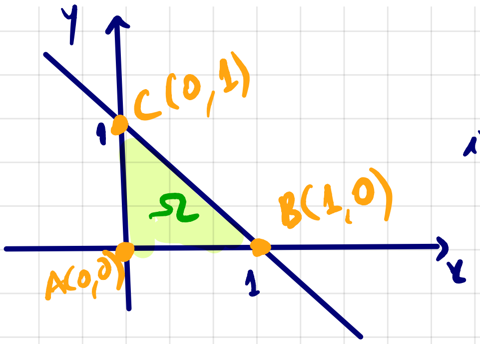
CÁLCULO IV 31/01/2024 - $\boxed{\text{AULA 13}}$

AULA DE EXERCÍCIOS

LISTA 03

3. Calcule $\displaystyle\iint_\Omega e^{\frac{x-y}{x+y}}\,dxdy$, onde Ω é o triângulo delimitado pelos eixos $x=0$, $y=0$ e pela reta $x+y=1$.



A região Ω é "boa" para integrar; mas a função $f(x,y) = e^{\frac{x-y}{x+y}}$ não é fácil de integrar.

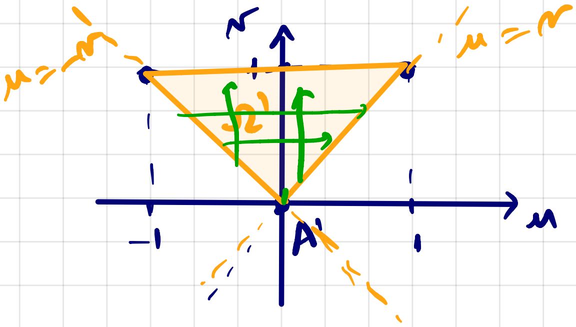
Por esta razão vamos efetuar uma mudança de variável

Escreva
$$\begin{cases} u = x-y \\ v = x+y \end{cases}$$
$$+$$
$$u+v = 2x \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$$

Então:
$$y = v - x$$
$$y = v - \frac{1}{2}u - \frac{1}{2}v \Rightarrow \boxed{y = -\frac{1}{2}u + \frac{1}{2}v}$$

$$\iint_\Omega f(x,y)\,dx\,dy = \iint_{\Omega'} f(u,v)\cdot|\det(J)(T)|\cdot du\,dv$$

$$(x,y) \xrightarrow{T} (u,v) = (x-y,\ x+y)$$
$$A(0,0) \longrightarrow (u,v) = (0-0,\ 0+0) = (0,0) = A'$$
$$B(1,0) \longrightarrow (u,v) = (1-0,\ 1+0) = (1,1) = B'$$
$$C(0,1) \longrightarrow (u,v) = (0-1,\ 0+1) = (-1,1) = C'$$



$$J(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

Assim, teremos:

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega'} f(u,v)\cdot|\det(J)(T)|\cdot du\,dv =$$

$$= \int_{v=0}^{v=1}\int_{u=-v}^{u=v} e^{\frac{u}{v}}\cdot\left|\frac{1}{2}\right|\cdot du\,dv = \frac{1}{2}\int_{v=0}^{v=1}\int_{u=-v}^{u=v} e^{u}\cdot e^{-v}\,du\,dv$$

$$= \frac{1}{2}\int_{v=0}^{v=1} e^{-v}\cdot\left(e^{u}\right)\Big|_{u=-v}^{u=v} dv = \frac{1}{2}\int_{v=0}^{v=1} e^{-v}\left(e^{v}-e^{-v}\right)dv$$

$$= \frac{1}{2}\int_{0}^{1}\left(e^{-v+v} - e^{-v-v}\right)dv = \frac{1}{2}\int_{0}^{1}\left(1 - e^{-2v}\right)dv$$

$$\frac{1}{2}\int_{0}^{1} dv - \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\int e^{-2v}(-2dv) = \frac{1}{2}v\Big|_{0}^{1} + \frac{1}{4}e^{-2v}\Big|_{0}^{1} =$$

