CÁLCULO 2 – Resolução de algumas questões da Lista 05. (Prof. Dr. M. Zahn)

Q1) (d) $\displaystyle\int \frac{(3x-1)\,dx}{\sqrt{x^2+x+1}} =$

Escrevendo $v = x^2+x+1$, obtemos $dv = (2x+1)\,dx$.

Assim; faremos:

$$3x-1 = \frac{3}{2}(2x+1) + \beta = 3x + \frac{3}{2} + \beta$$

PRIMEIRO AJUSTA O COEFICIENTE DO x

$$\frac{3}{2} + \beta = -1$$

$$\beta = -1 - \frac{3}{2} = -\frac{5}{2}$$

ou seja: $3x - 1 = \frac{3}{2}(2x+1) - \frac{5}{2}$. Portanto:

$$\int \frac{(3x-1)\,dx}{\sqrt{x^2+x+1}} = \int \frac{\frac{3}{2}(2x+1) - \frac{5}{2}}{\sqrt{x^2+x+1}}\,dx =$$

$$= \frac{3}{2}\int \frac{\overbrace{(2x+1)\,dx}^{dv}}{\sqrt{\underbrace{x^2+x+1}_{v}}} - \frac{5}{2}\int \frac{dx}{\sqrt{x^2+x+1}} = \frac{3}{2}\int \frac{dv}{\sqrt{v}} - \frac{5}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$v = x^2+x+1$
$\hookrightarrow dv = (2x+1)\,dx$
OK!

$$= \frac{3}{2}\int v^{-\frac{1}{2}}\,dv - \frac{5}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

VAMOS COMPLETAR UM QUADRADO PERFEITO PARA ESTE.

$$x^2+x+1 \equiv [x+a]^2+b$$
$$\equiv x^2+2ax+a^2+b$$
$$2a=1 \Leftrightarrow a=\frac{1}{2}$$
$$a^2+b=1 \Leftrightarrow \left(\frac{1}{2}\right)^2+b=1$$
$$\Leftrightarrow b=1-\frac{1}{4}=\frac{3}{4}$$

dimo:

$$x^2+x+1 \equiv \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\Leftrightarrow \frac{3}{2}\cdot\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-\frac{5}{2}\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}=$$

$$\int\frac{du}{\sqrt{u^2+a^2}}=\ln\left|u+\sqrt{u^2+a^2}\right|+c$$

$u=x+\frac{1}{2} \Rightarrow du=dx$ (ok!)

$$=\frac{3}{\cancel{2}}\,\frac{x^{\frac{1}{2}}}{\frac{1}{\cancel{2}}}-\frac{5}{2}\cdot\ln\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$$

$$=\boxed{3\sqrt{x}-\frac{5}{2}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c}$$

02) (a) $\displaystyle\int \frac{(4x-2)\,dx}{x^3-x^2-2x}$. (decomposição em frações parciais)

zeros do denominador:

$$x^3-x^2-2x=0 \Leftrightarrow x(x^2-x-2)=0$$

$$\begin{cases} x=0 \\ x^2-x-2=0 \Leftrightarrow x=\dfrac{1\pm\sqrt{1+8}}{2} \end{cases}$$

$$\Leftrightarrow x=\frac{1\pm 3}{2} \nearrow x=2 \quad \searrow -1$$

Assim:

$$x^3-x^2-2x=0 \Leftrightarrow x(x-(-1)).(x-2)=0$$

$$\Leftrightarrow x(x+1)(x-2)=0$$

Desta forma, efetuamos a seguinte decomposição em frações parciais:

$$\int \frac{(4x-2)\,dx}{x^3-x^2-2x} = \int \frac{(4x-2)\,dx}{x(x+1)(x-2)}, \text{ onde:}$$

$$\frac{4x-2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$\frac{4x-2}{x(x+1)\cancel{(x-2)}} = \frac{A(x+1)(x-2)+Bx(x-2)+Cx(x+1)}{x(x+1)\cancel{(x-2)}}$$

$$4x-2 = Ax^2-Ax-2A+Bx^2-2Bx+Cx^2+Cx$$

$$\Leftrightarrow \begin{cases} A+B+C=0 \\ -A-2B+C=4 \\ -2A \qquad = -2 \Rightarrow A=1 \end{cases} \rightsquigarrow \begin{cases} B+C=-1 \\ -2B+C=5 \end{cases}$$

$$\Rightarrow \begin{cases} C = -1 - B \\ -2B + C = 5 \end{cases} \rightsquigarrow \quad -2B - 1 - B = 5$$

$$-3B = 6 \Rightarrow \boxed{B = -2}$$

$$\Rightarrow C = -1 - B$$

$$C = -1 - (-2) \Rightarrow \boxed{C = 1}$$

Dessa forma, obtemos:

$$\int \frac{(4x-2)\,dx}{x^3 - x^2 - 2x} = \int \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2} \right) dx =$$

$$= \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{1}{x-2} \right) dx = \int \frac{dx}{x} - 2\int \frac{dx}{x+1} + \int \frac{dx}{x-2}$$

