15/12/23 – AULA 12

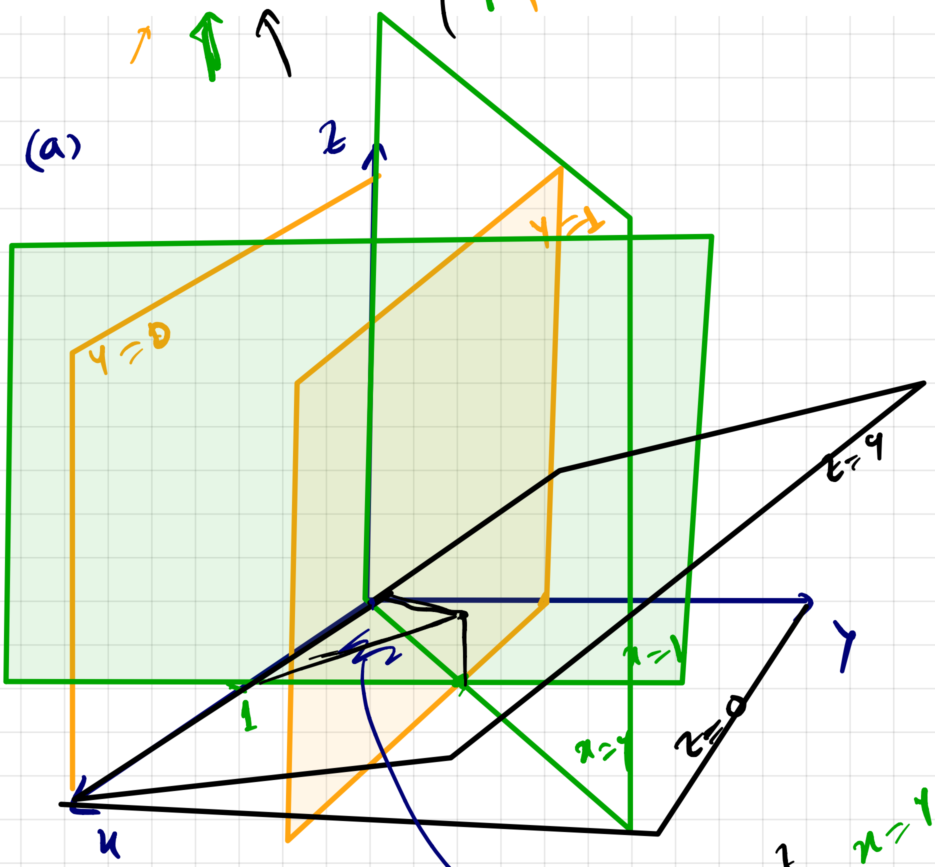
AULA DE EXERCÍCIOS:

L3

9. Escreva as outrs cinco integrais iteradas que sejam iguais a cada integral iterada dada:

(a) $\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy$ (b) $\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dz\,dy\,dx$

(a)



y = 0

y = 1.

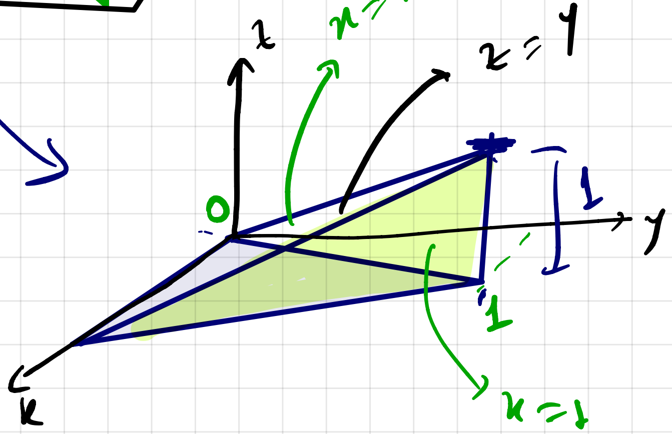
x = y

x = 1

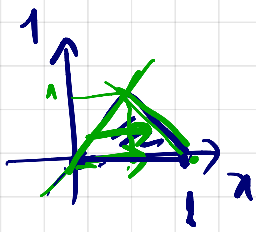
z = 0

z = y

x = y → face de trás.



