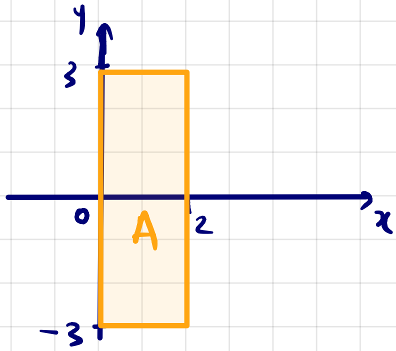
Cálculo IV – Resolução de questões da Lista 02.

01) (b) $\displaystyle\iint_A \frac{xy^2}{1+x^2}\,dx\,dy$; $A=\underbrace{[0,2]}_{x}\times\underbrace{[-3,3]}_{y}$.



$$\iint_A \frac{xy^2}{1+x^2}\,dx\,dy = \int_{y=-3}^{y=3}\left(\int_{x=0}^{x=2} \frac{x\,y^2}{1+x^2}\,dx\right)dy =$$

$$= \int_{y=-3}^{y=3} y^2\left(\frac{1}{2}\int_{x=0}^{x=2}\frac{2x\,dx}{1+x^2}\right)dy = \frac{1}{2}\int_{y=-3}^{y=3} y^2 .\ \ln|1+x^2|\Big|_{x=0}^{x=2} .\, dy =$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$u = 1+x^2 \Rightarrow du = 2x\,dx$$

$$= \frac{1}{2}.\int_{y=-3}^{y=3} y^2.(\ln 5 - \underbrace{\ln 1}_{=0})\,dy = \frac{\ln 5}{2}.\int_{-3}^{3} y^2\,dy = \frac{\ln 5}{2}.\ \frac{y^3}{3}\Big|_{-3}^{3} =$$

$$= \frac{\ln 5}{2}.\left(\frac{3^3}{3} - \frac{(-3)^3}{3}\right) = \frac{\ln 5}{2}.(9+9) = 9.\ln 5$$

02) (d) $$\int_{-1}^{1}\int_{x^2}^{1-x^2} 2x^2y^2\,dy\,dx = \int_{x=-1}^{x=1} 2x^2\left(\int_{y=x^2}^{y=1-x^2} y^2\,dy\right)dx =$$

$$= \int_{x=-1}^{x=1} 2x^2.\ \frac{y^3}{3}\Big|_{y=x^2}^{y=1-x^2} dx = \int_{x=-1}^{x=1} \frac{2}{3}x^2.\left[(1-x^2)^3 - (x^2)^3\right]dx$$

$$= \int_{x=-1}^{x=1} \frac{2}{3}.x^2.\left[(1-x^2)(1-x^2)^2 - x^6\right]dx =$$

$$= \frac{2}{3}\int_{-1}^{1} x^2\left[(1-x^2)(1-2x^2+x^4) - x^6\right] dx =$$

$$= \frac{2}{3}\int_{-1}^{1} x^2(1 - 2x^2 + x^4 - x^2 + 2x^4 - x^6 - x^6)\, dx =$$

$$= \frac{2}{3}\int_{-1}^{1} (x^2 - 3x^4 + 3x^6 - 2x^8)\, dx = \frac{2}{3}\left(\frac{x^3}{3} - \frac{3x^5}{5} + \frac{3x^7}{7} - \frac{2x^9}{9}\right)\Bigg|_{-1}^{1} =$$

$$= \frac{2}{3}\left(\cancel{\frac{1}{3}} - \cancel{\frac{3}{5}} + \cancel{\frac{3}{7}} - \cancel{\frac{2}{9}} - \cancel{\frac{1}{3}} + \cancel{\frac{3}{5}} - \cancel{\frac{3}{7}} + \cancel{\frac{2}{9}}\right) = 0$$