$w = -2v \Rightarrow dw = -2dv$

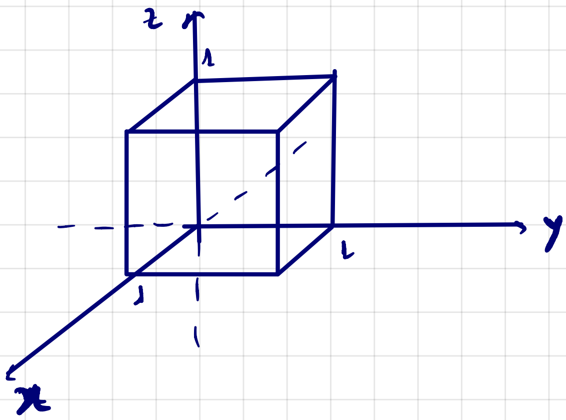
$$\frac{1}{2} - 0 + \frac{1}{4}\cdot\left(e^{-2} - e^{0}\right) = \frac{1}{2} + \frac{1}{4}\left(e^{-2} - 1\right)$$

$$= \frac{1}{2} + \frac{e^{-2}}{4} - \frac{1}{4} = \frac{1+e^{-2}}{4}$$

6. Em cada item a seguir, calcule a integral da função dada sobre o domínio D dado. Sempre que possível, faça um esboço do domínio D.

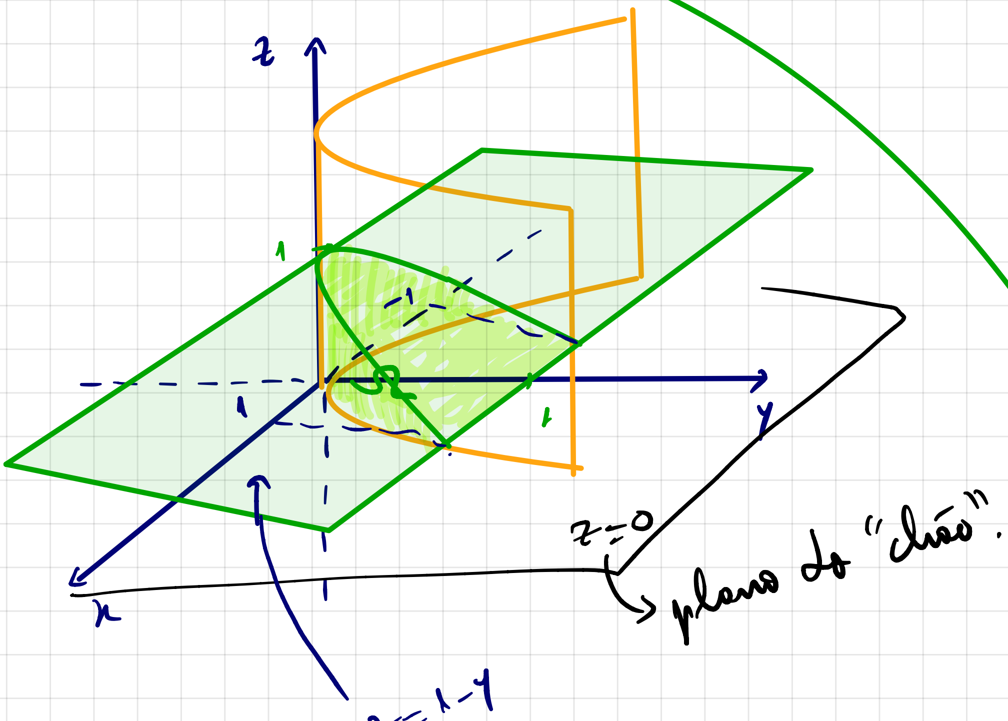
(a) $f(x,y,z) = xy^2z^3$ e $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.

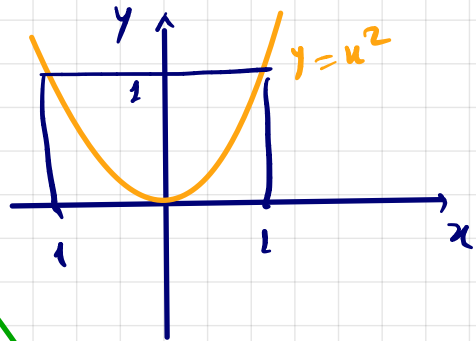
$$\iiint_D f = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} xy^2z^3 \, dz\, dy\, dx =$$

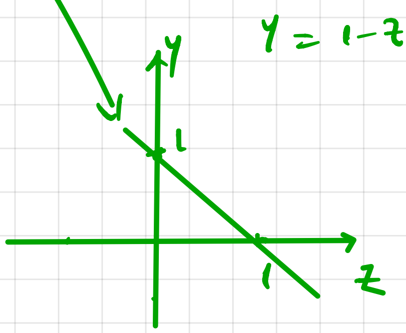


$$= \int_{x=0}^{x=1} x \left(\int_{y=0}^{y=1} y^2 \left(\int_{z=0}^{z=1} z^3 \, dz \right) dy \right) dx = \quad \ldots \text{ AGORA FICA SIMPLES.}$$