$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} \int_{z=0}^{z=y} f(x,y,z)\,dz\,dx\,dy =$$



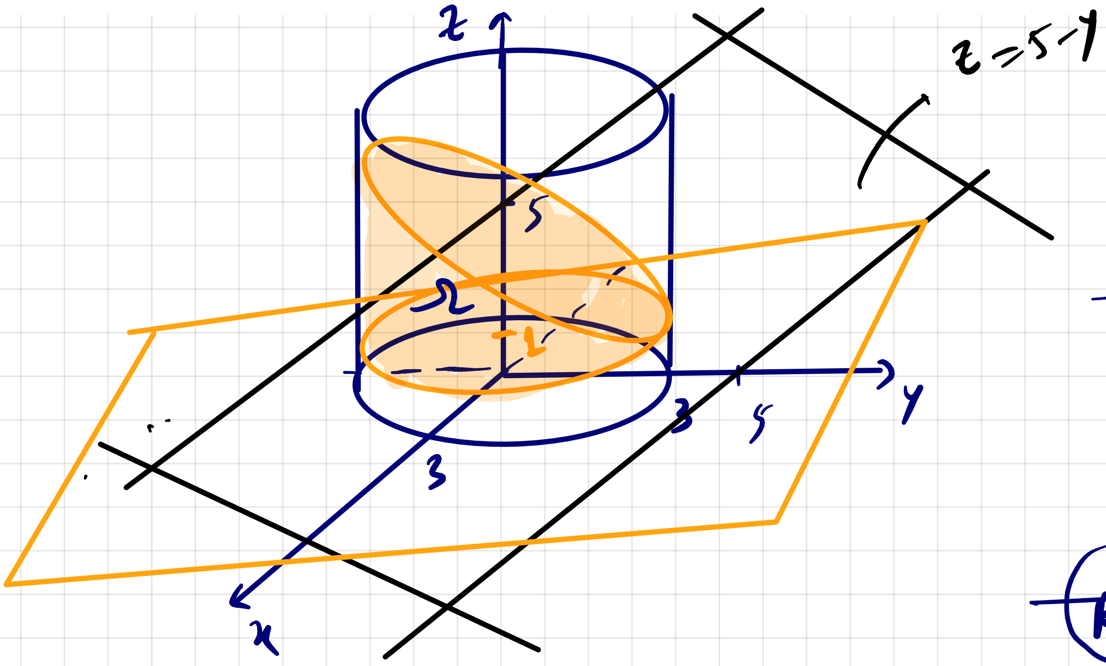
$$= \int_{z=0}^{z=1} \int_{x=0}^{x=1} \int_{y=x}^{y=-x+1} f(x,y,z)\,dy\,dx\,dz \; .$$

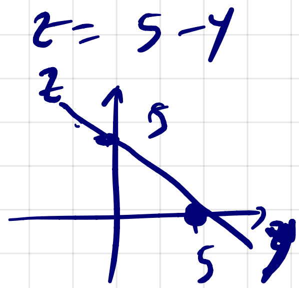
etc.