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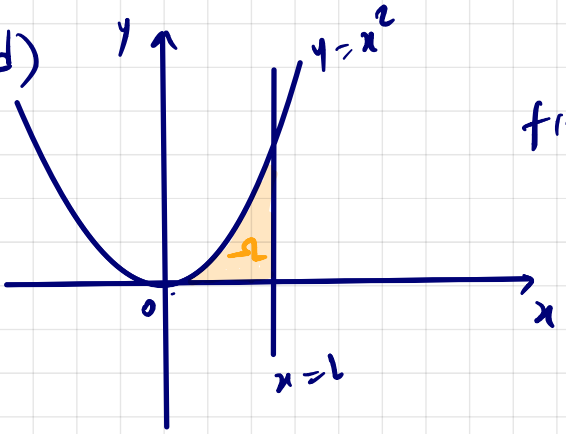
03) (d)



$$f(x,y) = x e^y$$

$$\iint_\Omega f(x,y)\, dA = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2} x . e^y\, dy\, dx = \int_{x=0}^{x=1} x . \; e^y\Big|_{y=0}^{y=x^2} dx = \int_0^1 x . (e^{x^2} - e^0)\, dx$$

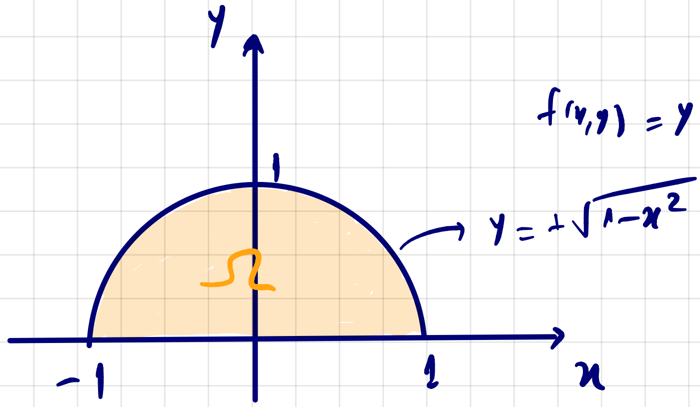
$$= \int_0^1 x\,(e^{x^2} - 1)\, dx = \frac{1}{2}\int_0^1 e^{x^2}(2x\, dx) - \int_0^1 x\, dx =$$

$$= \frac{1}{2} . e^{x^2}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}(e^1 - e^0) - \left(\frac{1}{2} - 0\right) =$$

$$\frac{1}{2}e - \frac{1}{2} - \frac{1}{2} = \frac{1}{2}e - 1$$

03) (e)

$f(x,y) = y$



$$\iint_{\Omega} f = \int_{x=-1}^{x=1} \left(\int_{y=0}^{y=\sqrt{1-x^2}} y \, dy \right) dx = \int_{x=-1}^{x=1} \frac{y^2}{2} \Bigg|_{y=0}^{y=\sqrt{1-x^2}} dx =$$

$$= \int_{x=-1}^{x=1} \frac{1}{2} \left((\sqrt{1-x^2})^2 - 0^2 \right) dx = \frac{1}{2} \int_{-1}^{1} (1-x^2) \, dx =$$

$$= \frac{1}{2} \left(x - \frac{x^3}{3} \right) \Bigg|_{-1}^{1} = \frac{1}{2} \left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3} \right) \right) =$$

$$= \frac{1}{2} \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \frac{1}{2} \left(2 - \frac{2}{3} \right) = \frac{1}{2} \left(\frac{4}{3} \right) = \frac{2}{3} \;//$$

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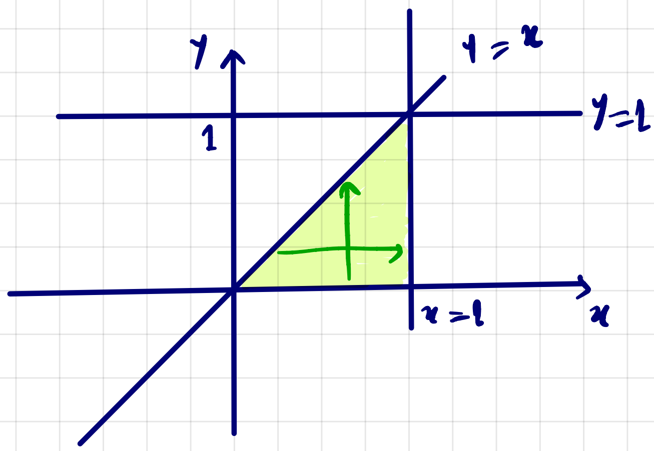
04) (a) $$\int_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} \, dx \, dy =$$



