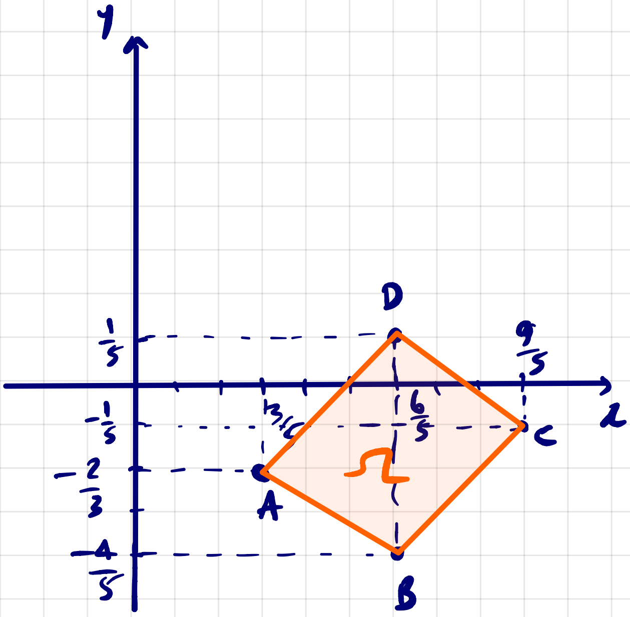
08/12/23 – AULA 10

04) Calcule $\displaystyle\iint_{\Omega} \frac{1}{x-y}\cdot\cos(2x+3y)\,dx\,dy$, sendo Ω a região do plano xy definida pelo quadrilátero ABCD de vértices $A\left(\frac{3}{5},-\frac{2}{5}\right)$; $B\left(\frac{6}{5};-\frac{4}{5}\right)$; $C\left(\frac{9}{5};-\frac{1}{5}\right)$ e $D\left(\frac{6}{5},\frac{1}{5}\right)$

SOLUÇÃO:



Escreva $\begin{cases} u = x - y \\ v = 2x+3y \end{cases}$ $\rightarrow$ $y = x - u$

$v = 2x + 3(x-u)$

$v = 2x + 3x - 3u$

$$\boxed{x = +\frac{3}{5}u + \frac{1}{5}v}$$

$y = x - u$

$y = +\frac{3}{5}u + \frac{1}{5}v - u$

$$\boxed{y = -\frac{2}{5}u + \frac{1}{5}v}$$

$$T(u,v) = (x,y) = \left(\frac{3}{5}u + \frac{1}{5}v, -\frac{2}{5}u + \frac{1}{5}v\right)$$

$$\det\left(J(T)(u,v)\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{vmatrix} = \frac{3}{25} - \left(-\frac{2}{25}\right) = \frac{1}{5}$$

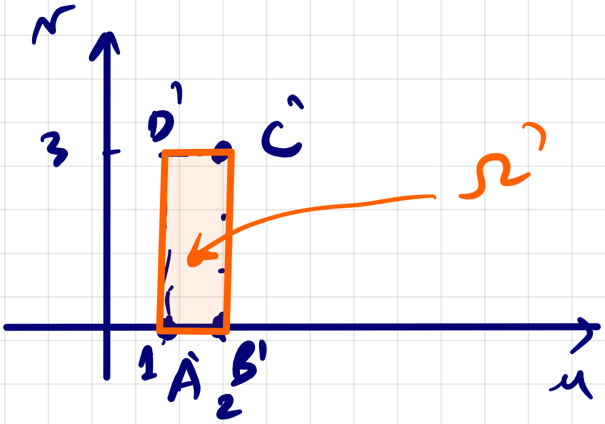
$$(x,y) \xrightarrow{T} (u,v) = (x-y,\ 2x+3y)$$

$$A: \left(\frac{3}{5}, -\frac{2}{5}\right) \xrightarrow{T} \left(\frac{3}{5}+\frac{2}{5}, \frac{6}{5}-\frac{6}{5}\right) = (1, 0) = T(A) := A'$$

$$B\left(\frac{6}{5}, -\frac{4}{5}\right) \xrightarrow{T} \left(\frac{6}{5}+\frac{4}{5}, \frac{12}{5}-\frac{12}{5}\right) = (2, 0) = T(B) = B'$$

$$C\left(\frac{9}{5}, -\frac{1}{5}\right) \xrightarrow{T} \left(\frac{9}{5}+\frac{1}{5}, \frac{18}{5}-\frac{3}{5}\right) = (2, 3) = T(C) = C'$$

$$D\left(\frac{6}{5}, \frac{1}{5}\right) \xrightarrow{T} \left(\frac{6}{5}-\frac{1}{5}, \frac{12}{5}+\frac{3}{5}\right) = (1, 3) = T(D) = D'$$