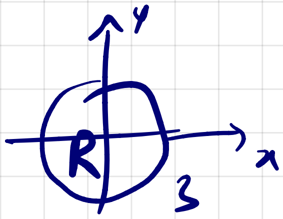
L3:

10. Use integral tripla para determinar o volume do sólido dado em cada item.

(a) O sólido limitado pelo cilindro $x^2 + y^2 = 9$ e pelos planos $y + z = 5$ e $z = 1$.
(Resp.: 36π u.v.)







$$V = \iiint_{\Omega} dV = \iint_R dx\,dy \int_{z=1}^{z=5-y} dz$$

VAMOS USAR COORD. POLARES NA REGIÃO R.

$$= \iint_R dx\,dy \; z\Big|_{z=1}^{z=5-y} = \iint_R (5-y-1)\,dx\,dy$$

$$= \iint_R (4-y)\,dx\,dy = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=3} (4-\rho\,\text{sen}\,\theta)\,\rho\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=3} (4\rho - \rho^2 \cdot \operatorname{sen}\theta)\, d\rho\, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(2\rho^2 - \frac{\rho^3}{3} \cdot \operatorname{sen}\theta \right) \Bigg|_{\rho=0}^{\rho=3} d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(18 - 9\operatorname{sen}\theta - (0 - 0) \right) d\theta =$$

$$= \int_{0}^{2\pi} (18 - 9\operatorname{sen}\theta)\, d\theta = (18\theta + 9\cos\theta) \Bigg|_{\theta=0}^{\theta=2\pi}$$

$$= 18 \cdot 2\pi + 9 \cdot \cos 2\pi - (\underbrace{18 \cdot 0}_{0} + 9 \cdot \cos 0)$$

$$= 36\pi + 9 - 9 = 36\pi \text{ u.v.}$$

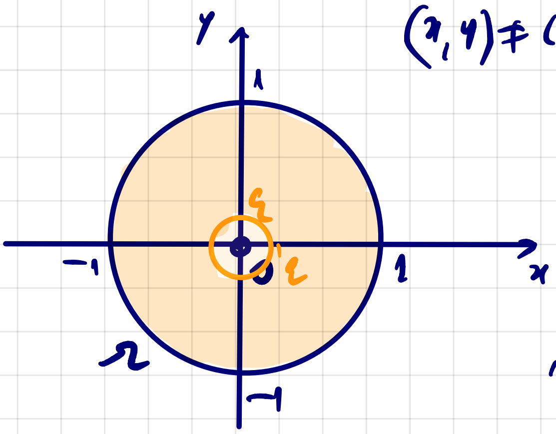
L3

1. Calcule as integrais impróprias abaixo.

(a) $\int_0^{\infty}\int_0^{\infty} e^{-x-y}\,dx\,dy$ (b) $\int_0^{\infty}\int_0^{\infty} x^2 e^{-x^2-y^2}\,dx\,dy$

(c) $\iint_{x^2+y^2\le 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}}$ (d) $\iint_{x^2+y^2\le 1} \ln\sqrt{x^2+y^2}\,dx\,dy$

(c)



$(x,y)\neq(0,0)\ \to$ a origem é um ponto de singularidade, pois

$$f(x,y) = \frac{x^2}{(x^2+y^2)^{\frac{7}{4}}}$$

não está definida neste ponto.

$\Omega = \{(x,y)\in\mathbb{R}^2 : \ x^2+y^2\le 1\}.$

Seja $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0,0)$.

Assim, temos:

$$\iint_\Omega f = \lim_{\varepsilon\to 0}\iint_{\Omega_\varepsilon} f = \lim_{\varepsilon\to 0}\iint_{\Omega_\varepsilon}\frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}} =$$

$$= \lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1}\frac{\rho^2\cos^2\theta\,.\,\rho\,d\rho\,d\theta}{(\rho^2)^{\frac{7}{4}}} = \lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1}\rho^{3-\frac{7}{2}}\,.\cos^2\theta\,d\rho\,d\theta$$

↑ COOR. POLARES

$x=\rho\cos\theta$

$x^2+y^2=\rho^2$

$$= \int_{\theta=0}^{\theta=2\pi}\cos^2\theta\,d\theta\,.\,\lim_{\varepsilon\to 0}\int_{\rho=\varepsilon}^{\rho=1}\rho^{-\frac{1}{2}}\,d\rho =$$

$$= \int_{\theta=0}^{\theta=2\pi}\frac{1+\cos 2\theta}{2}\,d\theta\,.\,\lim_{\varepsilon\to 0}\ 2\sqrt{\rho}\,\Big|_{\rho=\varepsilon}^{\rho=1} =$$

