CÁLCULO 2

16/11/23 — AULA 07

Na aula passada vimos várias regras de integração indefinida; onde $u = u(x)$; $v = v(x)$; $K, C \in \mathbb{R}$:

01) $\int 1\, dv = v + C$

02) $\int (u + v)\, dx = \int u\, dx + \int v\, dx$

03) $\int K \cdot v \cdot dv = K \cdot \int v\, dv$

04) $\int v^K \cdot dv = \dfrac{v^{K+1}}{K+1} + C \quad ; \quad K \neq -1.$

05) $\int \dfrac{dv}{v} = \ln |v| + C$

06) $\int e^v\, dv = e^v + C$

07) $\int \operatorname{sen} v \cdot dv = -\cos v + C$

08) $\int \cos v\, dv = \operatorname{sen} v + C$

09) $\int \tan v\, dv = -\ln |\cos v| + C$

↳ $\int \tan v\, dv = \int \dfrac{\operatorname{sen} v}{\cos v}\, dv = \int -\dfrac{dw}{w}$

$w = \cos v$

$\Rightarrow dw = -\operatorname{sen} v\, dv$

$\Rightarrow \operatorname{sen} v\, dv = -dw$

$= -\int \dfrac{dw}{w} =$

$= -\ln |w| + C$

10) $\int \sec x \cdot \tan x \, dx = \sec x + C$

11) $\int \csc x \cdot \cot x \, dx = -\csc x + C$

De fato; note que

$(\csc x)' = -\csc x . \cot x$

12) $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

13) $\int \csc x \, dx = \ln|\csc x - \cot x| + c$

De fato; para simplificar assuma que $\csc x - \cot x > 0$ (o caso onde $\csc x - \cot x < 0$ é análogo).

$$\frac{d}{dx}\Big(\ln(\underbrace{\csc x - \cot x}_{w}) + c\Big) = \frac{-\csc x . \cot x + \csc^2 x}{\csc x - \cot x} = \ldots$$

$(\cot x)' = -\csc^2 x$

$(\ln w)' = \frac{w'}{w}$

obs: $(\ln w)' =$

uso da regra da cadeia

$w = w(x)$

$g(x) = \ln x$

$f(x) = w(x)$

$(g \circ f)(x) = \ln(w(x))$. Então:

$$(g \circ f)'(x) = g'(f(x))\, f'(x)$$

$$= \frac{1}{f(x)} \cdot f'(x) = \frac{w'(x)}{w(x)}$$

$$= \frac{\csc v(\csc v - \cot v)}{\csc v - \cot v} = \csc v$$

14) $\int \sec^2 v \, dv = \tan v + C$

De fato, basta observar que:

$$(\tan v)' = \left(\frac{\operatorname{sen} v}{\cos v}\right)' = \frac{\cos v \cdot \cos v - \operatorname{sen} v(-\operatorname{sen} v)}{\cos^2 v} \cdot v'$$

$$\left(\frac{u}{v}\right)' = \frac{v \cdot u' - u \cdot v'}{v^2}$$

$$= \frac{\cos^2 v + \operatorname{sen}^2 v}{\cos^2 v} \cdot v' = \frac{1}{\cos^2 v} \cdot v'$$

($\cos^2 v + \operatorname{sen}^2 v = 1$)

$$= \sec^2 v \cdot v'$$

EXEMPLO: $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = ?$

$\int \sec^2 v \cdot dv$

$v = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow dv = \frac{1}{2} \cdot x^{\frac{1}{2} - 1} \cdot dx$

$dv = \frac{1}{2} \cdot x^{-\frac{1}{2}} dx$

$$\Rightarrow dv = \frac{dx}{2\sqrt{x}}$$

$$\Rightarrow \frac{dx}{\sqrt{x}} = 2dv$$

Logo, temos:

$$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}} \cdot dx = \int \sec^2\sqrt{x} \cdot \frac{dx}{\sqrt{x}} = \int \sec^2 v \cdot (2dv)$$

$$= 2\int \sec^2 v \cdot dv = 2 \cdot \tan v + C$$

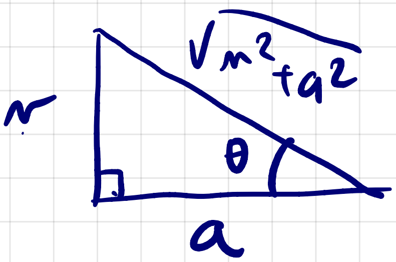
$$= 2 \cdot \tan\sqrt{x} + C$$

$$v = \sqrt{x}$$

15) $\int \csc^2 v \, dv = -\cot v + C$

(Se demonstra do mesmo modo que a fórmula anterior)

16) $\int \frac{dv}{v^2+a^2} = ?$, $a \in \mathbb{R}$, $a \neq 0$.



