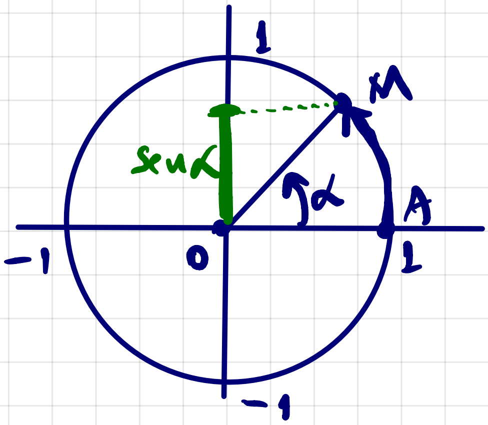
16/11/23 – AULA 07

Def.: Seja c o ciclo trigonométrico e $\alpha = \overset{\frown}{AM}$

Definimos:

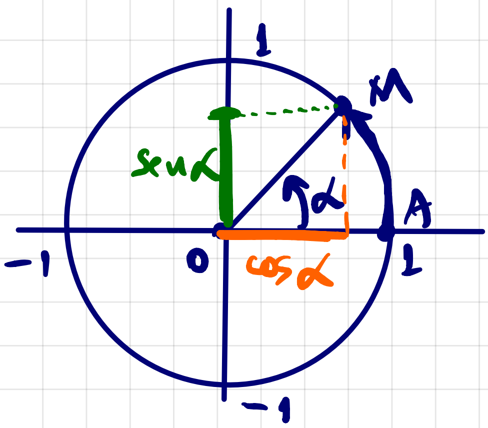
(a) SENO DE α: projeção do lado terminal $\overline{OM}$ do arco $\overset{\frown}{AM}$ sobre o eixo vertical.



NOTAÇÃO:

$$\operatorname{sen}\alpha = \operatorname{proj}_{OY} \overline{OM}$$

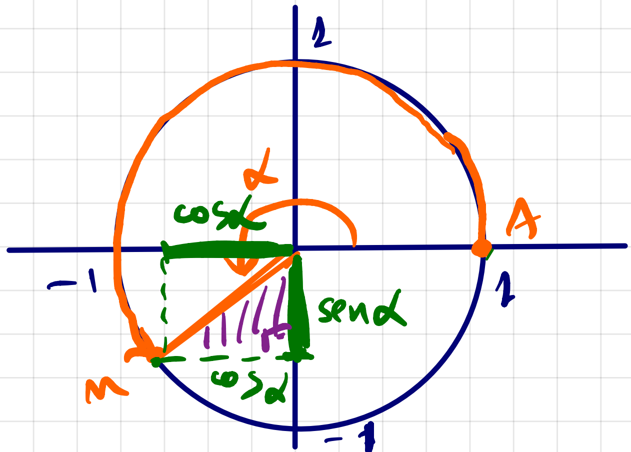
(b) COSSENO DE α: projeção do lado terminal $\overline{OM}$ do arco $\overset{\frown}{AM}$ sobre o eixo horizontal.

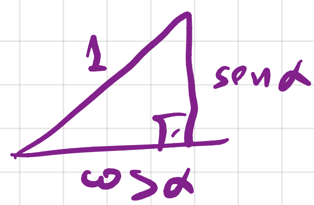


NOTAÇÃO:

$$\cos\alpha = \operatorname{proj}_{OX} \overline{OM}$$

Como $R=1$ no ciclo trigonométrico, $\forall \alpha \in \mathbb{R}$ temos $\operatorname{sen}^2\alpha + \cos^2\alpha = 1$.





