1. Apenas pelo esboço gráfico, determine as seguintes integrais definidas:

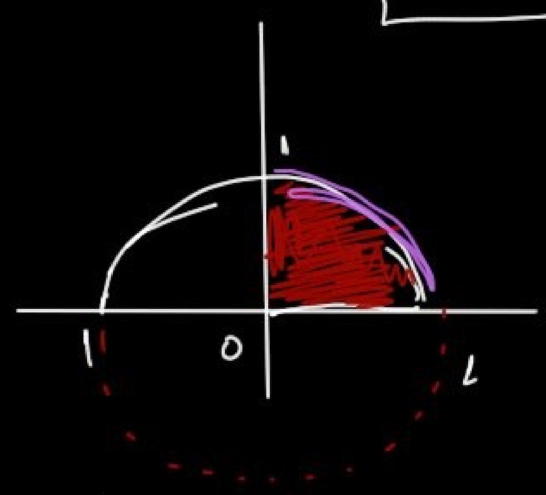
(a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^2 (x-1)dx$ (c) $\int_1^2 (x+2)dx$ (d) $\int_0^3 |x-2|dx$

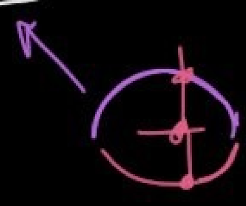
(a) $\int_0^1 \sqrt{1-x^2}\,dx$

$y = f(x) = \sqrt{1-x^2}$

$y^2 = 1-x^2$

$\boxed{x^2+y^2=1}$ → eq. de circunf. centrada na origem e raio 1.





$R=1$

$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4} \cdot A_O$ → área do círculo.

$A_O = \pi R^2$

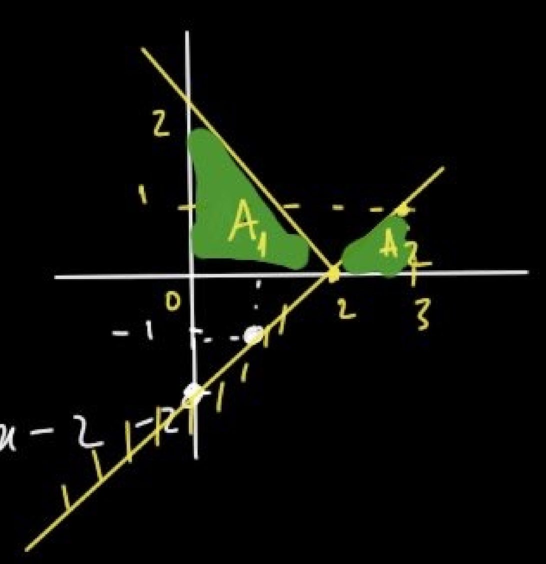
$A_O = \pi \cdot (1)^2 = \pi$

$\Rightarrow \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi$ //

(d) $\int_0^3 |x-2|\,dx = A_1 + A_2$, onde:

$A_1 = \frac{2\cdot 2}{2} = 2$

$A_2 = \frac{1\cdot 1}{2} = \frac{1}{2}$



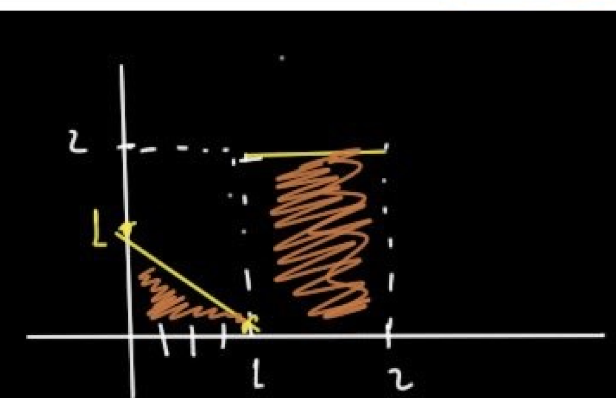
$x = 3$

$y = |3-2| = 1$

$\Rightarrow \int_0^3 |x-2|\,dx = 2 + \frac{1}{2} = \frac{5}{2}$ //

3. Idem para $f : [0,2] \to \mathbb{R}$ dada por

$$f(x) = \begin{cases} 1-x, & \text{se } 0 \le x \le 1 \\ 2, & \text{se } 1 < x \le 2 \end{cases}.$$