Assim, obtemos:

$$\int_{\Omega} \frac{1}{x-y}\cos(2x+3y)\,dx\,dy = \int_{\Omega'} \frac{1}{u}\cos v \cdot \underbrace{\left|\det\left(J(T)(u,v)\right)\right|}_{1/5} \cdot du\,dv$$

$$= \frac{1}{5} \int_{u=1}^{u=2} \int_{v=0}^{v=3} \frac{1}{u} \cos v \, dv \, du = \frac{1}{5} \int_{u=1}^{u=2} \frac{du}{u} \int_{v=0}^{v=3} \cos v \, dv =$$

$$\frac{1}{5} \ln|u| \Big|_{u=1}^{u=2} \cdot \operatorname{sen} v \Big|_{v=0}^{v=3} =$$

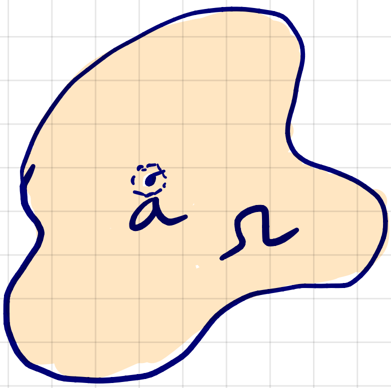
$$\frac{1}{5}\big(\ln 2 - \underbrace{\ln 1}_{0}\big)\cdot\big(\operatorname{sen} 3 - \underbrace{\operatorname{sen} 0}_{0}\big) = \frac{1}{5} \operatorname{sen} 3 \cdot \ln 2$$

INTEGRAIS IMPRÓPRIAS.

Do mesmo modo que em Cálculo II, temos integrais impróprias à várias variáveis.

A saber, são 2 tipos de impróprias:

1º: quando $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ possui uma singularidade no $\operatorname{int}(\Omega)$.



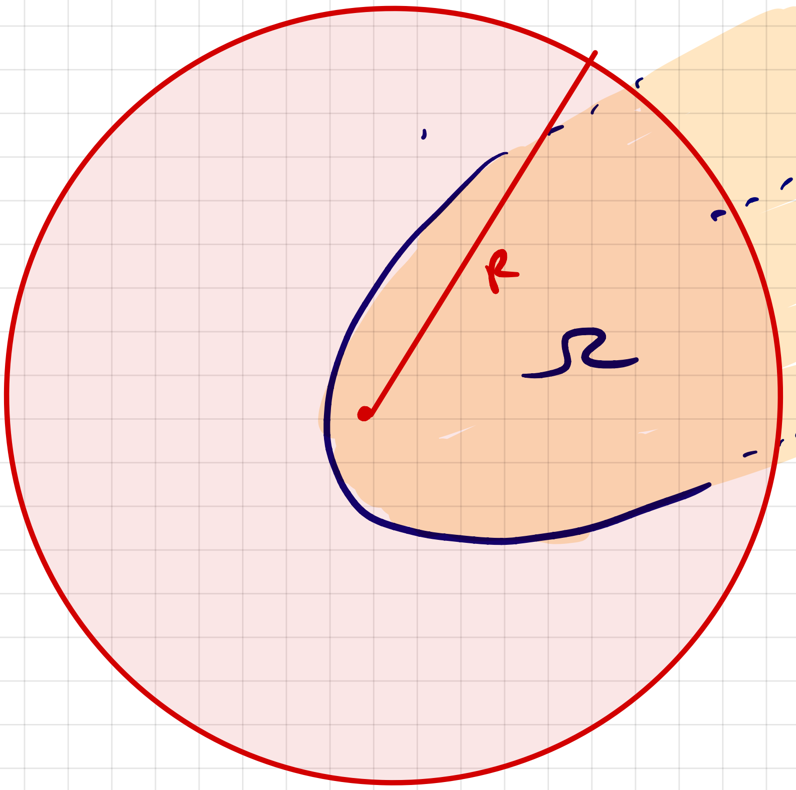
$\nexists f(a)$.

Neste caso, tome $\varepsilon > 0$ e considere a região $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(a)$.

Assim, $$\iint_{\Omega} f = \lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon} f$$

Se existir este limite, diremos que f é integrável e a integral converge para o valor encontrado no limite. Do contrário, diremos que a integral é divergente.