$\operatorname{sen}^2\alpha + \cos^2\alpha = 1, \forall \alpha$

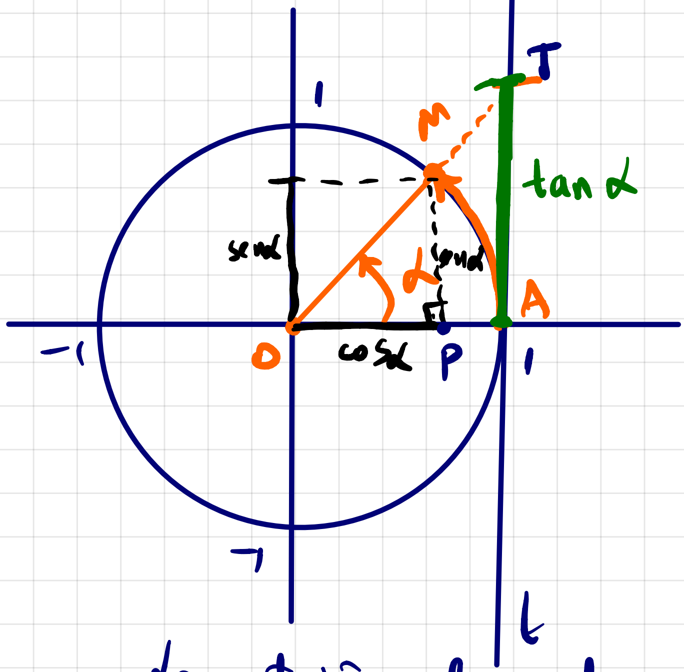
Obs: $-1 \leq \operatorname{sen}\alpha \leq 1$ e $-1 \leq \cos\alpha \leq 1$

$\Updownarrow$ $\Updownarrow$

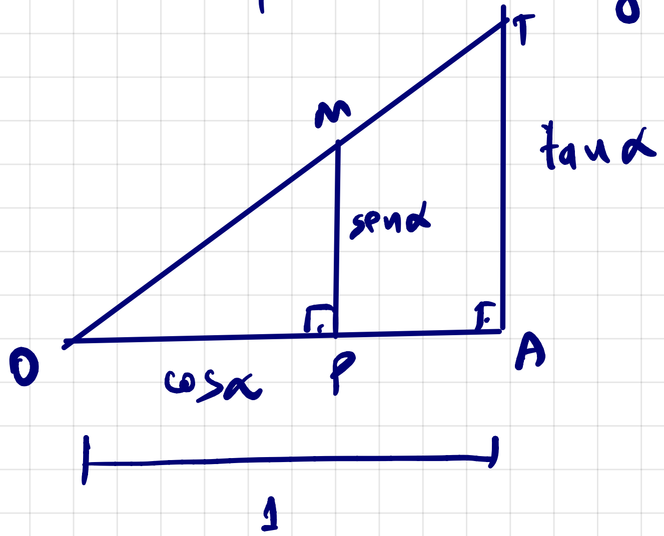
$|\operatorname{sen}\alpha| \leq 1$ $|\cos\alpha| \leq 1$

(c) TANGENTE DE α. Seja t a reta tangente ao ciclo c no ponto A (origem dos arcos).

A tangente de α é o prolongamento do lado terminal $\overline{OM}$ de $\alpha = \widehat{AM}$ sobre a reta t.



Por semelhança de triângulos, temos:

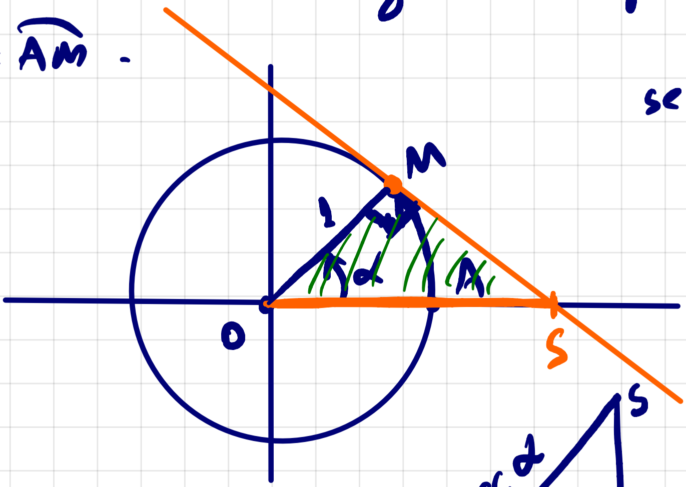


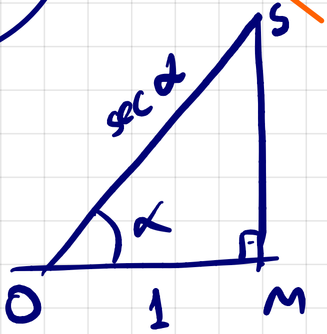
$$\frac{1}{\tan\alpha} = \frac{\cos\alpha}{\operatorname{sen}\alpha}$$