Observe que:

$\int_0^2 f = \int_0^1 f + \int_1^2 f$

• $\int_0^1 f = \int_0^1 (1-x)\,dx$.

Seja P_n a partição regular que divide o intervalo $[0,1]$ em n subintervalos de comprimento

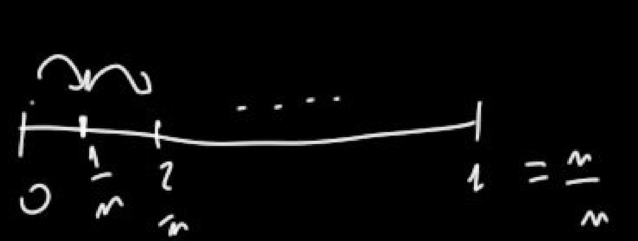
$\Delta x = t_i - t_{i-1} = \frac{1-0}{n} = \frac{1}{n}$.

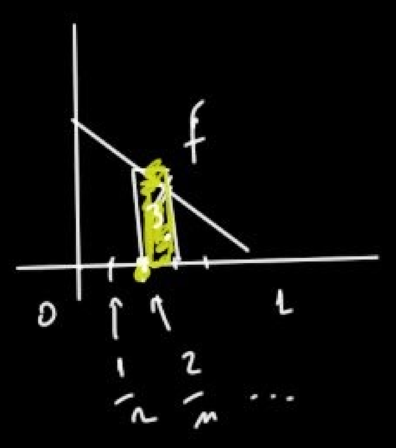
$a = t_0 = 0$; $t_1 = \frac{1}{n}$; $t_2 = 0 + 2\cdot\frac{1}{n} = 2/n$

$t_3 = 0 + 3\cdot\frac{1}{n} = \frac{3}{n}$

$\vdots$

$t_n = 0 + n\cdot\frac{1}{n} = 1 = b$





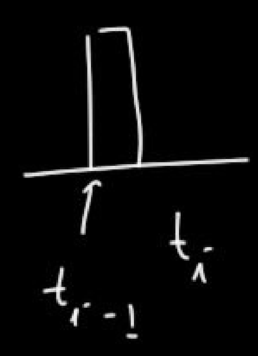
$\Delta x = \frac{1}{n}$ $\qquad f(x) = 1 - x$

$S(f;P) = \sum_{i=1}^{n} M_i \cdot \underbrace{(t_i - t_{i-1})}_{=\frac{1}{n}}$,

$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1\}$

onde $M_i = f\left(\frac{i}{n}\right) = 1 - \frac{i-1}{n}$

$f(x) = 1 - x$



$t_{i-1} = \frac{i-1}{n}$ $\qquad t_i = \frac{i}{n}$

$$S(f;P_n) = \sum_{i=1}^{n}\left(1 - \frac{i-1}{n}\right)\cdot\frac{1}{n} = \frac{1}{n}\cdot\sum_{i=1}^{n}\left(1 - \frac{i-1}{n}\right) =$$

$$= \frac{1}{n}\cdot\sum_{i=1}^{n} 1 - \frac{1}{n}\cdot\sum_{i=1}^{n}\frac{i-1}{n} =$$

$$= \cancel{\frac{1}{n}\cdot n} - \frac{1}{n^2}\cdot\sum_{i=1}^{n}(i-1)$$

$$= 1 - \frac{1}{n^2}\cdot\left[0 + 1 + 2 + 3 + \dots + (n-1)\right]$$

$$= 1 - \frac{1}{n^2}\cdot\underbrace{\frac{(\cancel{1} + (n-\cancel{1}))\cdot(n-1)}{2}}_{S_k = \frac{(a_1 + a_k)\cdot k}{2}} = 1 - \frac{1}{n^{\cancel{2}}}\cdot\frac{\cancel{n}\cdot(n-1)}{2}$$

