CÁLCULO 2   01/02/24 – AULA 14

Na aula passada estudamos integrais envolvendo potências de seno e cosseno, ou seja, integrais da forma

$$\int \operatorname{sen}^m x\,dx \;;\quad \int \cos^m x\,dx \quad e \quad \int \operatorname{sen}^m x \cdot \cos^p x\,dx,$$

com m e p pares e/ou ímpares.

Hoje vamos estudar as integrais racionais em termos de seno e cosseno.

INTEGRAIS DA FORMA $\int R(\operatorname{sen} x, \cos x)\,dx$; onde

$R(\operatorname{sen} x, \cos x)$ denota uma função racional em termos de $\operatorname{sen} x$ e de $\cos x$.

A ideia consiste em efetuar uma mudança de variável, onde a chave para a mudança é escrever

$$\boxed{z = \tan \frac{x}{2}}$$

Assim, dada $\int R(\operatorname{sen} x, \cos x)\,dx$,

precisamos escrever $\operatorname{sen} x$, $\cos x$ e dx em termos de z.

Da Trigonometria, dado um arco α, temos a fórmula de arco duplo.

$$\cos 2\alpha = \cos^2\alpha - \operatorname{sen}^2\alpha.$$

Como $\operatorname{sen}^2\alpha + \cos^2\alpha = 1$, então:

$$\cos 2\alpha = \cos^2\alpha - \operatorname{sen}^2\alpha = \cos^2\alpha - (1 - \cos^2\alpha)$$

$$\cos 2\alpha = \cos^2\alpha - 1 + \cos^2\alpha$$

$$\Rightarrow \cos 2\alpha = 2.\cos^2\alpha - 1.$$

Tomando $\alpha = \frac{x}{2}$, temos:

$$\cos x = 2.\cos^2\frac{x}{2} - 1 =$$

$$\cos\alpha = \frac{1}{\sec\alpha}$$

$$= 2.\frac{1}{\sec^2\frac{x}{2}} - 1 \qquad 2.\frac{1}{1+\tan^2\frac{x}{2}} - 1$$

$$1+\tan^2\alpha = \sec^2\alpha$$

$$\Rightarrow \cos x = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+z^2} - 1.$$

$$\cos x = \frac{2-(1+z^2)}{1+z^2} = \frac{2-1-z^2}{1+z^2}$$

$$\Longrightarrow \boxed{\cos x = \frac{1-z^2}{1+z^2}}$$

Disso, teremos:

$$\operatorname{sen} x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(\frac{1-z^2}{1+z^2}\right)^2} =$$

$$= \sqrt{1-\frac{(1-z^2)^2}{(1+z^2)^2}} = \sqrt{\frac{(1+z^2)^2-(1-z^2)^2}{(1+z^2)^2}}$$

$$= \frac{\sqrt{\cancel{1}+2z^2+\cancel{z^4}-\cancel{1}+2z^2-\cancel{z^4}}}{1+z^2} = \frac{\sqrt{4z^2}}{1+z^2} = \frac{2z}{1+z^2}$$

$$\Longrightarrow \boxed{\operatorname{sen} x = \frac{2z}{1+z^2}}$$

Resta obter a mudança de dx para um diferencial na variável z.

$$(\tan v)' = \sec^2 v \cdot v'$$

$$z = \tan\frac{x}{2}$$

$$\Rightarrow dz = \sec^2\frac{x}{2}\cdot\frac{1}{2}\cdot dx$$

$$\Rightarrow dz = \frac{1}{2}\Big(1+\underbrace{\tan^2\frac{x}{2}}_{z^2}\Big)\cdot dx \quad , \text{ ou seja:}$$

$$dz = \frac{1}{2}\cdot(1+z^2)\,dx \Rightarrow \boxed{dx = \frac{2dz}{1+z^2}}$$

EXEMPLOS:

01) $\displaystyle\int \frac{dx}{2+\operatorname{sen} x}$. $\left[R(\operatorname{sen}x, \cos x) = \frac{1}{2+\operatorname{sen}x}\right]$

Como $\operatorname{sen} x = \dfrac{2z}{1+z^2}$; $dx = \dfrac{2dz}{1+z^2}$, então:

$$\int \frac{dx}{2+\operatorname{sen}x} = \int \frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2(1+z^2)+2z}{\cancel{1+z^2}}} =$$

$$= \int \frac{2dz}{2+2z^2+2z} = \int \frac{\cancel{2}dz}{\cancel{2}(z^2+z+1)} =$$

$$= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$z^2+z+1 = (z+a)^2+b$$
$$= z^2+2az+a^2+b$$
$$\hookrightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$\left\{\begin{array}{l} a^2+b = 1 \\ \left(\frac{1}{2}\right)^2+b = 1 \\ b = 1-\frac{1}{4} \\ b = \frac{3}{4} \end{array}\right.$$

$$= \int \frac{dz}{\left(z+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \cdot \arctan\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$\int \frac{dv}{v^2+a^2} = \frac{1}{a} \cdot \arctan\left(\frac{v}{a}\right) + C$$

$$v = z + \frac{1}{2} \Rightarrow dv = 1 \cdot dz$$

$$= \frac{2}{\sqrt{3}} \cdot \arctan\left(\frac{\frac{2z+1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}}\right) + C = \frac{2}{\sqrt{3}} \arctan\left(\frac{2z+1}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \cdot \arctan\left(\frac{2 \cdot \tan\frac{x}{2} + 1}{\sqrt{3}}\right) + C.$$

$z = \tan\frac{x}{2}$

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02) $$\int \frac{\operatorname{sen} x \, dx}{2\operatorname{sen} x + \cos x} = ?$$

<u>solução</u>: Escreva $z = \tan\frac{x}{2}$. Então:

$$\operatorname{sen} x = \frac{2z}{1+z^2}; \quad \cos x = \frac{1-z^2}{1+z^2}; \quad dx = \frac{2\,dz}{1+z^2}$$

Dessa forma, temos:

$$\int \frac{\operatorname{sen} x\, dx}{2\operatorname{sen} x + \cos x} = \int \frac{\frac{2z}{1+z^2} \cdot \frac{2\,dz}{1+z^2}}{2 \cdot \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} = \int \frac{\frac{4z\,dz}{(1+z^2)^{\cancel{2}}}}{\frac{4z+1-z^2}{\cancel{1+z^2}}}$$

$$= \int \frac{4z}{1+z^2} \cdot \frac{1}{4z+1-z^2} \cdot dz = -4\int \frac{z\,dz}{(1+z^2)\cdot(z^2-4z-1)} =$$

zeros do denominador: [POIS PRECISAMOS DECOMPOR EM FRAÇÕES PARCIAIS]

$$z^2-4z-1=0$$

$$z = \frac{4 \pm \sqrt{16+4}}{2}$$

$$z = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

Disso: $z^2-4z-1 = (z-(2+\sqrt{5}))\cdot(z-(2-\sqrt{5}))$

$$= (z-2-\sqrt{5})(z-2+\sqrt{5})$$

Assim, temos:

$$\frac{z}{(1+z^2)(z^2-4z-1)} = \frac{Az+B}{z^2+1} + \frac{C}{z-2-\sqrt{5}} + \frac{D}{z-2+\sqrt{5}}$$

Agora, "basta" encontrar as constantes A, B, C e D através de um sistema linear (c.f. decomposição feita acima), para determiná-las.