2º: Quando Ω não é limitado



Neste caso, tome $R>0$ e considere
$$\Omega_R = \Omega \cap B_R(0).$$
e então, calcula-se
$$\lim_{R\to\infty} \iint_{\Omega_R} f.$$

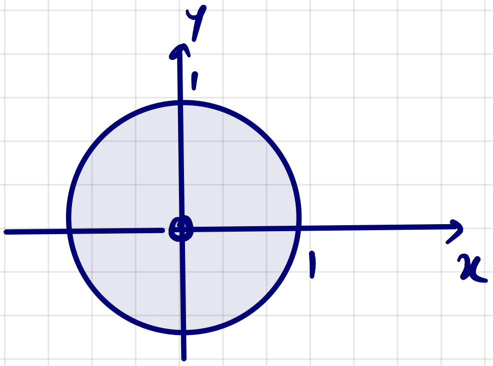
Se o limite existir, a integral dupla será convergente e converge para o limite. Do contrário, será divergente.

Obs: Usamos na explicação acima uma bola centrada na origem, mas pode ser qualquer figura que intercepte Ω e que quando seu raio ou alguma de suas dimensões for para o infinito, tal interseção resulte em toda a região Ω.

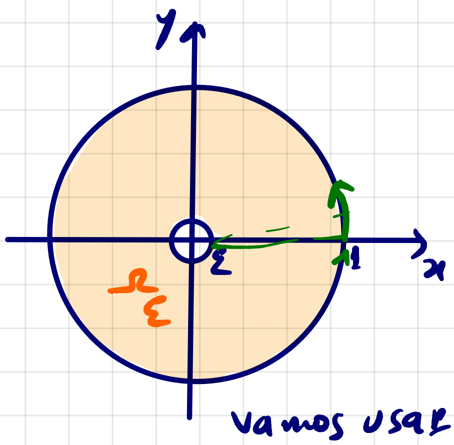
EXEMPLOS: Calcule cada integral imprópria a seguir:

(a) $$\iint_{x^2+y^2\leq 1} \frac{dx\,dy}{\sqrt{x^2+y^2}}.$$

SOLUÇÃO:



$(0,0)\in\Omega$, mas $\nexists f(0,0)$.



Tome $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0,0)$.

$$\iint_\Omega f = \lim_{\varepsilon\to 0}\iint_{\Omega_\varepsilon} f$$

Vamos usar coordenadas polares.

$$= \lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1}\frac{\cancel{\rho}\,d\rho\,d\theta}{\cancel{\sqrt{\rho^2}}} = \lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1} d\rho\,d\theta$$

$$= \lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi} d\theta\int_{\rho=\varepsilon}^{\rho=1} d\rho = \lim_{\varepsilon\to 0}\ \theta\Big|_0^{2\pi}\cdot\ \rho\Big|_\varepsilon^1 =$$

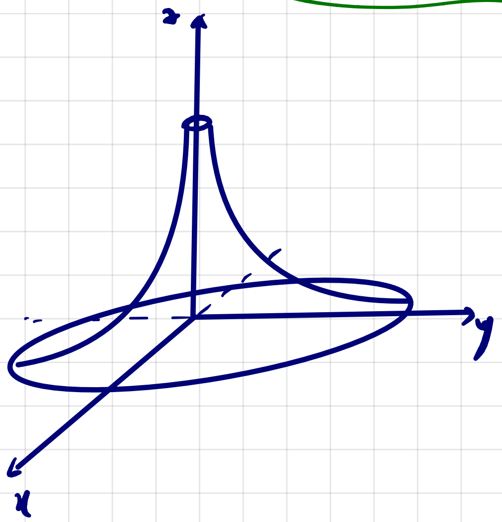
$$2\pi\cdot\lim_{\varepsilon\to 0}(1-\varepsilon) = \underline{\underline{2\pi}}$$

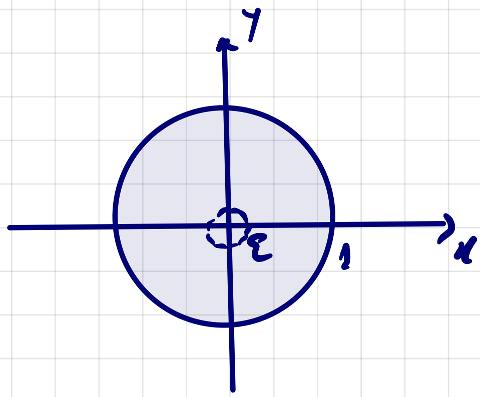
(b) $$\iint_{x^2+y^2\leq 1} -\ln(x^2+y^2)\,dx\,dy = ?$$

Solução:

$$\iint_{x^2+y^2\leq 1} -\ln(x^2+y^2)\,dx\,dy = \iint_{x^2+y^2\leq 1} \ln\frac{1}{x^2+y^2}\,dx\,dy$$