$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} \, dy \, dx =$$

$$= \int_{x=0}^{x=1} e^{x^2} \int_{y=0}^{y=x} dy \; dx = \int_{x=0}^{x=1} e^{x^2} \cdot y \Big|_{y=0}^{y=x} dx =$$

$$= \frac{1}{2}\int_{x=0}^{x=1} e^{x^2}(2x\,dx) = \frac{1}{2}\, e^{x^2}\Big|_{x=0}^{x=1} = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e-1)$$

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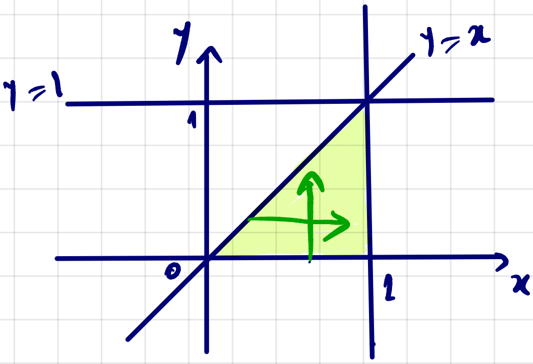
04) (b)

$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} \sqrt{1+x^2}\,dx\,dy =$$



$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x} \sqrt{1+x^2}\,dy\,dx = \int_{x=0}^{x=1} \sqrt{1+x^2}\int_{y=0}^{y=x} dx\,dy = \int_{x=0}^{x=1} \sqrt{1+x^2}\cdot x\Big|_{y=0}^{y=x} \cdot dy =$$

$$= \int_{x=0}^{x=1} \sqrt{1+x^2}\cdot x\cdot dx = \frac{1}{2}\int_0^1 (1+x^2)^{\frac{1}{2}}\cdot(2x\cdot dx) = \frac{1}{\cancel{2}}\,\frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}\Bigg|_0^1 =$$

$$= \frac{1}{3}(1+1)^{\frac{3}{2}} - \frac{1}{3}(1+0)^{\frac{3}{2}} = \frac{1}{3}\cdot\left[\sqrt{2^3} - 1\right] = \frac{1}{3}(2\sqrt{2}-1).$$

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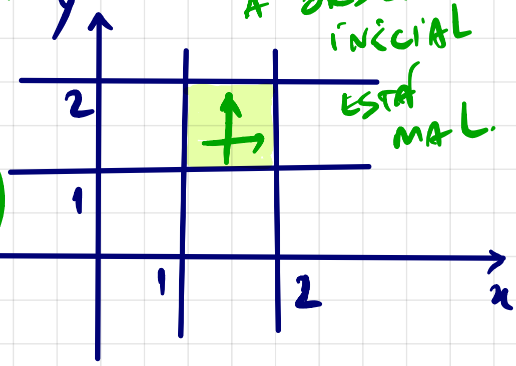
04) (c)

$$\int_{x=1}^{x=2}\int_{y=1}^{y=2} y\cdot e^{xy}\cdot dy\,dx =$$

ATENÇÃO, ARRUMAR NO ENUNCIADO, A ORDEM INICIAL ESTÁ MAL.

ORDEM INICIAL CORRETA.



$$= \int_{y=1}^{y=2}\int_{x=1}^{x=2} y\cdot e^{xy}\cdot dx\,dy = \int_{y=1}^{y=2}\left(\int_{x=1}^{x=2} e^{xy}\cdot(y\,dx)\right)dy =$$