$$\tan\theta = \frac{v}{a}$$

$$\Rightarrow v = a \cdot \tan\theta$$

$$\Rightarrow dv = a \cdot \sec^2\theta \cdot d\theta$$

$$\cos\theta = \frac{a}{\sqrt{x^2+a^2}} \Rightarrow \left(\sqrt{x^2+a^2}\right)^2 = \left(\frac{a}{\cos\theta}\right)^2$$

$$\boxed{x^2+a^2 = a^2 \cdot \sec^2\theta}$$

Assim:

$$\int \frac{dx}{x^2+a^2} = \int \frac{a \cdot \sec^2\theta \, d\theta}{a^2 \cdot \sec^2\theta} = \int \frac{1}{a} \cdot d\theta$$

$$= \frac{1}{a} \cdot \int 1 \cdot d\theta = \frac{1}{a} \cdot \theta + C =$$

$$\tan\theta = \frac{x}{a} \Rightarrow \theta = \arctan\left(\frac{x}{a}\right)$$

$$= \frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right) + C$$

De fato:

$$\frac{d}{dx}\left(\frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right) + C\right) =$$

$$= \frac{1}{a} \cdot \frac{\left(\frac{x}{a}\right)'}{1+\left(\frac{x}{a}\right)^2} + 0 = \frac{1}{a} \cdot \frac{\frac{1}{a} \cdot x'}{1 + \frac{x^2}{a^2}} \quad (=)$$

Obs.:

$$(\arctan w)' = \frac{w'}{1+w^2}$$

$$y = \arctan x$$

$$\Updownarrow$$

$$\tan y = x \quad \text{Derivando em } x:$$

$$\sec^2 y \cdot y' = x'$$

$$\Rightarrow y' = \frac{u'}{\sec^2 y} = \frac{u'}{1+\tan^2 y} = \frac{u'}{1+u^2}$$

$1+\tan^2 y = \sec^2 y$ DA TRIGONOMETRIA.

Ou seja, mostramos que, se $y = \arctan u$, então $y' = \frac{u'}{1+u^2}$

$$\Leftrightarrow \frac{1}{a} \cdot \frac{\frac{1}{a} \cdot u'}{\frac{a^2+u^2}{a^2}} = \frac{\frac{1}{a} u'}{\frac{u^2+a^2}{a}} = \frac{u'}{u^2+a^2}$$

Ex.: (a) $\int \frac{dx}{x^2+3} = ?$

TENTATIVA "FRUSTRADA":

$$\int \frac{du}{u}$$

$u = x^2+3$

$du = 2x\,dx$

VARIÁVEL QUE FALTA NO PROBLEMA

$$\int \frac{du}{u^2+a^2} \; ; \quad u = x, \quad du = dx.$$

$$a^2 = 3 \Rightarrow a = \sqrt{3}$$

Assim:

$$\int \frac{dx}{x^2+3} = \int \frac{dx}{x^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}} \cdot \arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

(b) $\int \frac{dx}{x^2+4x+9}$

Nestes casos, a saída é procurar completar um quadrado perfeito.

$$x^2+4x+9 = (x+2)^2+5$$

Assim:

$$\int \frac{dx}{x^2+4x+9} = \int \frac{dx}{(x+2)^2+5} = \int \frac{dx}{(x+2)^2+(\sqrt{5})^2}$$

$$r = x+2 \implies dr = 1.dx$$

$$= \int \frac{dr}{r^2+(\sqrt{5})^2} = \frac{1}{\sqrt{5}} \cdot \arctan\left(\frac{r}{\sqrt{5}}\right)+c$$

$$= \frac{1}{\sqrt{5}} \cdot \arctan\left(\frac{x+2}{\sqrt{5}}\right)+c$$

Verificação:

De fato:

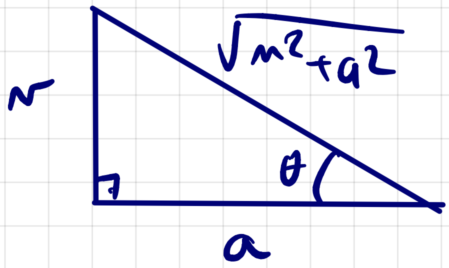
$$\frac{d}{dx}\left(\frac{1}{\sqrt{5}} \cdot \arctan\left(\frac{x+2}{\sqrt{5}}\right)+c\right) =$$

$$\frac{1}{\sqrt{5}} \cdot \left(\frac{\left(\frac{x+2}{\sqrt{5}}\right)'}{1+\left(\frac{x+2}{\sqrt{5}}\right)^2} + 0\right) =$$

$$\frac{1}{\sqrt{5}} \cdot \left[\frac{\frac{1}{\sqrt{5}}}{1 + \frac{x^2+4x+4}{5}} \right] = \frac{\frac{1}{5}}{\frac{5+x^2+4x+4}{5}} =$$

$$= \frac{1}{x^2+4x+9}$$

17) $\displaystyle\int \frac{dr}{\sqrt{r^2+a^2}} = ?$ $\quad a \neq 0.$



$\tan\theta = \frac{r}{a}$

$r = a \cdot \tan\theta$

$dr = a \cdot \sec^2\theta \cdot d\theta$

$$\cos\theta = \frac{a}{\sqrt{r^2+a^2}} \Rightarrow \frac{1}{a} \cdot \cos\theta = \frac{1}{\sqrt{r^2+a^2}}$$