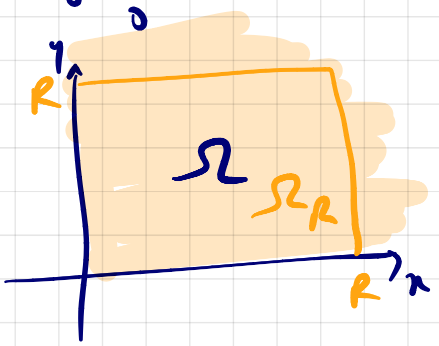
$$= \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{2} + \frac{1}{4} \cos 2\theta \cdot (2 d\theta) \right) \cdot \lim_{\varepsilon \to 0} \left(2\sqrt{1} - 2\sqrt{\varepsilon} \right)$$

$$\left(\frac{1}{2}\theta + \frac{1}{4} \operatorname{sen} 2\theta \right) \Big|_{\theta=0}^{\theta=2\pi} \cdot (2 - 0) =$$

$$= \left(\frac{1}{2} \cdot 2\pi + \frac{1}{4} \cdot \operatorname{sen} 4\pi - \left(0 + \frac{1}{4} \operatorname{sen} 0 \right) \right) \cdot 2$$

$$= \pi \cdot 2 = 2\pi$$

(a) $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y} \, dx \, dy$



$\Omega = \{ (x,y) \in \mathbb{R}^2 : x, y \geq 0 \}$

$\Omega_R = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq R \}$.

$$\iint_\Omega f = \lim_{R \to +\infty} \iint_{\Omega_R} f =$$

$$= \lim_{R \to +\infty} - \int_{y=0}^{y=R} \left(\int_{x=0}^{x=R} e^{-x-y} (-dx) \right) dy =$$

$$= \lim_{R \to +\infty} - \int_{y=0}^{y=R} \left(e^{-x-y} \right) \Big|_{x=0}^{x=R} dy =$$

$\int e^v dv = e^v + C$

$v = -x - y$

$\hookrightarrow dv = -dx$

$$= \lim_{R \to +\infty} - \int_{y=0}^{y=R} \left(e^{-R-y} - e^{-y}\right) dy =$$

$$= - \lim_{R \to +\infty} \int_{y=0}^{y=R} e^{-y} \cdot \left(e^{-R} - 1\right) dy =$$

$$= \lim_{R \to +\infty} \left(e^{-R} - 1\right) \cdot \int_{y=0}^{y=R} -e^{-y} dy =$$

$$\int e^{u} du \qquad u = -y \Rightarrow du = -dy$$

$$= \lim_{R \to +\infty} \left(e^{-R} - 1\right) \cdot e^{-y} \Big|_{y=0}^{y=R} =$$

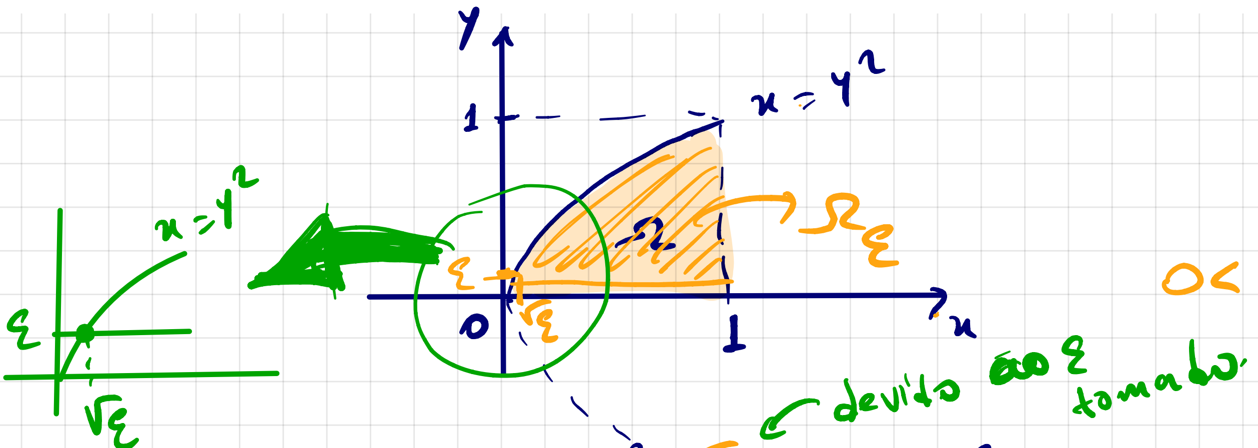
$$= \lim_{R \to +\infty} \left(e^{-R} - 1\right) \cdot \left(e^{-R} - e^{0}\right)$$

