CÁLCULO 1.

20/02/24 - AULA 18

No final da aula passada iniciamos o estudo sobre regras de derivação. Por exemplo, vimos:

01) $y = k \Rightarrow y' = 0$

02) $y = x \Rightarrow y' = 1$

03) $y = k \cdot x \Rightarrow y' = k$

04) $y = k \cdot v \Rightarrow y' = k \cdot v'$

05) $y = v^k \Rightarrow y' = k \cdot v^{k-1} \cdot v'$ $(k \in \mathbb{N})$

06) $y = u + v \Rightarrow y' = u' + v'$

Ex.: (a) $y = \underbrace{x^2}_{v^k} - 5\underbrace{x^3}_{v^k} + 6 \Rightarrow y' = ?$

$$y' = 2x^{2-1} \cdot 1 - 5 \cdot 3x^{3-1} \cdot 1 + 0$$

$$y' = 2x - 15x^2$$

(b) $y = (\underbrace{x^3 - 5x^4 + 3x}_{v})^6 \Rightarrow y' = ?$

$$(v^k)' = k v^{k-1} \cdot v'$$

$$y' = 6 \cdot (x^3 - 5x^4 + 3x)^{6-1} \cdot \underbrace{(x^3 - 5x^4 + 3x)'}_{v'}$$

$$y' = 6.(x^3-5x^4+3x)^5.(3x^{3-1}.1-5.4.x^{4-1}.1+3.1)$$

$$y' = 6(x^3-5x^4+3x)^5.(3x^2-20x^3+3)$$

(c) $y = (x^5-x)^3 \Rightarrow y' = ?$ $\qquad (v^k)' = k.v^{k-1}.v'$

$$y' = 3.(x^5-x)^{3-1}.(x^5-x)'$$

$$\boxed{y' = 3.(x^5-x)^2.(5x^4-1)}$$

Vejamos outras regras de derivação:

$y = u.v \Rightarrow y' = ?$; onde

$u = u(x)$; $v = v(x)$

Dado $x \in D(u.v)$ e seja $h \in \mathbb{R}$ tal que $x+h \in D(u.v)$. Então: , sendo $f(x) = (u.v)(x) = u(x).v(x)$

$$y' = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{(u.v)(x+h)-(u.v)(x)}{h} =$$

$$= \lim_{h\to 0} \frac{u.(x+h).v(x+h)-u(x).v(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\overbrace{u(x+h).v(x+h) - u(x).v(x+h)} + \overbrace{u(x).v(x+h) - u(x).v(x)}}{h}$$

$$= \lim_{h\to 0} \frac{v(x+h).[u(x+h) - u(x)]}{h} + \lim_{h\to 0} \frac{u(x).[v(x+h) - v(x)]}{h}$$

$$= \underbrace{\lim_{h\to 0} v(x+h)}_{v(x)} . \underbrace{\lim_{h\to 0} \frac{u(x+h) - u(x)}{h}}_{u'(x)} + \underbrace{\lim_{h\to 0} u(x)}_{u(x)} \underbrace{\lim_{h\to 0} \frac{v(x+h) - v(x)}{h}}_{v'(x)}$$

$$= v(x).u'(x) + u(x).v'(x)$$

CONCLUSÃO: 07) $\boxed{y = u.v \Rightarrow y' = u.v' + u'.v}$

EX.: $y = x^3 \Rightarrow y' = ?$ $\quad y' = 3x^2$

Por outro lado (claro, que um processo muito artificial):

$$y = x^3 = x^2 . x = u.v$$

$$\begin{cases} u = x^2 \Rightarrow u' = 2x \\ v = x \Rightarrow v' = 1 \end{cases}$$

$$\Rightarrow y' = u \cdot v' + u' \cdot v$$

$$= x^2 \cdot 1 + 2x \cdot x = x^2 + 2x^2 = 3x^2$$

outros exemplos para a derivada do produto:

(a) $y = (x^2 - 5x + 3) \cdot (x^5 - 7x^3 + 4x^2 - 1)$. $y' = ?$

$$y = u \cdot v \Rightarrow y' = u \cdot v' + u' \cdot v$$

$$\begin{cases} u = x^2 - 5x + 3 \Rightarrow u' = 2x - 5 \\ v = x^5 - 7x^3 + 4x^2 - 1 \Rightarrow v' = 5x^4 - 21x^2 + 8x \end{cases}$$

Assim, temos:

$$y' = u \cdot v' + u' \cdot v =$$

$$= (x^2 - 5x + 3) \cdot (5x^4 - 21x^2 + 8x) + (2x - 5) \cdot (x^5 - 7x^3 + 4x^2 - 1)$$

(b) $y = (x^2 - x)^5 \cdot (3x^2 - 9x)$. $y' = ?$

$$y = u \cdot v \Rightarrow y' = u \cdot v' + u' \cdot v$$

$$\begin{cases} u = (x^2 - x)^5 \Rightarrow u' = 5 \cdot (x^2 - x)^4 \cdot (x^2 - x)' \\ \qquad\qquad\qquad\quad u' = 5 (x^2 - x)^4 \cdot (2x - 1) \\ v = 3x^2 - 9x \Rightarrow v' = 6x - 9 \end{cases}$$

Portanto, teremos:

$$y' = u \cdot v' + u' \cdot v$$

$$= (x^2-x)^5 \cdot (6x-9) + 5 \cdot (x^2-x)^4 \cdot (2x-1)(3x^2-9x)$$

$$= (x^2-x)^4 \cdot \left[(x^2-x) \cdot (6x-9) + 5 \cdot (2x-1) \cdot (3x^2-9x) \right]$$

08) $\boxed{y = \dfrac{u}{v} \Rightarrow y' = \dfrac{v \cdot u' - u v'}{v^2}}$

De fato, seja $x \in D(f)$ e tome $h \in \mathbb{R}$ tal que $x+h \in D(f)$. Assim; para $f(x) = \frac{u}{v}(x) = \frac{u(x)}{v(x)}$:

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left[\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)} \right] =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{v(x) \cdot u(x+h) - u(x) \cdot v(x+h)}{v(x+h) \cdot v(x)} =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{v(x) \cdot u(x+h) - u(x) \cdot v(x) + u(x) \cdot v(x) - u(x) \cdot v(x+h)}{v(x+h) \cdot v(x)}$$

$$= \lim_{h \to 0} \underbrace{\frac{v(x)}{v(x+h) \cdot v(x)}}_{\frac{v(x)}{[v(x)]^2}} \underbrace{\frac{[u(x+h) - u(x)]}{h}}_{u'(x)} + \lim_{h \to 0} \underbrace{\frac{u(x)}{v(x+h) \cdot v(x)}}_{\frac{u(x)}{[v(x)]^2}} \underbrace{\frac{[v(x) - v(x+h)]}{h}}_{-v'(x)}$$

$$= \frac{v(x)}{[v(x)]^2} \cdot u'(x) - \frac{u(x)}{[v(x)]^2} \cdot v'(x) =$$

$$= \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

EXEMPLO: $y = \dfrac{x^2 - x}{2x+3} \Rightarrow y' = ?$

SOLUÇÃO: $Y = \dfrac{u}{v} \Rightarrow y' = \dfrac{v \cdot u' - u \cdot v'}{v^2}$

$$\begin{cases} u = x^2 - x \Rightarrow u' = 2x - 1 \\ v = 2x + 3 \Rightarrow v' = 2 \end{cases}$$