$S_k = \frac{(a_1+a_k)\cdot k}{2}$

SOMA DOS K TERMOS PRIMEIROS DE UMA P.A.

$$= 1 - \frac{n-1}{2n} = 1 - \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2n}$$

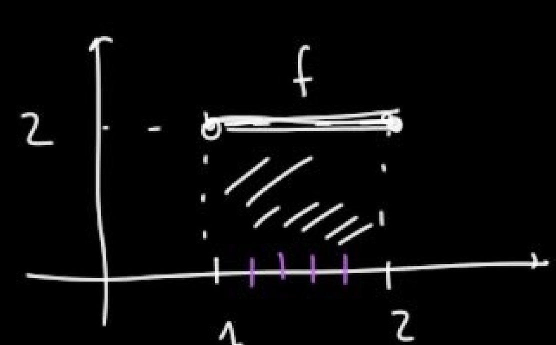
$$\Rightarrow S(f;P_n) = \frac{1}{2} + \frac{1}{2n}$$

$$\Rightarrow \int_0^1 f = \lim_{n\to\infty} S(f;P_n) = \lim_{n\to\infty}\left(\frac{1}{2} + \underbrace{\frac{1}{2n}}_{0}\right) = \frac{1}{2}$$

Analogamente se mostra que $\underline{\int_0^1} f = \frac{1}{2}$.

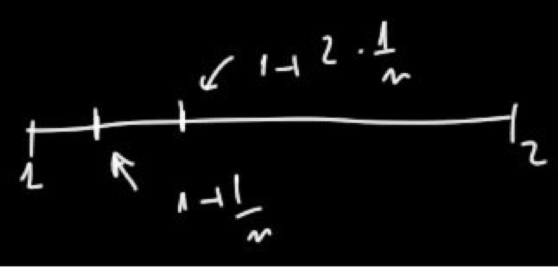
Portanto, $\int_0^1 f = \frac{1}{2}$ //

- $\int_1^2 f = \int_1^2 2\,dx$. $\qquad f(x) \equiv 2$



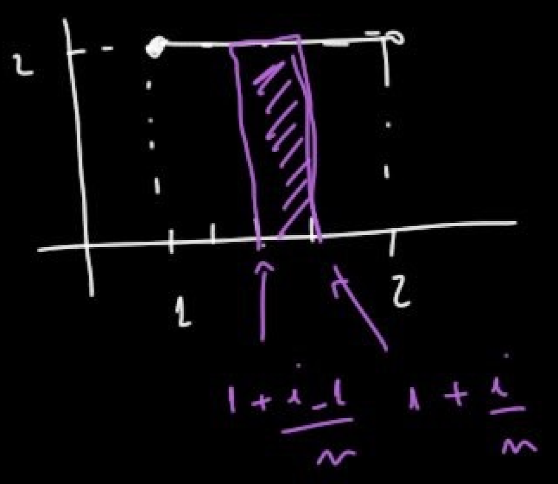
Seja P_n uma partição regular do intervalo $[1,2]$, que o divide em n subintervalos da forma $[t_{i-1}, t_i]$, de mesmo tamanho

$$\Delta t_i = t_i - t_{i-1} = \frac{2-1}{n} = \frac{1}{n}.$$



; onde: $\quad t_i = 1 + i\cdot\frac{1}{n}$

$i \in \{1, 2, 3, \dots, n\}$



Seja

$$S(f;P_n) = \sum_{i=1}^{n} M_i \cdot \underbrace{\Delta t_i}_{\frac{1}{n}}, \text{ onde}$$

$$M_i = m_i = 2$$

pois $f(x) = 2$ (constante)

$$\Rightarrow S(f;P_n) = \sum_{i=1}^{n} 2\cdot\frac{1}{n} = \frac{2}{n}\cdot\underbrace{\sum_{i=1}^{n} 1}_{=n} = \frac{2}{\cancel{n}}\cdot\cancel{n} = 2$$

$$\Rightarrow \overline{\int_1^2} f = \lim_{n\to\infty} S(f;P_n) = \lim_{n\to\infty} 2 = 2$$