$$= \lim_{R \to +\infty} \left(e^{-R} - 1\right)^2 = \lim_{R \to +\infty} \left(\frac{1}{e^R} - 1\right)^2 = 1$$

($\frac{1}{e^R} \to 0$)

L3:

2. Calcule $\iint_\Omega e^{\frac{x}{y}} dxdy$, onde $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le y^2, 0 \le y \le 1\}$.



$0 < \varepsilon < 1$.

$$\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 : \sqrt{\varepsilon} \le x \le y^2, \ \varepsilon \le y \le 1\}$$

$$\iint_{\Omega} f = \lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon} f =$$

$$= \lim_{\varepsilon \to 0} \int_{y=\varepsilon}^{y=1} y \left(\int_{x=\sqrt{\varepsilon}}^{x=y^2} e^{\frac{x}{y}} \frac{1}{y} dx \right) dy =$$

$$\int e^r dr$$

$$r = \frac{1}{y} \cdot x \Rightarrow dr = \frac{1}{y} dx$$

$$= \lim_{\varepsilon \to 0} \int_{y=\varepsilon}^{y=1} y \left(e^{\frac{x}{y}} \right) \Big|_{x=\sqrt{\varepsilon}}^{x=y^2} dy$$

$$= \lim_{\varepsilon \to 0} \int_{y=\varepsilon}^{y=1} y \cdot \left(e^{\frac{y^2}{y}} - e^{\frac{\sqrt{\varepsilon}}{y}} \right) dy = \lim_{\varepsilon \to 0} \int_{y=\varepsilon}^{y=1} \left(y \cdot e^{y} - y \cdot e^{\frac{\sqrt{\varepsilon}}{y}} \right) dy$$

integrar ambos por partes...

(...)

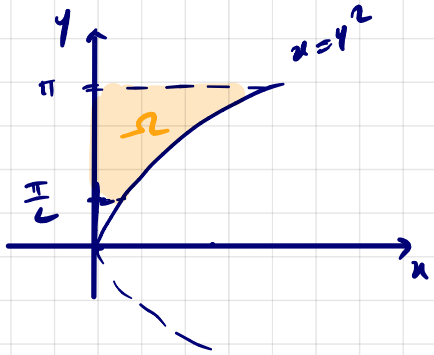
LISTA 02

2. Calcule as integrais iteradas:

(a) $\int_0^1\int_0^{x^2}(x+2y)dydx$ (b) $\int_0^1\int_x^{2-x}(x^2-y)dydx$ (c) $\int_1^2\int_y^2 xydxdy$

(d) $\int_{-1}^1\int_{x^2}^{1-x^2}2x^2y^2dydx$ (e) $\int_{\frac{\pi}{2}}^{\pi}\int_0^{y^2}\text{sen}\,\frac{x}{y}dxdy$