$du = dx$; OK!

$$= \ln|x| - 2\ln|x+1| + \ln|x-2| + C$$

$$= \ln|x| + \ln|x+1|^{-2} + \ln|x-2| + C =$$

$$= \ln \frac{|x \cdot (x-2)|}{(x+1)^2} + C$$

02) (m) $\int \frac{(4x^2+2x+8)\,dx}{x(x^2+2)^2} = ?$

A decomposição será:

$$\frac{4x^2+2x+8}{x(x^2+2)^2} \equiv \frac{A}{x} + \frac{Bx+C}{(x^2+2)^2} + \frac{Dx+E}{x^2+2}$$

$$\equiv \frac{A(x^2+2)^2 + (Bx+C).x + (Dx+E).x.(x^2+2)}{x(x+2)^2} \Leftarrow$$

$$4x^2+2x+8 \equiv Ax^4+4Ax^2+4A+Bx^2+Cx+Dx^4+2Dx^2+Ex^3+2Ex$$

$$\begin{cases} A + D = 0 \\ E = 0 \\ 4A + B + D = 4 \\ C + E = 2 \\ 4A = 8 \Longrightarrow A = 2 \end{cases}$$

$\longrightarrow C + 0 = 2 \Rightarrow C = 2$

$$\begin{cases} A + D = 0 \\ 4A + B + D = 4 \end{cases} \longrightarrow 2 + D = 0 \Rightarrow D = -2$$

$4.(2) + B + (-2) = 4$

$8 + B - 2 = 4 \rightarrow B = 4 - 6 \Rightarrow B = -2$

Assim, obtemos:

$$\int \frac{(4x^2+2x+8)\,dx}{x(x^2+2)^2} = \int \left(\frac{A}{x} + \frac{Bx+C}{(x^2+2)^2} + \frac{Dx+E}{x^2+2} \right) dx$$

$$= \int \left(\frac{2}{x} + \frac{(-2x+2)}{(x^2+2)^2} + \frac{-2x+0}{x^2+2} \right) dx =$$

$$2\int \frac{dx}{x} + \int \frac{(-2x+2)}{(x^2+2)^2} - \int \frac{\overbrace{2x\,dx}^{dr}}{\underbrace{x^2+2}_{r}}$$

$r = x^2+2 \Rightarrow dr = 2x\,dx$

$$= 2\ln|x| - \int \frac{dr}{r} - 2\int \frac{x-1}{(x^2+2)^2}\,dx =$$

$$= 2\ln|x| - \ln|r| - 2\int \frac{x\,dx}{(x^2+2)^2} + 2\int \frac{dx}{(x^2+2)^2}$$

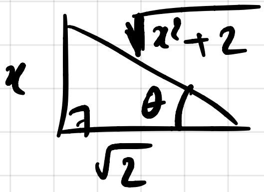
$r = x^2+2 \Rightarrow dr = 2x\,dx$

$\hookrightarrow x\,dx = \frac{dr}{2}$

$$= 2\ln|x| - \ln|x^2+2| - \cancel{2}\int \frac{\frac{dr}{\cancel{2}}}{r^2} + 2\underbrace{\int \frac{dx}{(x^2+2)^2}}$$

SUBSTITUIÇÃO TRIGONOM.

$$= 2\ln|x| - \ln(x^2+2) - \int r^{-2}\,dr + 2\int \frac{dx}{(x^2+2)^2} \; =$$



$$\tan\theta = \frac{x}{\sqrt{2}} \Rightarrow x = \sqrt{2}\tan\theta$$
$$\hookrightarrow dx = \sqrt{2}\cdot\sec^2\theta\, d\theta$$

$$\operatorname{sen}\theta = \frac{x}{\sqrt{x^2+2}}$$

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{x^2+2}} \Rightarrow \sqrt{x^2+2} = \sqrt{2}\cdot\sec\theta$$
$$\Rightarrow (x^2+2) = 2\sec^2\theta.$$