Do mesmo modo conclui-se que $\underline{\int_1^2} f = 2$

Portanto, $\int_1^2 f = 2$

Por fim, concluímos que

$$\int_0^2 f = \int_0^1 f + \int_1^2 f = \frac{1}{2} + 2 = \frac{5}{2}$$

10. Se f é integrável e m e M são tais que $m \le f(x) \le M$, $\forall x \in [a,b]$ (i.e., f é limitada em $[a,b]$), mostre que
$$m(b-a) \le \int_a^b f \le M(b-a).$$

$$m \le f(x) \le M \quad \forall x \in [a,b]$$

Por propriedade da integral definida, temos que [devido à propriedade (ii) da aula 03]

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$m\underbrace{\int_a^b 1\,dx}_{b-a} \le \int_a^b f(x)\,dx \le M\cdot\underbrace{\int_a^b 1\,dx}_{=b-a}$$

ÁREA: $(b-a)\cdot 1$

$$m\cdot(b-a) \le \int_a^b f \le M\cdot(b-a)$$

11. Se f é integrável (e então limitada) em $[a,b]$, mostre que
$$\left|\int_a^b f\right| \le (b-a)\sup_{x\in[a,b]}|f(x)|.$$

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Como f é limitada, pois é integrável, então $\exists \sup_{x\in[a,b]}|f(x)|$ e é tal que

$$|f(x)| \le \sup_{x\in[a,b]}|f(x)| := M > 0.$$

$$\left|\int_a^b f\right| \le \int_a^b |f| \le \int_a^b M\,dx = M\cdot\underbrace{\int_a^b 1\,dx}_{b-a} = M\cdot(b-a) = (b-a)\cdot\sup_{x\in[a,b]}|f(x)|$$

$$|f| \le M = \sup|f|$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \le (b-a)\cdot\sup_{x\in[a,b]}|f(x)|.$$

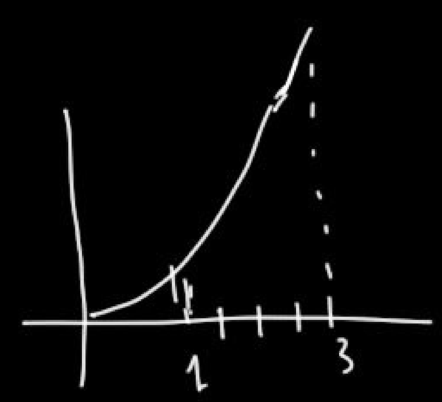
9. Se a função $f : [a,b] \to \mathbb{R}$ é integrável em $[a,b]$, definimos o *valor médio* VM de f no domínio $[a,b]$ por