$v = xy \Rightarrow dv = y\,dx$

$$= \int_{y=1}^{y=2} e^{xy}\Big|_{x=1}^{x=2} \cdot dy = \int_{y=1}^{y=2} (e^{2y} - e^{y})\,dy = \frac{1}{2}\int_1^2 e^{2y}(2dy) - \int_1^2 e^{y}dy$$

$$= \left(\frac{1}{2}e^{2y} - e^{y}\right)\Big|_1^2 = \frac{1}{2}e^4 - e^2 - \frac{1}{2}e^2 + e^1$$

$$= \frac{1}{2}e^4 - \frac{3}{2}e^2 + e = \frac{1}{2}\cdot e\,(e^2 - 3e + 2)$$

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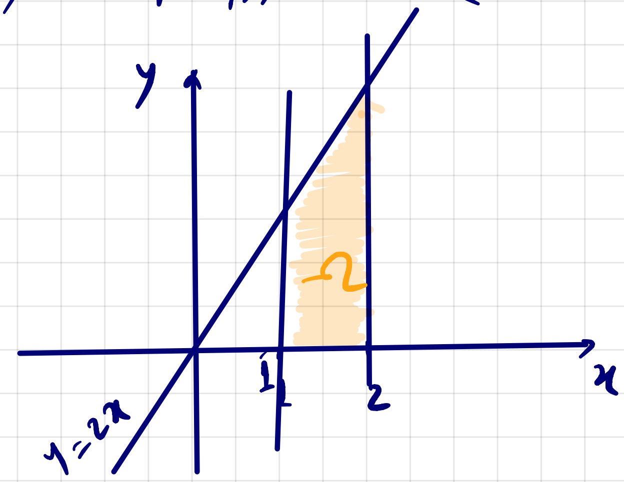
05) (a) $\iint_{\Omega} \frac{2y}{x^3+2}\,dA$; $\Omega = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2 ; \ 0 \leq y \leq 2x\}$



$$\iint_{\Omega} \frac{2y}{x^3+2}\,dA = \int_{x=1}^{x=2}\left(\int_{y=0}^{y=2x} \frac{2y}{x^3+2}\cdot dy\right)dx = \int_{x=1}^{x=2} \frac{2}{x^3+2}\left(\int_{y=0}^{y=2x} y\,dy\right)dx$$

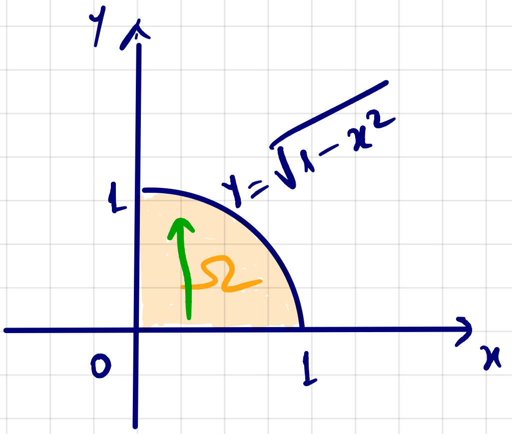
$$= \int_{x=1}^{x=2} \frac{\cancel{2}}{x^3+2}\cdot \frac{y^2}{\cancel{2}}\Big|_{y=0}^{y=2x} dx = \int_1^2 \frac{4x^2}{x^3+2}\,dx =$$