$a\ln b = \ln b^a$





$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0,0).$$

Novamente, usaremos coordenadas polares:

$$\iint_{\Omega} -\ln(x^2+y^2)\,dx\,dy = \lim_{\varepsilon\to 0}\iint_{\Omega_\varepsilon} -\ln\rho^2\cdot\rho\,d\rho\,d\theta$$

$$= \lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi}\left(\int_{\rho=\varepsilon}^{\rho=1} -2\cdot\ln\rho\cdot\rho\,d\rho\right)d\theta =$$

$\ln a^m = m.\ln a$

$$= -2\cdot\lim_{\varepsilon\to 0}\int_{\theta=0}^{\theta=2\pi} d\theta\cdot\underbrace{\int_{\rho=\varepsilon}^{\rho=1}\ln\rho\cdot\rho\,d\rho}_{\text{TEMOS QUE INTEGRAR POR PARTES.}}$$

$$\int \ln\rho \cdot \rho \, d\rho = \int u \, dv = u \cdot v - \int v \, du$$

$$\left[\begin{array}{l} u = \ln\rho \Rightarrow du = \frac{d\rho}{\rho} \\ dv = \rho \, d\rho \Rightarrow v = \frac{\rho^2}{2} \end{array}\right.$$

$$\Rightarrow \int \ln\rho \cdot \rho \, d\rho = \frac{\rho^2}{2} \cdot \ln\rho - \int \frac{\rho^{\cancel{2}}}{2} \cdot \frac{d\rho}{\cancel{\rho}}$$

$$= \frac{\rho^2}{2} \cdot \ln\rho - \frac{1}{2} \int \rho \, d\rho$$

$$= \frac{\rho^2}{2} \ln\rho - \frac{1}{2} \frac{\rho^2}{2} + C$$

$$= \frac{\rho^2}{2} \left(\ln\rho - \frac{1}{2} \right) + C$$

$$\circledast = -2 \lim_{\varepsilon \to 0} \theta \Big|_{\theta=0}^{\theta=2\pi} \left[\frac{\rho^2}{2} \left(\ln\rho - \frac{1}{2} \right) \right] \Bigg|_{\rho=\varepsilon}^{\rho=1} =$$

$$= -4\pi \lim_{\varepsilon \to 0} \frac{\rho^2}{2} \left(\ln\rho - \frac{1}{2} \right) \Big|_{\rho=\varepsilon}^{\rho=1}$$

$$= -4\pi \lim_{\varepsilon \to 0} \left[\frac{1}{2} \Big(\underbrace{\ln 1}_{= 0} - \frac{1}{2} \Big) - \frac{\varepsilon^2}{2} \left(\ln\varepsilon - \frac{1}{2} \right) \right]$$

$$= -4\pi \lim_{\varepsilon \to 0} \Big[-\frac{1}{4} - \underbrace{\frac{\varepsilon^2}{2} \cdot \ln\varepsilon}_{\substack{0 \cdot \infty \\ (\text{INDET.})}} + \frac{\varepsilon^2}{4} \Big]$$

$$= -4\pi \left(-\frac{1}{4} \right) + \frac{\cancel{4}\pi}{\cancel{2}} \cdot \lim_{\varepsilon \to 0} \varepsilon^2 \cdot \ln\varepsilon - \pi \underbrace{\lim_{\varepsilon \to 0} \varepsilon^2}_{0}$$

$$= \pi + 2\pi \cdot \lim_{\varepsilon \to 0} \frac{\ln\varepsilon}{\frac{1}{\varepsilon^2}} = \frac{\pi}{2} + \pi \cdot \lim_{\varepsilon \to 0} \frac{\frac{1}{\varepsilon}}{-\frac{2}{\varepsilon^3}} =$$