7. Use integral tripla para determinar o volume do sólido limitado pelo cilindro $y = x^2$ e pelos planos $z = 0$ e $y + z = 1$.







$$V = \iiint_{\Omega} dV = \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} \int_{z=0}^{z=1-y} dz\,dy\,dx =$$

$$= \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} z \Big|_{z=0}^{z=1-y} dy\,dx = \int_{x=-1}^{x=1} \left(\int_{y=x^2}^{y=1} (1-y)\,dy \right) dx =$$

$$= \int_{x=-1}^{x=1} \left(y - \frac{y^2}{2} \right) \Big|_{y=x^2}^{y=1} dx = \int_{-1}^{1} \left(1 - \frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx =$$

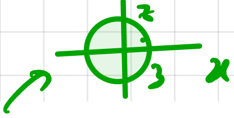
$$= \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx = \left(\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10} \right) \Big|_{-1}^{1} =$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{10} - \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{10} \right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10}$$

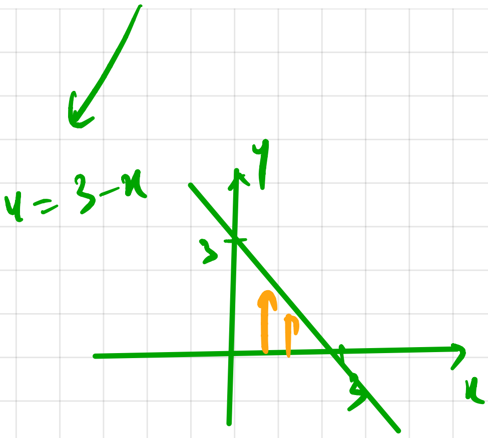
$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{15 - 10 + 3}{15} = \frac{8}{15}$$

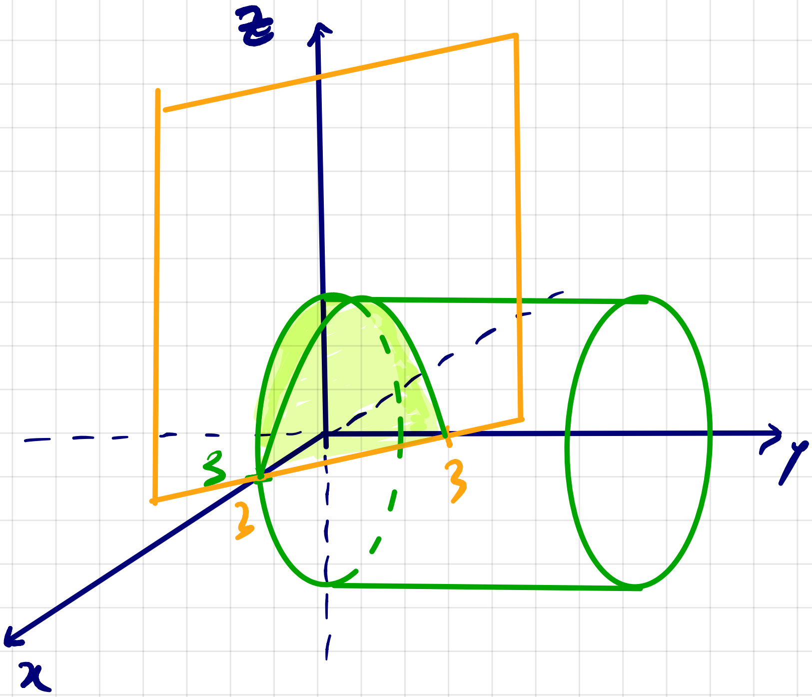
LISTA 03 – A1 → não fazer. (deve ter algum problema)