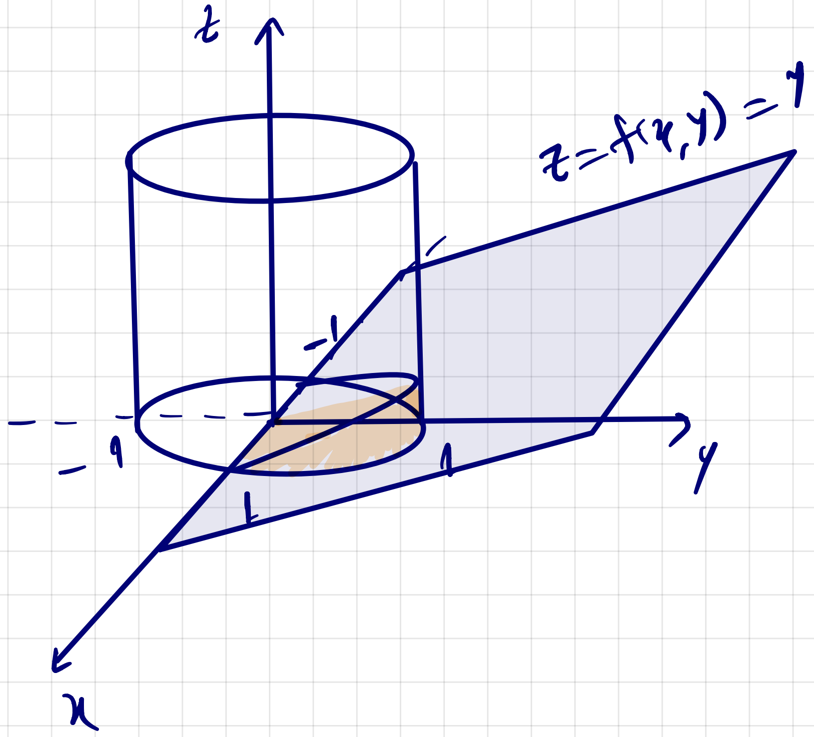
$\rightarrow v = x^3+2$ ↗ $dv = 3x^2dx$

$$= \frac{4}{3}\int_1^2 \frac{3x^2}{x^3+2}\,dx = \frac{4}{3}\cdot \ln|x^3+2|\Big|_1^2 =$$

$$= \frac{4}{3}\cdot \ln|10| - \ln 3 = \frac{4}{3}\ln\frac{10}{3}$$

08) Achar o volume do sólido limitado pelo cilindro $x^2+y^2=1$ e pelos planos $y=z$; $x=0$, $z=0$, no primeiro quadrante.





$$V = \iint_\Omega f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} y\,dy\,dx = \int_{x=0}^{x=1} \frac{y^2}{2}\Big|_{y=0}^{y=\sqrt{1-x^2}} dx =$$

$$= \frac{1}{2}\int_0^1 \left((\cancel{\sqrt{1-x^2}})^{\cancel{2}} - 0^2\right) dx = \frac{1}{2}\int_0^1 (1-x^2)\,dx =$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{2}\left(1 - \frac{1}{3} - 0\right) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

Resp.: $\frac{1}{3}$ unidades de volume (u.v.)

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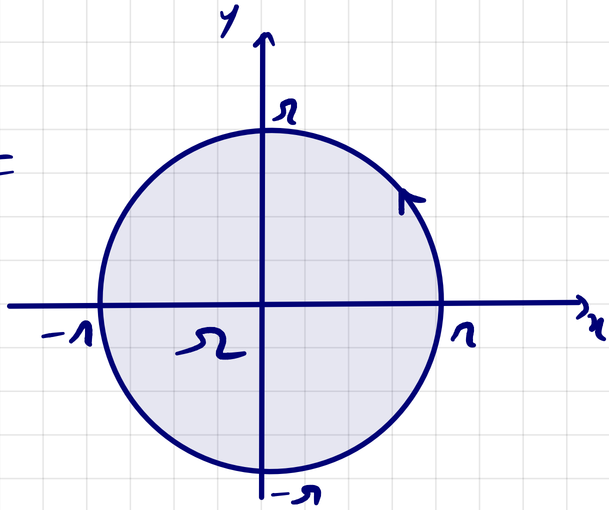
09) (c) $$\iint\limits_{x^2+y^2\le r^2} x^2 \cdot e^{-(x^2+y^2)^2}\,dx\,dy =$$



$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=r} \rho^2\cos^2\theta \cdot e^{-\rho^4}\cdot \rho\,d\rho\,d\theta =$$

$x = \rho\cos\theta$.

$x^2+y^2 = \rho^2$

$$= \int_{\theta=0}^{\theta=2\pi} \cos^2\theta\,d\theta \cdot \int_{\rho=0}^{\rho=r} e^{-\rho^4}\cdot\rho^3\,d\rho =$$

$u = -\rho^4 \Rightarrow du = -4\rho^3\,d\rho$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1+\cos 2\theta}{2}\,d\theta \cdot \left(-\frac{1}{4}\right)\int_{\rho=0}^{\rho=r} e^{-\rho^4}(-4\rho^3\cdot d\rho) =$$

$$\frac{1}{2}\int_0^{2\pi} d\theta + \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi}\cos 2\theta\,(2d\theta)\cdot\left(-\frac{1}{4}\right)\cdot e^{-\rho^4}\Big|_{\rho=0}^{\rho=1} =$$