$$\boxed{\tan\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha}}$$

(d) SECANTE DE α Traçando uma reta tangente ao ciclo c no ponto M, a distância da origem dos eixos O ao ponto de intercepto com o eixo horizontal representa a secante de $\alpha = \widehat{AM}$.

$\sec\alpha = \overline{OS}$





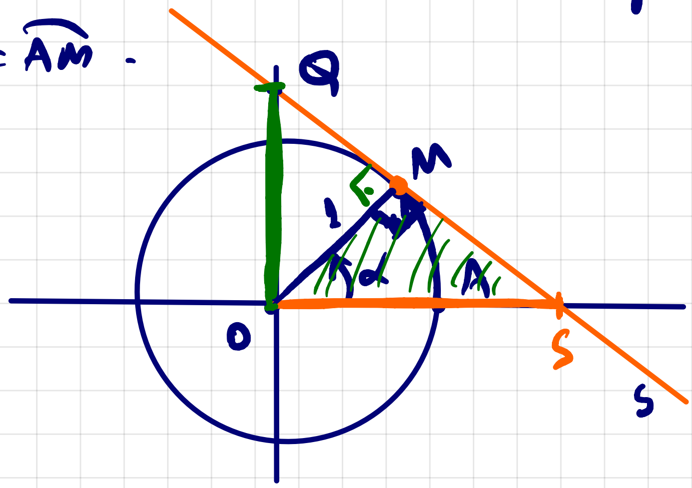
$\cos\alpha = \dfrac{1}{\sec\alpha}$

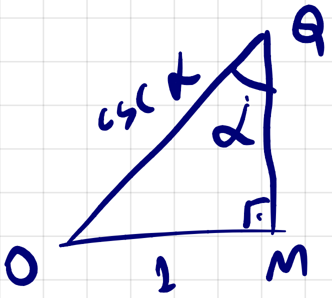
$\Updownarrow$

$\sec\alpha = \dfrac{1}{\cos\alpha}$

(e) COSSECANTE DE α Traçando uma reta tangente ao ciclo c no ponto M, a distância da origem dos eixos O ao ponto de intercepto com o eixo VERTICAL representa a cossecante $\alpha = \widehat{AM}$.

$\csc\alpha = \overline{OQ}$



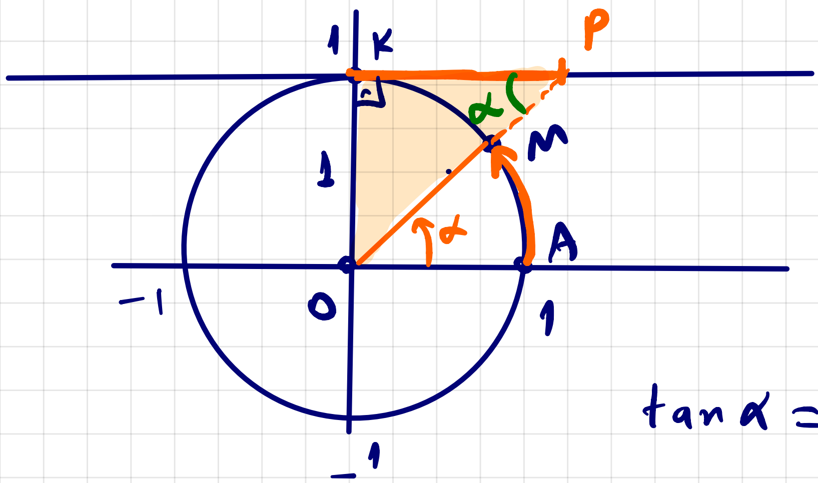


$$\text{sen}\,\alpha = \frac{1}{\csc\alpha}$$

$$\Updownarrow$$

$$\csc\alpha = \frac{1}{\text{sen}\,\alpha}$$

(f) COTANGENTE DE α: Traçando uma reta tangente a ciclo c no ponto K(0,1), a cotangente de $\alpha = \widehat{AM}$ é a medida do segmento obtido pelo intercepto desta tangente com o prolongamento do lado terminal $\overline{OM}$ do arco α.



