05/12/23 – AULA 09

INTEGRAIS DA FORMA $\int \frac{mx+n}{ax^2+bx+c}\,dx$ E $\int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$

Em ambos os casos temos um polinômio de 1º grau no numerador e um polinômio de 2º grau no denominador, este podendo estar em um radicando.

Chamando $r = ax^2+bx+c$, então teremos

$$dr = (2ax+b)dx$$

Assim, a idéia consiste em transformar numerador $mx+n$ em $2ax+b$, multiplicando $mx+n$ e somando por constantes adequadas. Vejamos exemplos:

01) $\int \frac{(3x+1)\,dx}{x^2+2x-9}$.

SOLUÇÃO:

$$r = x^2+2x-9$$
$$\Rightarrow dr = (2x+2)dx$$

Então;

$$3x+1 = \frac{3}{2}(2x+2) - 2$$

PARA APARECER O FATOR DIFERENCIAL

Então,

$$\int \frac{(3x+1)\,dx}{x^2+2x-9} = \int \frac{\frac{3}{2}(2x+2)-2}{x^2+2x-9}\,dx$$

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$= \frac{3}{2}\underbrace{\int \frac{(2x+2)\,dx}{x^2+2x-9}}_{} - 2\cdot\int \frac{dx}{x^2+2x-9}$$

$$\int \frac{dv}{v} = \ln|v| + C$$

COMPLETAMOS QUADRADO PERFEITO

$$= \frac{3}{2}\ln|x^2+2x-9| - 2\underbrace{\int \frac{dx}{(x+1)^2-10}}_{}$$

$$\int \frac{dv}{v^2-a^2}$$

$$a = \sqrt{10}$$

$$= \frac{1}{2a}\ln\left|\frac{v-a}{v+a}\right| + C$$

$$v = x+1$$
$$dv = dx$$

$$= \frac{3}{2}\ln|x^2+2x-9| - \cancel{2}\cdot\frac{1}{\cancel{2}\cdot\sqrt{10}}\ln\left|\frac{x+1-\sqrt{10}}{x+1+\sqrt{10}}\right| + C$$

$$= \frac{3}{2}\ln|x^2+2x-9| - \frac{1}{\sqrt{10}}\cdot\ln\left|\frac{x+1-\sqrt{10}}{x+1+\sqrt{10}}\right| + C.$$

02) $\displaystyle\int \frac{(1-2x)\,dx}{3x^2-5x+7}$

$u = 3x^2-5x+7 \Rightarrow du = (6x-5)\,dx$

$1-2x = -\frac{1}{3}(6x-5) - \frac{2}{3}$

$+\frac{5}{3} + \alpha = 1 \leadsto \alpha = 1 - \frac{5}{3} = -\frac{2}{3}$

Assim, teremos:

$$\int \frac{(1-2x)\,dx}{3x^2-5x+7} = \int \frac{-\frac{1}{3}(6x-5) - \frac{2}{3}}{3x^2-5x+7}\cdot dx =$$

$$= -\frac{1}{3}\int \frac{(6x-5)\,dx}{3x^2-5x+7} - \frac{2}{3}\int \frac{dx}{3x^2-5x+7}$$

$\int \frac{du}{u} = \ln|u| + c$

COMPLETAR QUADRADO PERFEITO

$$= -\frac{1}{3}\cdot\ln|3x^2-5x+7| - \frac{2}{3}\int \frac{dx}{3x^2-5x+7} =$$

$3x^2-5x+7 = 3.[(x-a)^2+b]$

$3x^2-5x+7 = 3.[x^2-2ax+a^2+b]$

$3x^2-5x+7 = 3x^2-6ax+3a^2+3b$

$$-5 = -6a \Rightarrow \boxed{a = \frac{5}{6}}$$

$$3a^2 + 3b = 7$$

$$3 \cdot \left(\frac{5}{6}\right)^2 + 3b = 7$$

$$\cancel{3} \cdot \frac{25}{\cancel{36}} + 3b = 7$$

$$\frac{25}{12} + 3b = 7$$

$$3b = 7 - \frac{25}{12}$$

$$3b = \frac{84 - 25}{12}$$

$$b = \frac{59}{36}$$

Ou seja, obtemos:

$$3x^2 - 5x + 7 = 3 \cdot \left[\left(x - \frac{5}{6}\right)^2 + \frac{59}{36}\right]$$

$$= -\frac{1}{3} \cdot \ln|3x^2 - 5x + 7| - \frac{2}{3} \int \frac{dx}{3\left[\left(x - \frac{5}{6}\right)^2 + \frac{59}{36}\right]}$$

$$= -\frac{1}{3} \cdot \ln|3x^2 - 5x + 7| - \frac{2}{9} \int \frac{dx}{\underbrace{\left(x - \frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2}}$$

$$\int \frac{dv}{v^2 + a^2} = \frac{1}{a} \arctan\left(\frac{v}{a}\right) + c$$

$$v = x - \frac{5}{6} \Rightarrow dv = dx$$

$$= -\frac{1}{3}\ln|3x^2-5x+7| - \frac{2}{9}\cdot\frac{1}{\frac{\sqrt{59}}{6}}\cdot\arctan\left(\frac{x-\frac{5}{6}}{\frac{\sqrt{59}}{6}}\right)+C$$

$$= -\frac{1}{3}\ln|3x^2-5x+7| - \frac{2}{\cancel{9}_{\,3}}\times\frac{\cancel{6}^{\,2}}{\sqrt{59}}\arctan\left(\frac{\frac{6x-5}{\cancel{6}}}{\frac{\sqrt{59}}{\cancel{6}}}\right)+C$$

$$= -\frac{1}{3}\ln|3x^2-5x+7| - \frac{4}{3\sqrt{59}}\cdot\arctan\left(\frac{6x-5}{\sqrt{59}}\right)+C.$$

03) $\displaystyle\int\frac{(4x+7)\,dx}{\sqrt{x^2+3x+1}}$

$$u = x^2+3x+1 \Rightarrow du = (2x+3)\,dx$$

$$4x+7 = 2.(2x+3)+1$$

Assim, temos:

$$\int\frac{(4x+7)\,dx}{\sqrt{x^2+3x+1}} = \int\frac{2.(2x+3)+1}{\sqrt{x^2+3x+1}}\,dx =$$

$$= 2.\int\frac{(2x+3)\,dx}{\sqrt{x^2+3x+1}} + \int\frac{dx}{\sqrt{x^2+3x+1}}$$

$$= 2.\underbrace{\int(x^2+3x+1)^{-\frac{1}{2}}.(2x+3)\,dx}_{\int u^{-\frac{1}{2}}.du} + \int\frac{dx}{\sqrt{x^2+3x+1}}$$

$$\begin{cases} x^2+3x+1 = (x+a)^2+b \\ x^2+3x+1 = x^2+2ax+a^2+b \\ 2a=3 \Rightarrow a=\frac{3}{2} \\ a^2+b=1 \\ \left(\frac{3}{2}\right)^2+b=1 \\ b=1-\frac{9}{4}=-\frac{5}{4} \end{cases}$$

$$x^2+3x+1 = \left(x+\frac{3}{2}\right)^2-\frac{5}{4}$$

Disso, segue que:

$$= 2\cdot\frac{(x^2+3x+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \int \frac{dv}{\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}} =$$

$$v = x+\frac{3}{2} \Rightarrow dv = dx$$

$$\int \frac{dv}{\sqrt{v^2-a^2}} = \ln\left|v+\sqrt{v^2-a^2}\right|+c$$

$$= 2\cdot\frac{\sqrt{x^2+3x+1}}{\frac{1}{2}} + \ln\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right|+c$$

$$= 4\sqrt{x^2+3x+1} + \ln\left|x+\frac{3}{2}+\sqrt{x^2+3x+1}\right|+c.$$

QUESTÕES DA PROVA: (ALGUMAS)