(e) $$\int_{y=\frac{\pi}{2}}^{y=\pi}\int_{x=0}^{x=y^2}\text{sen}\,\frac{x}{y}\,dx\,dy = \int_{y=\frac{\pi}{2}}^{y=\pi} y\cdot\int_{x=0}^{x=y^2}\text{sen}\,\underbrace{\frac{x}{y}}_{v}\cdot\underbrace{\left(\frac{1}{y}dx\right)}_{dv}dy \;=$$



$$\int \text{sen}\,v\,dv = -\cos v + C$$
$$v = \frac{x}{y} \Rightarrow dv = \frac{1}{y}dx$$

$$= \int_{y=\frac{\pi}{2}}^{y=\pi} y\cdot\left(-\cos\frac{x}{y}\right)\Bigg|_{x=0}^{x=y^2} dy = \int_{y=\frac{\pi}{2}}^{y=\pi} y\cdot\left(-\cos\frac{y^2}{y}+\underbrace{\cos 0}_{1}\right)dy$$

$$= \int_{y=\frac{\pi}{2}}^{y=\pi}(-y\cos y + y)dy = -\int_{\frac{\pi}{2}}^{\pi}y\cos y\,dy + \int_{\frac{\pi}{2}}^{\pi}y\,dy \;=$$

↑ POR PARTES.

- $\int y\cos y\,dy = \int u\,dv = u\cdot v - \int v\,du$

$$\begin{cases} u = y \Rightarrow du = dy \\ dv = \cos y\,dy \Rightarrow v = \text{sen}\,y \end{cases}$$

$$\int y\cos y\,dy = y\cdot\text{sen}\,y - \int \text{sen}\,y\,dy = y\,\text{sen}\,y + \cos y + C$$

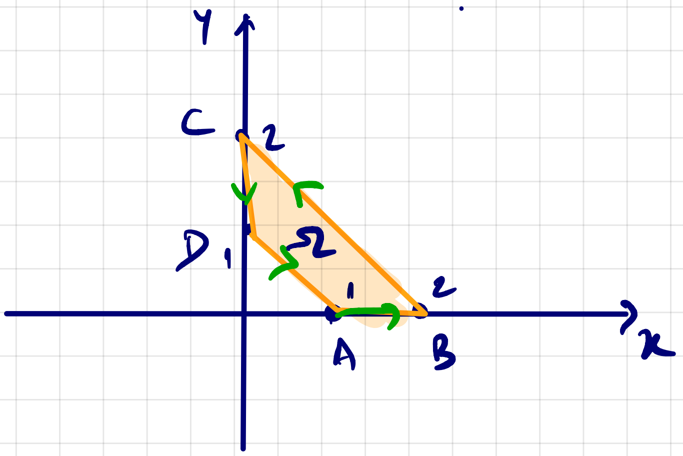
$$\overset{\circ}{=} -(y\,\text{sen}\,y + \cos y)\Big|_{y=\frac{\pi}{2}}^{y=\pi} + \frac{y^2}{2}\Big|_{y=\frac{\pi}{2}}^{y=\pi} =$$

$$-\Big(\underbrace{\pi\,\text{sen}\,\pi}_{0} + \underbrace{\cos\pi}_{-1} - \frac{\pi}{2}\cdot\underbrace{\text{sen}\frac{\pi}{2}}_{1} - \underbrace{\cos\frac{\pi}{2}}_{0}\Big) + \frac{\pi^2}{2} - \frac{\pi^2}{8}$$

$$= -\left(-1 - \frac{\pi}{2}\right) + \frac{3\pi^2}{8} = 1 + \frac{\pi}{2} + \frac{3\pi^2}{8}$$

L2

16. Calcule $\displaystyle\iint_\Omega \cos\left(\frac{y-x}{y+x}\right)$, onde Ω é a região trapezoidal com vértices em $A(1,0)$, $B(2,($
$C(0,2)$ e $D(0,1)$.