$$= \left(\frac{\theta}{2}+\frac{1}{4}\,\operatorname{sen}2\theta\right)\Big|_{\theta=0}^{\theta=2\pi}\cdot\left(-\frac{1}{4}\left(e^{-1^4}-e^{0}\right)\right) =$$

$$=\left(\pi+\underbrace{\frac{1}{4}\cdot\operatorname{sen}4\pi}_{=0} - 0 - \underbrace{\frac{1}{4}\operatorname{sen}0}_{=0}\right)\cdot\left(+\frac{1}{4}\left(1-e^{-1^4}\right)\right)=$$

$$=\frac{\pi}{4}\cdot\left(1-e^{-1^4}\right)$$

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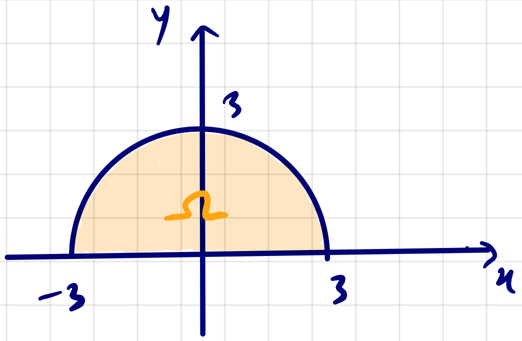
11) $$\iint_\Omega \cos(x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\theta=\pi}\int_{\rho=0}^{\rho=3}\cos\rho^2\cdot\rho\,d\rho\,d\theta =$$



$$=\int_{\theta=0}^{\theta=\pi}d\theta\cdot\frac{1}{2}\int_{\rho=0}^{\rho=3}\cos\rho^2\,(2\rho\,d\rho) =$$

$$=\frac{1}{2}\cdot\theta\Big|_0^{\pi}\cdot\left(\operatorname{sen}\rho^2\right)\Big|_0^{3} = \frac{\pi}{2}\cdot\left(\operatorname{sen}9-\operatorname{sen}0\right)=\frac{\pi}{2}\cdot\operatorname{sen}9\,.$$

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12) $\displaystyle\iint_\Omega \arctan \frac{y}{x} \cdot dA$.

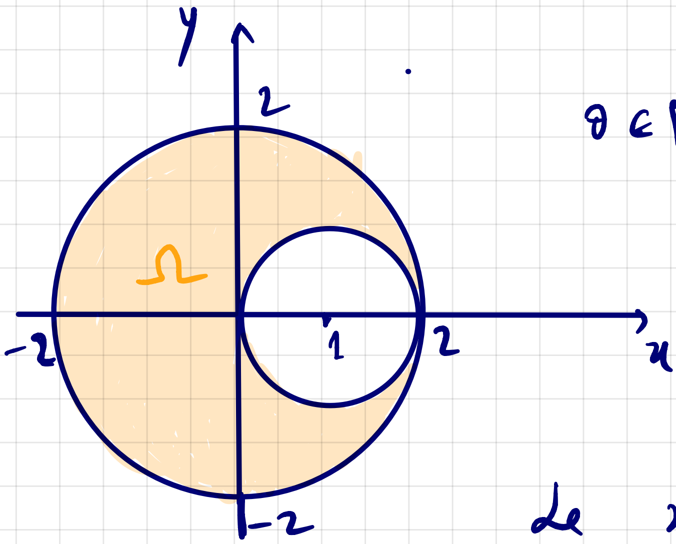
Ω : região entre os círculos $x^2+y^2=4$ e $x^2+y^2=2x$

$$\Updownarrow$$

$$x^2-2x+y^2=0$$

$$x^2-2x+1+y^2=1$$

$$(x-1)^2+y^2=1.$$



$\theta \in [0, 2\pi]$

variação de ρ:

de $x^2+y^2=2x$ até $x^2+y^2=4$

$\rho^{\cancel{2}} = 2\cancel{\rho}\cos\theta$ até $\rho^2=4$

$\rho = 2\cos\theta$ até $\rho = 2$.