Assim, temos:

$$\int \frac{dr}{\sqrt{r^2+a^2}} = \int a \cdot \sec^2\theta \cdot d\theta \cdot \frac{1}{a} \cdot \cos\theta$$

$$= \int \frac{\cos\theta \cdot d\theta}{\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \int \sec\theta \, d\theta =$$

$$= \ln|\sec\theta + \tan\theta| + C$$

FÓRMULA 12

Como $\tan\theta = \frac{r}{a}$ e $\cos\theta = \frac{a}{\sqrt{r^2+a^2}}$, e disso

$$\sec\theta = \frac{\sqrt{r^2+a^2}}{a};$$

então, temos:

$$\int \frac{dr}{\sqrt{r^2+a^2}} = \ln|\sec\theta + \tan\theta| + C =$$

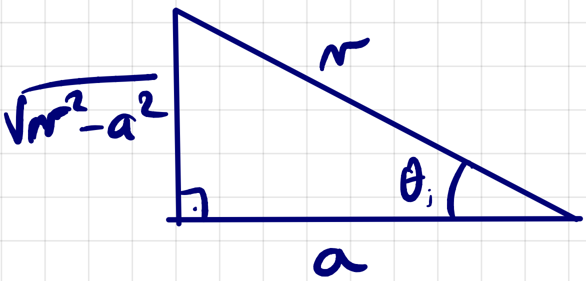
$$= \ln\left| \frac{\sqrt{r^2+a^2}}{a} + \frac{r}{a} \right| + C$$

$$= \ln\left| r + \sqrt{r^2+a^2} \right| + C$$

Portanto, temos a fórmula:

17) $$\int \frac{dr}{\sqrt{r^2+a^2}} = \ln\left| r + \sqrt{r^2+a^2} \right| + C.$$

18) $$\int \frac{dr}{r^2-a^2} = ?$$



$$\tan\theta = \frac{\sqrt{r^2-a^2}}{a}$$

$$\sqrt{r^2-a^2} = a\cdot\tan\theta$$

$$\Rightarrow r^2-a^2 = a^2\tan^2\theta.$$

$$\cos\theta = \frac{a}{r} \Rightarrow r = \frac{a}{\cos\theta}$$

$$r = a \cdot \sec\theta$$

$$\Rightarrow dr = a \cdot \sec\theta \cdot \tan\theta \cdot d\theta$$

Logo, teremos:

$$\int \frac{dr}{r^2 - a^2} = \int \frac{a \cdot \sec\theta \cdot \tan\theta\, d\theta}{a^2 \cdot \tan^2\theta} = \int \frac{\sec\theta\, d\theta}{a \cdot \tan\theta} =$$

$$= \frac{1}{a} \cdot \int \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\operatorname{sen}\theta}\, d\theta = \frac{1}{a} \cdot \int \csc\theta\, d\theta =$$

FÓRMULA 13

$$= \frac{1}{a} \cdot \ln|\csc\theta - \cot\theta| + C \quad ;$$

como $\operatorname{sen}\theta = \dfrac{\sqrt{r^2 - a^2}}{r}$ e $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{a}{\sqrt{r^2 - a^2}}$,

então:

$$\int \frac{dr}{r^2 - a^2} = \frac{1}{a} \cdot \ln|\csc\theta - \cot\theta| + C$$

$$= \frac{1}{a} \cdot \ln\left|\frac{r}{\sqrt{r^2 - a^2}} - \frac{a}{\sqrt{r^2 - a^2}}\right| + C$$

$$= \frac{1}{a} \cdot \ln\left|\frac{r - a}{\sqrt{r^2 - a^2}}\right| + C$$

conclusão:

$$\int \frac{dv}{v^2-a^2} = \frac{1}{a} \cdot \ln\left|\frac{v-a}{\sqrt{v^2-a^2}}\right| + c$$

Ex.: $\int \frac{dx}{x^2+x} = \int \frac{dx}{(x+\frac{1}{2})^2-\frac{1}{4}} =$

$$x^2+x = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} = x^2 + 2\cdot\frac{1}{2}\cdot x + \frac{1}{4} - \frac{1}{4}$$

$$= \int \frac{dx}{(x+\frac{1}{2})^2 - (\frac{1}{2})^2} = \int \frac{dv}{v^2-a^2} =$$

$$v = x + \frac{1}{2} \Rightarrow dv = dx$$

$$= \frac{1}{a} \cdot \ln\left|\frac{v-a}{\sqrt{v^2-a^2}}\right| + c$$

$$= \frac{1}{\frac{1}{2}} \cdot \ln\left|\frac{x+\frac{1}{2}-\frac{1}{2}}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}}\right| + c$$

$$= 2\cdot\ln\left|\frac{x}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\right| + c = 2\cdot\ln\left|\frac{x}{\sqrt{x^2+x}}\right| + c$$