Escreva
$$u = -x + y$$
$$v = x + y$$
$$+$$
$$u + v = 2y$$

$$\Rightarrow \boxed{y = \frac{1}{2}u + \frac{1}{2}v}$$

$$\Rightarrow x = v - y = v - \frac{1}{2}u - \frac{1}{2}v$$

$$\Rightarrow \boxed{x = -\frac{1}{2}u + \frac{1}{2}v}$$

$$T(u,v) = (x,y) = \left(-\frac{1}{2}u + \frac{1}{2}v,\ \frac{1}{2}u + \frac{1}{2}v\right)$$

$$J(T(u,v)) = ?$$

$$\det\left(J(T(u,v))\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

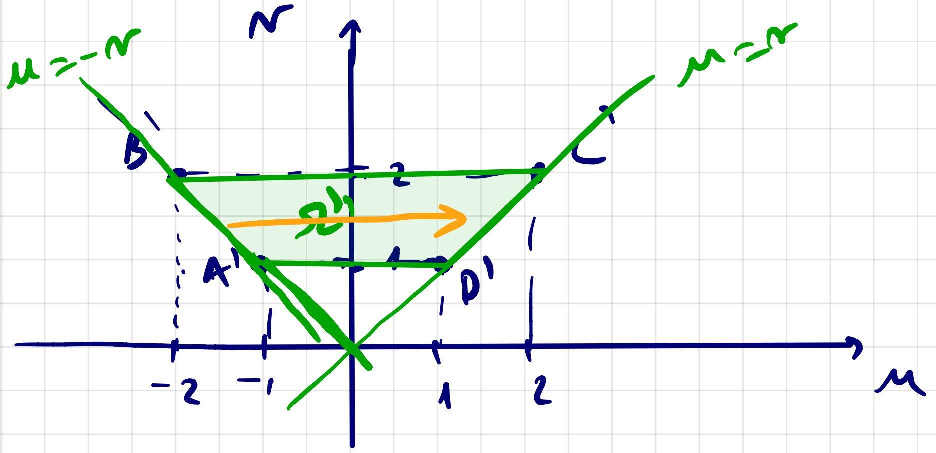
$$(x,y) \xmapsto{uv} (u,v) = (y-x, y+x)$$

$$A(1,0) \xmapsto{uv} (0-1, 0+1) = (-1,1) = A'$$

$$B(2,0) \xmapsto{uv} (0-2, 0+2) = (-2,2) = B'$$

$$C(0,2) \xmapsto{uv} (2-0, 2+0) = (2,2) = C'$$

$$D(0,1) \xmapsto{uv} (1-0, 1+0) = (1,1) = D'$$



Então $\iint_{\Omega} \cos\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_{\Omega'} \cos\left(\frac{u}{v}\right) \left|\det J(T)\right| . du\,dv$

$$= \int_{v=1}^{v=2} \int_{u=-v}^{u=v} \cos\frac{u}{v} \cdot \left|-\frac{1}{2}\right| . du\,dv =$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \int_{u=-r}^{u=r} \cos\left(\underbrace{\frac{u}{r}}_{w}\right) \cdot \underbrace{\left(\frac{1}{r}\,du\right)}_{dw}\, dr =$$

$$\int \cos w \, dw$$

$$w = \frac{u}{r} \Rightarrow dw = \frac{1}{r}\,du$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \cdot \operatorname{sen}\left(\frac{u}{r}\right)\Bigg|_{u=-r}^{u=r} dr =$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \cdot \left(\operatorname{sen} \frac{r}{r} - \operatorname{sen}\left(-\frac{r}{r}\right)\right) dr$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \cdot (\operatorname{sen} 1 + \operatorname{sen} 1)\, dr =$$

$$\frac{1}{2} \cdot 2 \cdot \operatorname{sen} 1 \cdot \int_1^2 r\, dr = \operatorname{sen} 1 \cdot \frac{r^2}{2}\Bigg|_1^2 =$$

$$= \operatorname{sen} 1 \cdot \left(2 - \frac{1}{2}\right) = \frac{3}{2} \cdot \operatorname{sen} 1 .$$