Assim, temos:

$$\iint_\Omega \arctan\frac{y}{x}\, dA = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=2\cos\theta}^{\rho=2} \theta \cdot \rho\, d\rho\, d\theta = \int_{\theta=0}^{\theta=2\pi} \theta \cdot \left(\int_{\rho=2\cos\theta}^{\rho=2} \rho\, d\rho\right) d\theta =$$

Note que $\tan\theta = \frac{y}{x}$

$\Rightarrow \theta = \arctan\frac{y}{x}$

$$= \int_{\theta=0}^{\theta=2\pi} \theta \cdot \frac{\rho^2}{2}\Bigg|_{\rho=2\cos\theta}^{\rho=2} d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{\theta}{2} \cdot (4-4\cos^2\theta)\, d\theta = 2 \cdot \int_0^{2\pi} \theta \cdot (1-\cos^2\theta)\, d\theta =$$

$$= 2\int_0^{2\pi} \theta \cdot \operatorname{sen}^2\theta \cdot d\theta = \cancel{2}\int_0^{2\pi} \theta \cdot \left(\frac{1-\cos 2\theta}{\cancel{2}}\right) d\theta =$$

$$= \int_0^{2\pi} (\theta - \theta \cdot \cos 2\theta)\, d\theta = \int_0^{2\pi} \theta\, d\theta - \underbrace{\int_0^{2\pi} \theta \cdot \cos 2\theta\, d\theta}_{\text{INTEGRAR POR PARTES.}} =$$

$$\bullet \int \theta \cos 2\theta\, d\theta = \int u\, dv = u \cdot v - \int v\, du$$

$$\left[\begin{array}{l} u = \theta \Rightarrow du = d\theta \\ dv = \cos 2\theta\, d\theta \Rightarrow v = \frac{1}{2} \operatorname{sen} 2\theta \end{array} \right.$$

$$\Rightarrow \int \theta \cos 2\theta\, d\theta = \frac{\theta}{2} \cdot \operatorname{sen} 2\theta - \int \frac{1}{2} \operatorname{sen} 2\theta \cdot d\theta$$

$$= \frac{\theta}{2} \operatorname{sen} 2\theta - \frac{1}{4} \cdot \int \operatorname{sen} 2\theta \cdot (2\, d\theta)$$

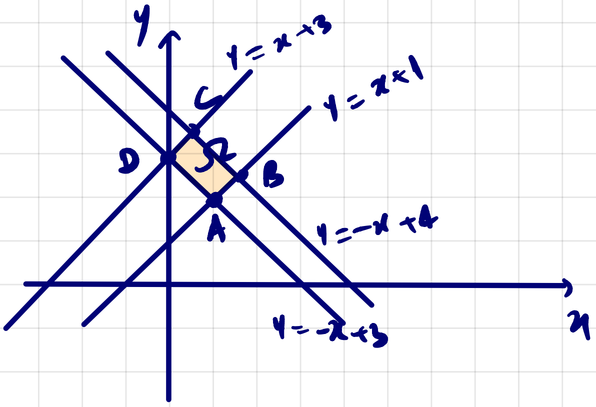
$$= \frac{\theta}{2} \cdot \operatorname{sen} 2\theta + \frac{1}{4} \cdot \cos 2\theta + C$$

$$= 2\pi - \left(\frac{\theta}{2} \operatorname{sen} 2\theta + \frac{1}{4} \cos 2\theta \right) \Bigg|_0^{2\pi} = 2\pi - \frac{2\pi}{2} \cdot \overbrace{\operatorname{sen} 4\pi}^{=0} - \frac{1}{4} \overbrace{\cos 4\pi}^{=1} +$$

$$+ 0 + \frac{1}{4} \underbrace{\cos 0}_{=1} =$$

$$= 2\pi - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} = \underline{\underline{2\pi}}$$

13) $\displaystyle\iint_{\Omega} \frac{(x+y)^7}{y-x}\,dx\,dy$; Ω limitada pelas retas $y = x+3$; $y = x+1$; $y = -x+3$ e $y = -x+4$.