$$VM = \frac{\int_a^b f}{b-a}.$$

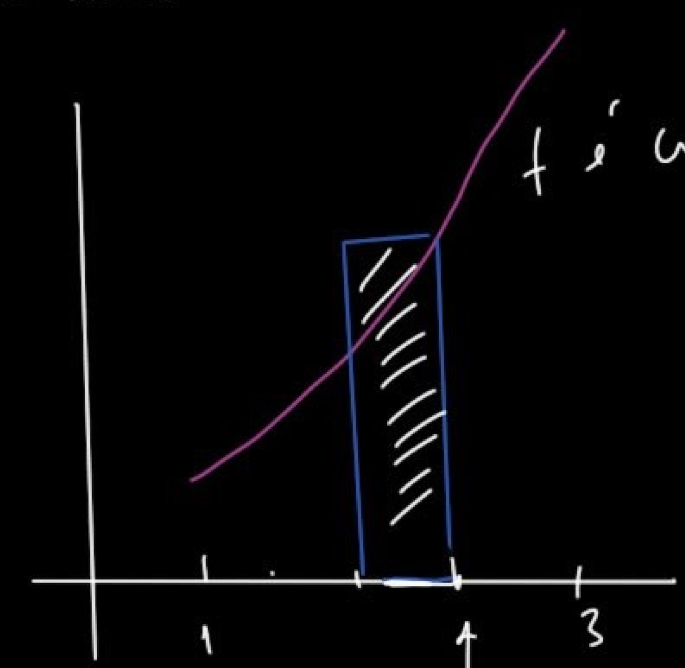
Encontre o valor médio para $f : [1,3] \to \mathbb{R}$ dada por $f(x) = x^2$.

$$V.M. = \frac{\int_a^b f}{b-a}$$

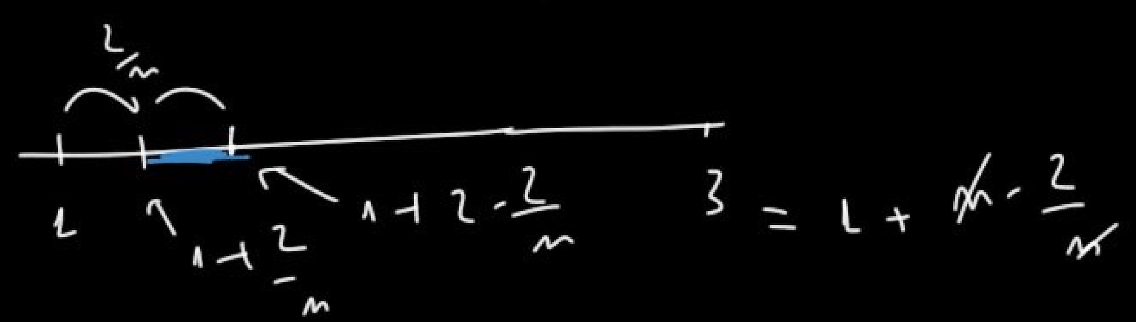
$$\int_1^3 x^2\,dx.$$



Seja P_n a partição regular que divide $[1,3]$ em n subintervalos de comprimento $\Delta t_i = \frac{3-1}{n} = \frac{2}{n}$



f é crescente



$$t_i = 1 + i\cdot\frac{2}{n}, \quad \forall i \in \{1,2,\dots,n\}$$

$$t_i = 1 + i\cdot\frac{2}{n}$$

$$M_i := \sup_{[t_{i-1},t_i]} f(x) = f(t_i) = \left(1+\frac{2i}{n}\right)^2 = 1 + \frac{4i}{n} + \frac{4i^2}{n^2}$$

Assim, teremos: $$S(f,P_n) = \sum_{i=1}^{n} M_i\cdot\underbrace{(t_i - t_{i-1})}_{\Delta t_i = \frac{2}{n}} = \sum_{i=1}^{n}\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)\cdot\frac{2}{n} =$$

$$= \sum_{i=1}^{n}\left(\frac{2}{n} + \frac{8i}{n^2} + \frac{8i^3}{n^3}\right) = \frac{2}{n}\cdot\underbrace{\sum_{i=1}^{n} 1}_{n} + \frac{8}{n^2}\underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} + \frac{8}{n^3}\cdot\underbrace{\sum_{i=1}^{n} i^3}_{\left[\frac{n(n+1)}{2}\right]^2}$$

$$= \frac{2\cancel{n}}{\cancel{n}} + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^{\cancel{3}}}\cdot\frac{\cancel{n^2}(n+1)^2}{4}$$

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