$$\cot\alpha = \overline{KP}$$

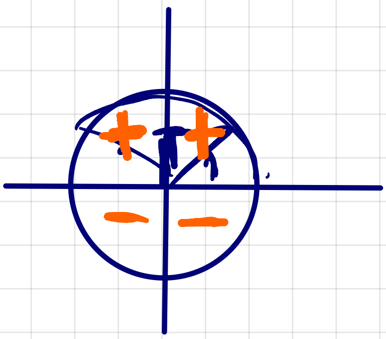
$$\tan\alpha = \frac{1}{\cot\alpha}$$

$$\Updownarrow$$

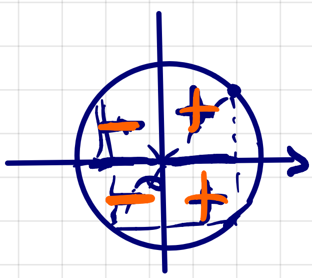
$$\cot\alpha = \frac{1}{\tan\alpha}.$$

SINAIS:

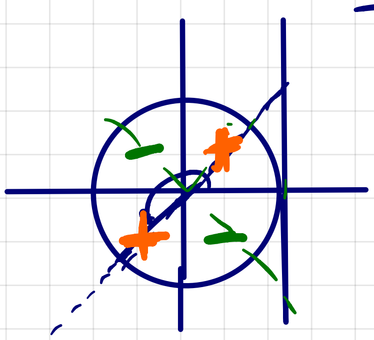
SENO:/CSC



COSSENO/SEC

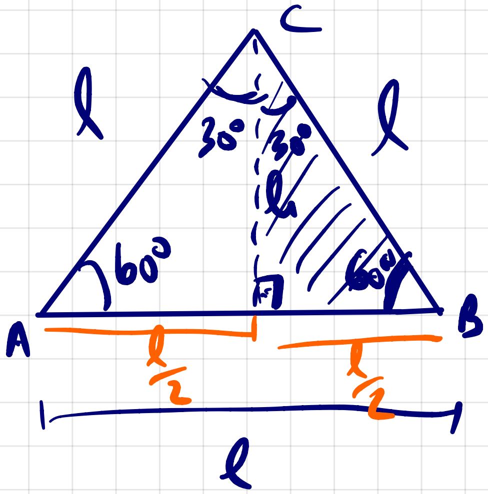


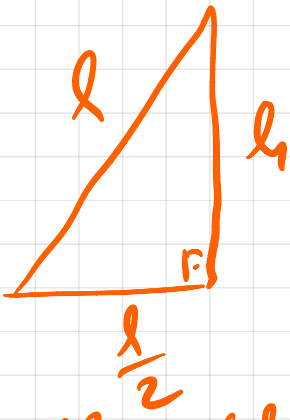
TANGENTE:/COT.



Valores: 30°; 60° e 45° — ARCOS NOTÁVEIS:

Considere o triângulo equilátero ABC, de lado l.





$$l^2 = \left(\frac{l}{2}\right)^2 + h^2$$

$$l^2 = \frac{l^2}{4} + h^2$$

$$h^2 = l^2 - \frac{l^2}{4}$$

$$h^2 = \frac{3l^2}{4} \Rightarrow \underline{h = \frac{l\sqrt{3}}{2}}$$

Assim:

$$\text{sen}\,30° = \frac{\frac{l}{2}}{l} = \frac{\cancel{l}}{2} \times \frac{1}{\cancel{l}} = \frac{1}{2}$$

$$\cos 30° = \frac{h}{l} = \frac{\frac{\cancel{l}\sqrt{3}}{2}}{\cancel{l}} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\frac{l}{2}}{h} = \frac{\frac{\cancel{l}}{\cancel{2}}}{\frac{\cancel{l}\sqrt{3}}{\cancel{2}}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\text{sen}\,60° = \cos(90° - 60°) = \cos 30° = \frac{\sqrt{3}}{2}$$