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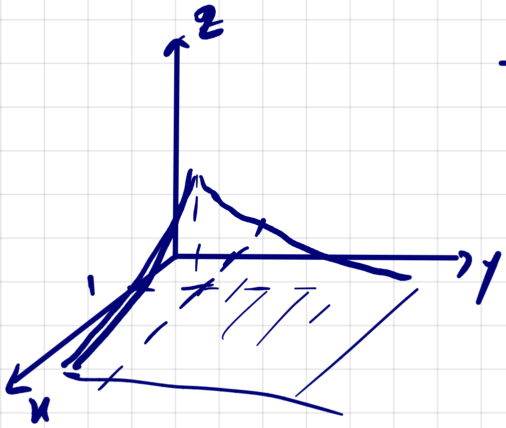
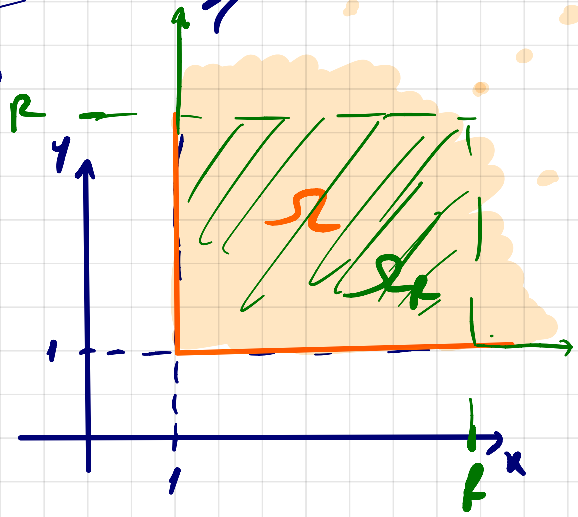
$$= \pi + 2\pi \cdot \underbrace{\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \left(-\frac{\varepsilon^3}{2}\right)}_{\to 0} = \pi$$

(c) $$\int_1^{+\infty}\int_1^{+\infty} \frac{dx\,dy}{x^2y^2}.$$





$$\Omega_R = [1,R]\times[1,R]$$

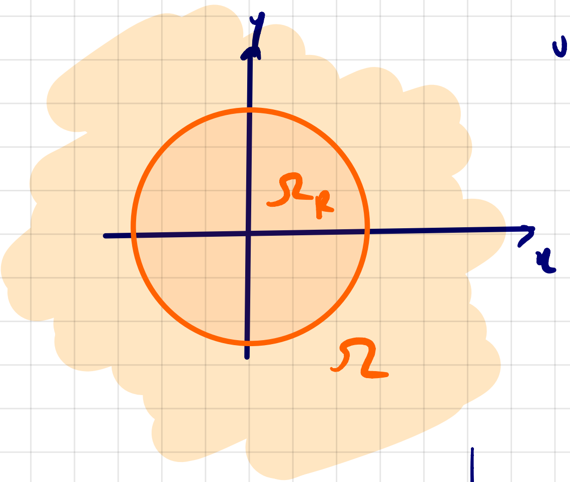
$$\iint_{\Omega} f = \lim_{R\to+\infty} \iint_{\Omega_R} f = \lim_{R\to+\infty} \int_{x=1}^{x=R}\int_{y=1}^{y=R} \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot dy\,dx =$$

$$= \lim_{R\to+\infty} \int_{x=1}^{x=R} x^{-2}\,dx \cdot \int_{y=1}^{y=R} y^{-2}\,dy = \lim_{R\to+\infty} \left(\frac{x^{-1}}{-1}\right)\Bigg|_1^R \cdot \left(\frac{y^{-1}}{-1}\right)\Bigg|_1^R$$

$$= \lim_{R\to+\infty} \left(-\frac{1}{R} - \left(-\frac{1}{1}\right)\right)\cdot\left(-\frac{1}{R} - \left(-\frac{1}{1}\right)\right) =$$

$$= \lim_{R\to+\infty} \left(\underbrace{-\frac{1}{R}}_{\to 0} + 1\right)^2 = 1$$

(d) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy$



usando coordenadas polares:

$$\iint_{\Omega} f = \lim_{R\to+\infty} \iint_{\Omega_R} f =$$

$$= \lim_{R\to+\infty} -\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=R} e^{-\rho^2}(-2\rho)\,d\rho\,d\theta$$

$$\begin{cases} \rho^2 = x^2+y^2 \\ dx\,dy = \rho\,d\rho\,d\theta \end{cases}$$

$$= -\frac{1}{\cancel{2}} \lim_{R\to+\infty} \cancel{2}\pi \cdot e^{-\rho^2}\Big|_{\rho=0}^{\rho=R}$$

$\int e^{u}du$

$u = -\rho^2$

$du = -2\rho d\rho$

$$= -\pi \cdot \lim_{R\to+\infty} \left(e^{-R^2} - e^{0}\right)$$

$$= -\pi \cdot \lim_{R\to+\infty} \Big(\underbrace{\frac{1}{e^{R^2}}}_{\to 0} - 1 \Big) = \pi$$

(e) INTEGRAL DE POISSON : $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}dx$

usando o exemplo anterior:

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{+\infty} e^{-x^2}dx \cdot \int_{-\infty}^{+\infty} e^{-y^2}dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}dx\,dy = \pi$$

pelo exemplo anterior.

Logo: $\displaystyle\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \pi$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$