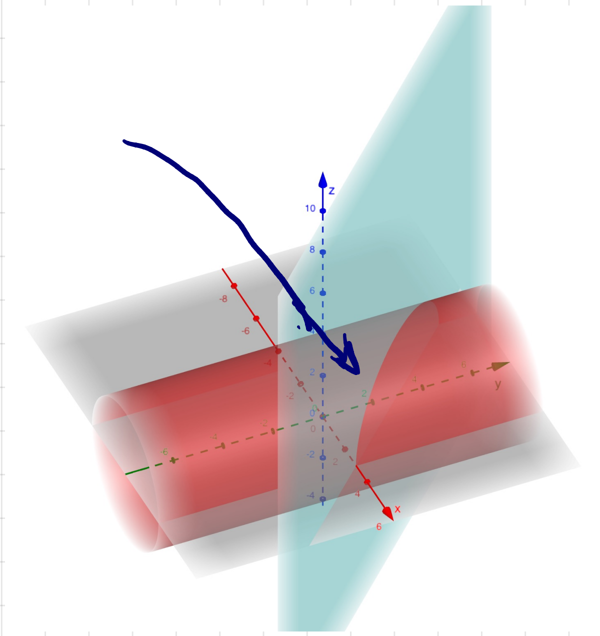
L3

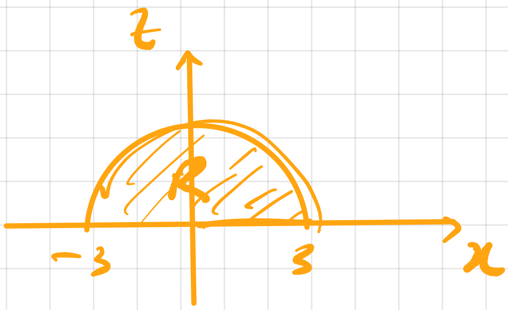


12. Calcule $\iiint_S (xz+3z)dV$, onde S é a região limitada pelo cilindro $x^2+z^2=9$ e pelos planos $x+y=3$, $z=0$, $y=0$ e acima do plano xy. (Resp.: $\frac{648}{5}$)









$$\iiint_S (xz+3z)\,dV = \iint_R \int_{y=0}^{y=3-x} (xz+3z)\,dy\,\underbrace{dx\,dz}_{} =$$

VAMOS USAR COORD. POLARES

$$= \int_{\theta=0}^{\theta=\pi}\int_{\rho=0}^{\rho=3}\left(\int_{y=0}^{y=3-x}(xz+3z)\,dy\right)\cdot\rho\,d\rho\,d\theta =$$

$$= \int_{\theta=0}^{\theta=\pi}\int_{\rho=0}^{\rho=3}(xz+3z)\cdot y\,\Big|_{y=0}^{y=3-x}\rho\,d\rho\,d\theta =$$

$$\int_{\theta=0}^{\theta=\pi}\int_{\rho=0}^{\rho=3} z(x+3)\cdot\left[3-x-0\right]\rho\, d\rho\, d\theta =$$

[NOTE QUE $x=\rho\cos\theta$
$z=\rho\,\text{sen}\,\theta$.
i.e. z substituí o "papel" do y]

$$=\int_{\theta=0}^{\theta=\pi}\left(\int_{\rho=0}^{\rho=3}\rho\,\text{sen}\,\theta(\rho\cos\theta+3)\cdot(\rho\cos\theta-3)\cdot(-1)\cdot\rho\cdot d\rho\right)d\theta$$

$$=-\int_{\theta=0}^{\theta=\pi}\left(\int_{\rho=0}^{\rho=3}\rho^2\,\text{sen}\,\theta\cdot(\rho^2\cos^2\theta-9)\cdot d\rho\right)d\theta =$$

$$=\int_{\theta=0}^{\theta=\pi}\left(\int_{\rho=0}^{\rho=3}(-\rho^4\cos^2\theta\cdot\text{sen}\,\theta+9\rho^2\,\text{sen}\,\theta)\,d\rho\right)d\theta =$$

$$=\int_{\theta=0}^{\theta=\pi}\left(-\cos^2\theta\cdot\text{sen}\,\theta\cdot\frac{\rho^5}{5}+\text{sen}\,\theta\cdot 3\rho^3\right)\Bigg|_{\rho=0}^{\rho=3} d\theta =$$

$$=\int_{\theta=0}^{\theta=\pi}\left[-\cos^2\theta\cdot\text{sen}\,\theta\left(\frac{243}{5}-0\right)+\text{sen}\,\theta\,(81-0)\right]d\theta$$

$$=\frac{243}{5}\int_0^{\pi}\underbrace{(\cos\theta)^2(-\text{sen}\,\theta\, d\theta)}_{u^k du}+81\cdot\int_0^{\pi}\text{sen}\,\theta\, d\theta$$

$$=\left[\frac{243}{5}\cdot\frac{\cos^3\theta}{3}+81\cdot(-\cos\theta)\right]\Bigg|_0^{\pi} =$$

$$\frac{243}{15} \cdot (-1)^3 + 81 - \frac{243}{15} (1)^3 - (-81)$$

$$= -\frac{243}{15} \times 2 + 81 \times 2 = -\frac{486}{15} + 162 = \frac{-486 + 2430}{15}$$

$$= \frac{1944}{15} \overset{\llcorner 3}{=} \frac{648}{5}$$