Então

$$\begin{cases} u = y - x \\ v = y + x \end{cases}$$

$+$

$u + v = 2y \Rightarrow y = \frac{1}{2}u + \frac{1}{2}v$

$$\begin{cases} v = \frac{1}{2}u + \frac{1}{2}v + x \\ \Rightarrow x = -\frac{1}{2}u + \frac{1}{2}v \end{cases}$$

$$T(u,v) = (x,y) = \left(-\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u + \frac{1}{2}v\right)$$

$$\det(J(T)(x,y)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ +\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

VÉRTICES DE Ω:

- $A(x,y)$; onde $\begin{cases} y = x+1 \\ y = -x+3 \end{cases} \Leftrightarrow x+1 = -x+3 \Leftrightarrow 2x = 2 \Leftrightarrow x = 1$

  Então: $y = 1+1 = 2$

  Logo: $A(1,2)$

- $B(x,y)$; onde $\begin{cases} y = x+1 \\ y = -x+4 \end{cases} \Leftrightarrow x+1 = -x+4 \Leftrightarrow 2x = 3 \Leftrightarrow x = \frac{3}{2}$.

  Então: $y = \frac{3}{2} + 1 = \frac{5}{2}$

  Logo: $B\left(\frac{3}{2}, \frac{5}{2}\right)$

- $C(x,y)$; onde: $\begin{cases} y = x+3 \\ y = -x+4 \end{cases} \Leftrightarrow x+3 = -x+4 \Leftrightarrow 2x = 1 \Leftrightarrow x = \frac{1}{2}$

  Então: $y = \frac{1}{2} + 3 = \frac{7}{2}$

  Logo: $C\left(\frac{1}{2}, \frac{7}{2}\right)$

• $D(x,y)$; onde: $\begin{cases} y = x+3 \\ y = -x+3 \end{cases} \Leftrightarrow x+3 = -x+3 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$

Então: $y = 3$

Logo: $D(0,3)$

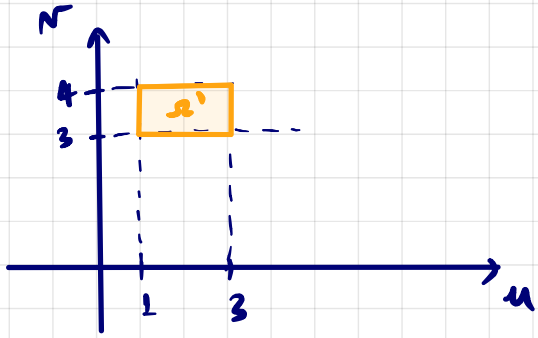
Assim, sendo $u = y-x$ e $v = y+x$, teremos, via T:

• $A(1,2) \rightsquigarrow T(A) = (u,v) = (2-1,\ 2+1) = (1,3)$

• $B\left(\frac{3}{2}, \frac{5}{2}\right) \rightsquigarrow T(B) = (u,v) = \left(\frac{5}{2} - \frac{3}{2},\ \frac{5}{2} + \frac{3}{2}\right) = (1,4)$

• $C\left(\frac{1}{2}, \frac{7}{2}\right) \rightsquigarrow T(C) = (u,v) = \left(\frac{7}{2} - \frac{1}{2},\ \frac{7}{2} + \frac{1}{2}\right) = (3,4)$

• $D(0,3) \rightsquigarrow T(D) = (u,v) = (3-0,\ 3+0) = (3,3)$



Assim, obtemos, através da mudança de variáveis:

$$\iint_{\Omega} \frac{(x+y)^7}{y-x}\, dx\, dy = \iint_{\Omega'} \frac{v^7}{u} \cdot \left|\det J(T)(u,v)\right| \cdot du\, dv =$$

$$= \int_{v=3}^{v=4} \int_{u=1}^{u=3} \frac{v^7}{u} \cdot \left|-\frac{1}{2}\right| du\, dv = \frac{1}{2} \int_{v=3}^{v=4} v^7 \cdot dv \cdot \int_{u=1}^{u=3} \frac{du}{u} =$$

$$\frac{1}{2}\, \frac{v^8}{8}\Big|_3^4 \cdot \ln|u|\Big|_{u=1}^{u=3} =$$

$$\frac{1}{16} \cdot (4^8 - 3^8) \cdot (\ln 3 - \underbrace{\ln 1}_{0}) = \frac{1}{16} \cdot (4^8 - 3^8) \cdot \ln 3.$$