cdot dv = u \cdot v - \int v \, du$

$u = \ln x \Rightarrow du = \frac{1}{x} \cdot dx.$

$dv = \frac{1}{(x+1)^2} dx \Rightarrow v = \int \frac{dx}{(x+1)^2} = \int (x+1)^{-2} \cdot dx$

$\int v^k$

$v = x+1$

$= \frac{(x+1)^{-1}}{-1} = -\frac{1}{x+1}$

$dv = dx$

Disso, temos:

$$\int \frac{\ln x}{(x+1)^2} dx = \ln x \cdot \left(-\frac{1}{x+1}\right) - \int -\frac{1}{x+1} \cdot \frac{1}{x} dx$$

$$= -\frac{1}{x+1} \cdot \ln x + \int \frac{dx}{x^2+x} =$$

$x^2 + x + 0 = (x+a)^2 + b$

$x^2 + 2ax + a^2 + b$

$\Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$

$a^2 + b = 0$

$\frac{1}{4} + b = 0 \Rightarrow b = -\frac{1}{4}$

$\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$

$$= -\frac{1}{x+1}\cdot \ln x + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} =$$

$$\hookrightarrow \int \frac{dm}{m^2-a^2} = \frac{1}{2a}\cdot \ln\left|\frac{m-a}{m+a}\right| + c$$

$$m = x+\frac{1}{2} \Rightarrow dm = dx$$

ok!

$$= -\frac{1}{x+1}\cdot \ln x + \frac{1}{\cancel{2}\cdot\frac{1}{\cancel{2}}}\cdot \ln\left|\frac{x+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}}{x+\frac{1}{2}+\frac{1}{2}}\right| + c$$

$$= -\frac{1}{x+1}\cdot \ln x + \ln\left|\frac{x}{x+1}\right| + c.$$

LISTA 05.

01) (c) $\int \frac{(2x-4)dx}{\sqrt{x^2-12x+4}}$

$$m = x^2-12x+4 \Rightarrow dm = (2x-12)dx$$

$$\int \frac{(2x-4)dx}{\sqrt{x^2-12x+4}} = \int \frac{1(2x-12)+8}{\sqrt{x^2-12x+4}} =$$

$$= \int \frac{(2x-12)dx}{\sqrt{x^2-12x+4}} + 8\cdot \int \frac{dx}{\sqrt{x^2-12x+4}}$$

$$= \int \underbrace{(x^2-12x+4)^{-\frac{1}{2}}\cdot(2x-12)dx}_{} + 8\cdot \int \frac{dx}{\sqrt{(x-6)^2-32}}$$

$$\int m^k dm = \frac{m^{k+1}}{k+1} + c$$

$$m = x^2-12x+4 \Rightarrow dm = (2x-12)dx$$

$$= \frac{(x^2-12x+4)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 8 \cdot \int \frac{dx}{\sqrt{(x-6)^2-(\sqrt{32})^2}}$$

$$\int \frac{dv}{\sqrt{v^2-a^2}} = \ln|v+\sqrt{v^2-a^2}|+c$$

$v = x-6 \Rightarrow dv = dx$

ok!

$$= \frac{\sqrt{x^2-12x+4}}{\frac{1}{2}} + 8 \cdot \ln|x-6+\sqrt{(x-6)^2-32}|+c$$

$$= 2 \cdot \sqrt{x^2-12x+4} + 8 \cdot \ln|x-6+\sqrt{x^2-12x+4}|+c.$$

Lista 07.

01) (g) $\int \frac{2-\sin x}{2+\cos x}dx$.

$z = \tan \frac{x}{2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$\operatorname{sen} x = \frac{2z}{1+z^2}$

$dx = \frac{2dz}{1+z^2}$

$$\int \frac{2-\operatorname{sen} x}{2+\cos x}dx = \int \frac{2-\frac{2z}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} \cdot \frac{2dz}{1+z^2} =$$

$$= \int \frac{\frac{2(1+z^2)-2z}{\cancel{1+z^2}}}{\frac{2(1+z^2)+1-z^2}{\cancel{1+z^2}}} \cdot \frac{2dz}{1+z^2} = \int \frac{2+2z^2-2z}{2+2z^2+1-z^2} \cdot \frac{2dz}{1+z^2}$$

$$\int \frac{(4z^2-4z+4)\,dz}{(z^2+3)\cdot(z^2+1)}$$

(DECOMP. EM FRAÇÕES PARCIAIS)

$$\begin{cases} z^2+3=0 \Leftrightarrow z^2=-3 \notin \mathbb{R} \\ z^2+1=0 \Leftrightarrow z^2=-1 \notin \mathbb{R}. \end{cases}$$

$$\frac{4z^2-4z+4}{(z^2+3)\cdot(z^2+1)} = \frac{Az+B}{z^2+3} + \frac{Cz+D}{z^2+1} =$$

$$= \frac{(Az+B)(z^2+1)+(Cz+D)\cdot(z^2+3)}{(z^2+3)(z^2+1)}$$

$$4z^2-4z+4 = Az^3+Az+Bz^2+B+Cz^3+3zC+Dz^2+3D$$

$$\begin{cases} A + C = 0 \\ B + D = 4 \\ A + 3C = -4 \\ B + 3D = 4 \end{cases} \quad \begin{cases} A+C=0 \rightsquigarrow C=-A \\ A+3C=-4 \end{cases}$$

$$\rightarrow A-3A=-4$$

$$-2A=-4$$

$$A=2$$

$$C=-2$$

$$- \begin{cases} \cancel{B}+D=4 \\ \cancel{B}+3D=4 \end{cases}$$

$$-2D=0$$

$$D=0$$

$$B+D=4 \Rightarrow B=4$$

$$\int \frac{(4z^2-4z+4)\,dz}{(z^2+3)\cdot(z^2+1)} = \int \frac{2z+4}{z^2+3}\,dz + \int \frac{-2z+0}{z^2+1}\,dz$$

$$= \int \frac{2z\,dz}{z^2+3} + 4\int \frac{dz}{z^2+3} - \int \frac{2z\,dz}{z^2+1}$$

$\int \frac{dv}{v} = \ln|v| + C$

$v = z^2+3 \Rightarrow dv = 2z\,dz$

$\int \frac{dv}{v^2+a^2} = \frac{1}{a}\arctan\left(\frac{v}{a}\right) + C$

$(a = \sqrt{3})$

$\int \frac{dv}{v} = \ln|v| + C$

$v = z^2+1$

$dv = 2z\,dz$

$$= \ln|z^2+3| + 4 \cdot \frac{1}{\sqrt{3}} \cdot \arctan\left(\frac{z}{\sqrt{3}}\right) + \ln|z^2+1| + C$$

$z = \tan\frac{x}{2}$

$$= \ln\left(\tan^2\frac{x}{2}+3\right) + \frac{4}{3}\cdot\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + \ln\left(\tan^2\frac{x}{2}+1\right) + C$$