05) (c) $\displaystyle\int \frac{dx}{\csc x \sqrt[3]{\cos x}} = \int \underbrace{(\cos x)^{-\frac{1}{3}}}_{v} \cdot \underbrace{\operatorname{sen} x \, dx}_{-dv} =$

$\int v^k \, dv$

$v = \cos x \Rightarrow dv = -\operatorname{sen} x \, dx$

$\operatorname{sen} x \, dx = -dv$

$$= \int v^{-\frac{1}{3}} (-1 dv) = -\int v^{-\frac{1}{3}} dv =$$

$$= -\frac{v^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$$

$$= -\frac{v^{\frac{2}{3}}}{\frac{2}{3}} + C = -\frac{3}{2} v^{\frac{2}{3}} + C$$

$$= -\frac{3}{2} (\cos x)^{\frac{2}{3}} + C$$

QUESTÃO 06: $\displaystyle\int_0^1 \frac{dx}{x^2+4x+3}$

$$\int \frac{dx}{x^2+4x+3} = \int \frac{dx}{(x+2)^2-1} = \int \frac{dv}{v^2-a^2} =$$

$x^2+4x+3 = (\underbrace{x+2}_{v})^2 - 1$

$v = x+2$

$dv = dx$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c = \frac{1}{2\cdot 1}\ln\left|\frac{x+2-1}{x+2+1}\right| + c$$

$$\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + c.$$

Assim:

$$\int_0^1 \frac{dx}{x^2+4x+3} = \left(\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|\right)\Bigg|_0^1 =$$

$$= \frac{1}{2}\ln\left|\frac{1+1}{1+3}\right| - \frac{1}{2}\ln\left|\frac{0+1}{0+3}\right| =$$

$$= \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{3} =$$

$$\boxed{\begin{array}{l} \ln\frac{a}{b} = \ln a - \ln b \\ \ln\frac{1}{2} = \ln 1 - \ln 2 \end{array}}$$

$$= \frac{1}{2}\cdot\Big(\underbrace{\ln 1}_{0} - \ln 2 - \big(\underbrace{\ln 1}_{0} - \ln 3\big)\Big)$$

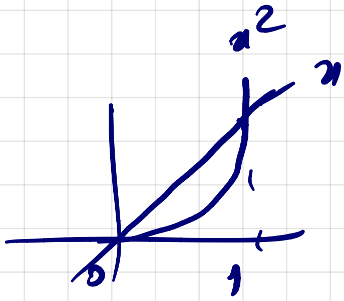
$$= \frac{1}{2}(-\ln 2 + \ln 3) = \frac{1}{2}\ln\frac{3}{2}$$

QUESTÃO 04:

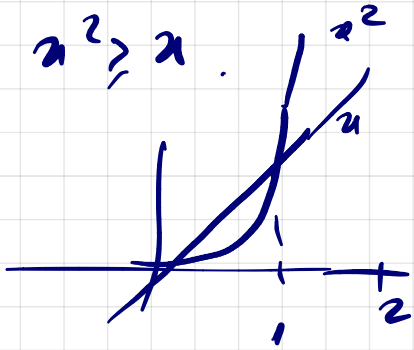
$$\int_0^1 x\,dx \geq \int_0^1 x^2\,dx:$$

pois, $\forall x \in [0,1]$, $x \geq x^2$

$$\Rightarrow \int_0^1 x \geq \int_0^1 x^2$$



$$\int_1^2 x\,dx \le \int_1^2 x^2\,dx$$, γιατί , $\forall x \in [1,2]$: $x^2 \geq x$.



$$\Rightarrow \int_1^2 x^2 \geq \int_1^2 x$$