$$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{(2\sec^2\theta)^2} = \int \frac{\sqrt{2}}{4}\cdot\frac{\cancel{\sec^2\theta}}{\sec^{\cancel{4}}\theta}\, d\theta =$$

$$= \frac{\sqrt{2}}{4}\cdot\int \cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\int \frac{1+\cos 2\theta}{2}\, d\theta =$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{\sqrt{2}}{4}\cdot\int \frac{1}{2}\, d\theta + \frac{\sqrt{2}}{4}\cdot\frac{1}{2}\int \cos 2\theta\, d\theta =$$

$$u = 2\theta$$
$$du = 2\, d\theta$$
$$d\theta = \frac{du}{2}$$

$$\frac{\sqrt{2}}{8}\int d\theta + \frac{\sqrt{2}}{8}\cdot\int \cos(u)\cdot\frac{du}{2}$$

$$= \frac{\sqrt{2}}{8}\,\theta + \frac{\sqrt{2}}{16}\cdot\operatorname{sen} u + C = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{16}\cdot\operatorname{sen} 2\theta + C$$

Como $\tan\theta = \frac{x}{\sqrt{2}} \Rightarrow \theta = \arctan\left(\frac{x}{\sqrt{2}}\right)$

Além disso, da trigonometria:

$$\operatorname{sen}2\theta = 2\operatorname{sen}\theta\cdot\cos\theta = 2\cdot\frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}} = \frac{2\sqrt{2}\,x}{x^2+2}$$

$$\Rightarrow \int \frac{dx}{(x^2+2)^2} = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{16}\operatorname{sen}2\theta + C =$$

$$= \frac{\sqrt{2}}{8} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}}{16} \cdot 2 \cdot \frac{\sqrt{2}\, x}{x^2+2} + C =$$

$$= \frac{\sqrt{2}}{8} \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{4} \cdot \frac{x}{x^2+2} + C$$

Juntando, temos finalmente, obter:

$$= 2\ln|x| - \ln(x^2+2) - \int r^{-2}\,dr + 2\int \frac{dx}{(x^2+2)^2} =$$

$$= 2\ln|x| - \ln(x^2+2) - \frac{r^{-1}}{-1} + 2\cdot\frac{\sqrt{2}}{8}\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{2\cdot x}{4(x^2+2)} + C$$

$$= 2\ln|x| - \ln(x^2+2) + \frac{1}{x^2+2} + \frac{\sqrt{2}}{4}\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{2(x^2+2)} + C$$

02) (m) $\displaystyle\int \frac{dx}{x^3+x^2+x}$.

$$x^3+x^2+x = 0 \Leftrightarrow x(x^2+x+1) = 0$$

$$\Leftrightarrow \begin{cases} x = 0 \\ x^2+x+1 = 0 \Leftrightarrow x = \dfrac{-1 \pm \sqrt{1-4}}{2} \notin \mathbb{R} \end{cases}$$

ou seja, $x(x^2+x+1)$ é a forma IRREDUTÍVEL PARA x^3+x^2+x

Assim, teremos:

$$\frac{1}{x^3+x^2+x} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1} = \frac{A(x^2+x+1) + x(Bx+C)}{x(x^2+x+1)}$$

$$\Leftrightarrow 1 \equiv Ax^2 + Ax + A + Bx^2 + Cx$$

$$\Leftrightarrow \begin{cases} A + B = 0 \\ A + C = 0 \\ A = 1 \end{cases} \rightarrow \begin{matrix} B = -1 \\ C = -1 \end{matrix}$$

Logo, temos:

$$\frac{1}{x^3+x^2+x} \equiv \frac{1}{x} + \frac{-x-1}{x^2+x+1} \; ; \quad \text{e então:}$$

$$\int \frac{dx}{x^3+x^2+x} = \int \left(\frac{1}{x} - \frac{x+1}{x^2+x+1} \right) dx =$$

$$\int \frac{dx}{x} - \int \frac{(x+1)\,dx}{x^2+x+1} = \int \frac{dx}{x} - \int \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1}\, dx =$$

$$u = x^2 + x + 1$$
$$\hookrightarrow du = 2x + 1\, dx$$

$$= \ln|x| - \frac{1}{2} \int \frac{2x+1}{x^2+x+1}\, dx - \frac{1}{2} \int \frac{dx}{x^2+x+1}$$

completamos um quadrado perfeito.

$$= \ln|x| - \frac{1}{2} \ln|\underbrace{x^2+x+1}_{>0}| - \frac{1}{2} \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\cdot \arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\cdot \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$