Apenas para reduzir os cálculos, vamos continuar a resolução do problema, mantendo estas constantes sem determiná-los. (Fica como exercício)

Continuando o cálculo, teremos:

$$= -4\int \frac{z\,dz}{(1+z^2)(z^2-4z-1)} =$$

$$= -4\cdot\left(\int \frac{Az+B}{z^2+1}dz + C\cdot\int\underbrace{\frac{dz}{z-2-\sqrt{5}}}_{\int\frac{dv}{v}} + D\cdot\int\underbrace{\frac{dz}{z-2+\sqrt{5}}}_{\int\frac{dw}{w}}\right)$$

$$= -4A\underbrace{\int\frac{z\,dz}{z^2+1}}_{\int\frac{dv}{v}} - 4\cdot B\int\frac{dz}{z^2+1} - 4C\ln|z-2-\sqrt{5}| -$$

$$-4D.\ln|z-2+\sqrt{5}|$$

$$\int\frac{du}{u^2+a^2} = \frac{1}{a}\,\text{arctan}\left(\frac{u}{a}\right)+c$$

$v = z^2+1 \Rightarrow dv = 2z\,dz$

$$z\,dz = \frac{dv}{2}$$

$$= -\overset{2}{\cancel{4}}A\int\frac{\frac{dv}{\cancel{2}}}{v} - 4B.\frac{1}{1}.\,\text{arctan}\left(\frac{z}{1}\right) -$$

$$-4C\ln|z-2-\sqrt{5}| - 4D\ln|z-2+\sqrt{5}| + C$$

$$= -2A\ln(z^2+1) - 4B\,\text{arctan}(z) - 4C\ln|z-2-\sqrt{5}|$$

$$-4D\ln|z-2+\sqrt{5}| + C$$

Por fim, como $z = \tan\frac{x}{2}$, então:

$$= -2A\ln\left(\tan^2\frac{x}{2}+1\right) - 4B\arctan\left(\overbrace{\tan\frac{x}{2}}^{=\frac{x}{2}}\right) - 4C\ln\left|\tan\frac{x}{2} - 2 - \sqrt{5}\right|$$

$$-4D\ln\left|\tan\frac{x}{2} - 2 + \sqrt{5}\right| + C$$

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03) $\displaystyle\int \frac{2\tan x\,dx}{2+3\cos x}$

Escreve $z = \tan\frac{x}{2}$. Assim:

$$\operatorname{sen} x = \frac{2z}{1+z^2}\,; \quad \cos x = \frac{1-z^2}{1+z^2}\,; \quad dx = \frac{2dz}{1+z^2}$$

$$\Rightarrow \underset{\sim}{\tan x} = \frac{\operatorname{sen} x}{\cos x} = \frac{\frac{2z}{\cancel{1+z^2}}}{\frac{1-z^2}{\cancel{1+z^2}}} = \underset{\sim}{\frac{2z}{1-z^2}}$$

Disso, temos:

$$\int \frac{2\tan x\,dx}{2+3\cos x} = \int \frac{2\cdot\frac{2z}{1-z^2}\cdot\frac{2dz}{1+z^2}}{\frac{2}{1}+3\cdot\left(\frac{1-z^2}{1+z^2}\right)} =$$

$$= \int \frac{\frac{8z\,dz}{(1-z^2)\cancel{(1+z^2)}}}{\frac{2+2z^2+3-3z^2}{\cancel{1+z^2}}} = \int \frac{8z\,dz}{1-z^2}\cdot\frac{1}{-z^2+5} =$$

$$\int \frac{8z\,dz}{(z^2-1)(z^2-5)} =$$

$z^2-1 = (z+1)(z-1)$

$z^2-5 = (z+\sqrt{5})(z-\sqrt{5})$

$$\frac{8z}{(z^2-1)(z^2-5)} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{z+\sqrt{5}} + \frac{D}{z-\sqrt{5}}$$

$$= \frac{A\cdot(z-1)\cdot(z^2-5) + B\cdot(z+1)\cdot(z^2-5) + C\cdot(z^2-1)\cdot(z-\sqrt{5}) + D\cdot(z^2-1)(z+\sqrt{5})}{(z+1)(z-1)\underbrace{(z+\sqrt{5})(z-\sqrt{5})}_{z^2-5}}$$

(...)

exercício ...