Disso, obtemos:

$$y' = \frac{(2x+3) \cdot (2x-1) - (x^2 - x) \cdot 2}{(2x+3)^2}$$

$$y' = \frac{4x^2 - \cancel{2x} + 6x - 3 - 2x^2 + \cancel{2x}}{(2x+3)^2}$$

$$y' = \frac{2x^2 + 6x - 3}{(2x+3)^2}$$

09) $\boxed{y = \ln v \quad \Rightarrow y' = \frac{v'}{v}}$

DEMONSTR.: Dado $x \in D(f)$ e $h \in \mathbb{R}$ tal que $x+h \in D(f)$, então, para $f(x) = \ln v(x)$, temos:

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\ln v(x+h) - \ln v(x)\right] =$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \ln \frac{v(x+h)}{v(x)} = \lim_{h\to 0} \ln \left(\frac{v(x+h)}{v(x)}\right)^{\frac{1}{h}} =$$

$$= \ln \lim_{h\to 0} \left(1 + \frac{v(x+h)}{v(x)} - 1\right)^{\frac{1}{h}} = \ln \lim_{h\to 0} \left(1 + \frac{v(x+h) - v(x)}{v(x)}\right)^{\frac{1}{h}}$$

$$= \ln \left[\underbrace{\lim_{h\to 0} \left(1 + \frac{v(x+h) - v(x)}{v(x)}\right)^{\frac{v(x)}{v(x+h) - v(x)}}}_{\to e}\right]^{\frac{v(x+h) - v(x)}{v(x)} \cdot \frac{1}{h}} =$$

$$= \ln e^{\lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \cdot \frac{1}{v(x)}} =$$

$$= \lim_{h \to 0} \underbrace{\frac{v(x+h) - v(x)}{h}}_{v'(x)} \cdot \frac{1}{v(x)} = \frac{v'(x)}{v(x)}$$

EXEMPLOS:

(a) $y = \ln(3x^2 - 4x + 2) \Rightarrow y' = ?$

$$y = \ln v \Rightarrow y' = \frac{v'}{v}.$$

$$v = 3x^2 - 4x + 2 \Rightarrow v' = 6x - 4$$

Portanto, $y' = \frac{v'}{v} \Rightarrow y' = \frac{6x - 4}{3x^2 - 4x + 2}$.

(b) $y = \ln x^3 \Rightarrow y' = ?$

$$y' = \frac{v'}{v} \quad ; \quad v = x^3 \Rightarrow v' = 3x^2$$

$$\Rightarrow y' = \frac{v'}{v} = \frac{3\cancel{x^2}}{x^{\cancel{3}}} = \frac{3}{x}.$$

(c) $y = \ln \sqrt{\frac{x-1}{x+2}}$. $\quad y' = ?$

Neste caso, convém efetuar algumas transformações logarítmicas a fim de deixar o problema mais simples.

Note que:

$$f(x) = \ln\sqrt{\frac{x-1}{x+2}} = \ln\left(\frac{x-1}{x+2}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{x-1}{x+2}\right)$$

$$= \frac{1}{2}\left[\ln(x-1) - \ln(x+2)\right]$$

$$\Rightarrow f'(x) = \frac{1}{2}\cdot\left[\ln(x-1) - \ln(x+2)\right]'$$

$$= \frac{1}{2}\left[\left(\ln(x-1)\right)' - \left(\ln(x+2)\right)'\right]$$

$$(\ln u)' = \frac{u'}{u}$$

$$= \frac{1}{2}\cdot\left[\frac{1}{x-1} - \frac{1}{x+2}\right] = \frac{1}{2}\,\frac{x+2-(x-1)}{(x-1)(x+2)} =$$

$$= \frac{1}{2}\,\frac{\cancel{x}+2-\cancel{x}+1}{(x-1)\cdot(x+2)} = \frac{3}{2(x-1)(x+2)}$$

Obs: Esta regra ajuda a extender a regra

$$y = u^k \Rightarrow y' = k\,u^{k-1}\cdot u', \quad \forall k \in \mathbb{R}.$$

De fato, sendo $y = u^k$, então

$$\ln y = \ln u^k$$

$$\Rightarrow \ln y = k \cdot \ln r$$

Derivando em x, obtemos:

$$\frac{y'}{y} = k \cdot \frac{r'}{r}$$

$$\Rightarrow y' = k \cdot \frac{r'}{r} \cdot y \qquad (y = r^k)$$

$$y' = k \cdot \frac{r'}{\cancel{r}} \cdot r^{\cancel{k}^{k}}$$

$$y' = k \cdot r^{k-1} \cdot r' \qquad , \forall k \in \mathbb{R}.$$

Ex.: $y = \sqrt[3]{x^2} - \sqrt{x}. \Rightarrow y' = ?$

$$y = x^{\frac{2}{3}} - x^{\frac{1}{3}}. \quad \text{Disso,}$$

$$y' = \frac{2}{3} \cdot x^{\frac{2}{3}-1} \cdot 1 - \frac{1}{3} \cdot x^{\frac{1}{3}-1} \cdot 1$$

$$y' = \frac{2}{3} \cdot x^{-\frac{1}{3}} - \frac{1}{3} \cdot x^{-\frac{2}{3}}$$

$$y' = \frac{2}{3} \cdot \frac{1}{x^{\frac{1}{3}}} - \frac{1}{3} \cdot \frac{1}{x^{\frac{2}{3}}}$$

$$y' = \frac{2}{3\sqrt[3]{x}} - \frac{1}{3\sqrt[3]{x^2}}$$

10) $\boxed{y = e^{u} \Rightarrow y' = e^{u} \cdot u'}$

De fato:

$$y = e^{u} \iff \ln y = \ln e^{u} = u \underbrace{\ln e}_{=1}$$

$$\iff u = \ln y$$

Derivando em x, vem:

$$u' = \frac{y'}{y} \Rightarrow y' = y \cdot u'$$

$$\boxed{y' = e^{u} \cdot u'}$$

Ex.: (a) $f(x) = e^{3x^2 - x}$. $\quad f'(x) = ?$

$$y = e^{u} \quad . \quad u = 3x^2 - x \Rightarrow u' = 6x - 1$$

$$\Rightarrow f'(x) = e^{u} \cdot u'$$

$$f'(x) = e^{3x^2 - x} \cdot (6x - 1)$$

(b) $y = e^{3x}$. $\quad y' = ?$

$$u = 3x \Rightarrow u' = 3$$

$$y' = e^{u} \cdot u' = e^{3x} \cdot 3 = 3 \cdot e^{3x}.$$

11) $\boxed{y = \operatorname{sen} u \Rightarrow y' = \cos u \cdot u'}$

12) $\boxed{y = \cos u \Rightarrow y' = -\operatorname{sen} u \cdot u'}$

Ex: (a) $y = \operatorname{sen}(3x^2 - x + 1)$. $y' = ?$

$u = 3x^2 - x + 1 \Rightarrow u' = 6x - 1.$

$y = \operatorname{sen} u \Rightarrow y' = \cos u \cdot u'$

$y' = \cos(3x^2 - x + 1) \cdot (6x - 1)$

$y' = (6x - 1) \cdot \cos(3x^2 - x + 1)$

(b) $y = \ln \cos(x^2 - x)$. $y' = ?$

$y = \ln u \Rightarrow y' = \dfrac{u'}{u}$

$u = \cos(x^2 - x) \Rightarrow u' = -\operatorname{sen}(x^2 - x) \cdot (2x - 1)$

Portanto, obtemos:

$$y' = \frac{u'}{u} = \frac{-\operatorname{sen}(x^2 - x) \cdot (2x - 1)}{\cos(x^2 - x)} =$$

$$= -(2x - 1) \cdot \frac{\operatorname{sen}(x^2 - x)}{\cos(x^2 - x)} = (1 - 2x) \tan(x^2 - x)$$