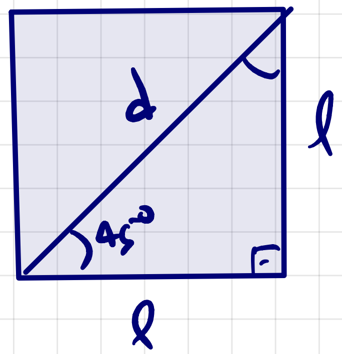
ou:

$$\text{sen}\,60° = \frac{h}{l} = \frac{\frac{\cancel{l}\sqrt{3}}{2}}{\cancel{l}} = \frac{\sqrt{3}}{2}$$

$$\underset{\sim}{\cos 60°} = \operatorname{sen} 30° = \underset{\sim}{\frac{1}{2}}$$

$$\underset{\sim}{\tan 60°} = \frac{h}{\frac{l}{2}} = \frac{\frac{l\sqrt{3}}{2}}{\frac{l}{2}} = \underset{\sim}{\sqrt{3}}$$

PARA O ARCO DE 45° CONSIDERAMOS UM QUADRADO DE LADO l:



$$d^2 = l^2 + l^2$$

$$d^2 = 2l^2$$

$$\boxed{d = l\sqrt{2}}$$

$$\underset{\sim}{\operatorname{sen} 45°} = \frac{l}{d} = \frac{\cancel{l}}{\cancel{l}\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \underset{\sim}{\frac{\sqrt{2}}{2}}$$

$$\underset{\sim}{\cos 45°} = \operatorname{sen}(90° - 45°) = \operatorname{sen} 45° = \underset{\sim}{\frac{\sqrt{2}}{2}}$$

$$\underset{\sim}{\tan 45°} = \frac{l}{l} = \underset{\sim}{1}$$

Relações:

$$\tan \alpha = \frac{1}{\cot \alpha} \; ; \quad \tan \alpha = \frac{\operatorname{sen} \alpha}{\cos \alpha} \quad \Big\} \quad \cot \alpha = \frac{\cos \alpha}{\operatorname{sen} \alpha}$$

Relação fundamental: $\operatorname{sen}^2 \alpha + \cos^2 \alpha = 1$. $(*)$

Dividindo por $\cos^2 \alpha \neq 0$, temos:

$$\frac{\operatorname{sen}^2\alpha}{\cos^2\alpha} + \underbrace{\frac{\cancel{\cos^2\alpha}}{\cancel{\cos^2\alpha}}}_{1} = \frac{1}{\cos^2\alpha}$$

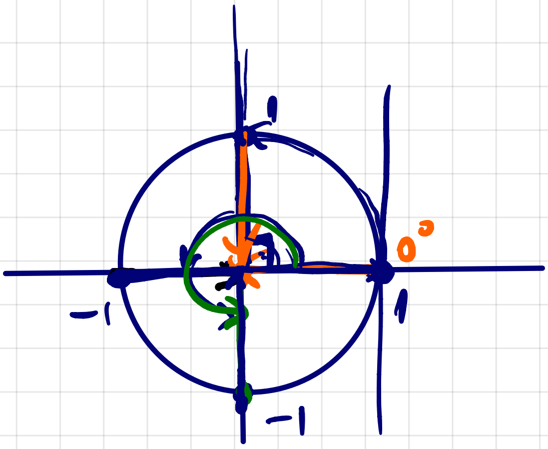
$$\tan^2\alpha + 1 = \sec^2\alpha \implies \boxed{1+\tan^2\alpha = \sec^2\alpha}$$

Dividindo (*) por $\operatorname{sen}^2\alpha \neq 0$, obtemos

$$\frac{\cancel{\operatorname{sen}^2\alpha}}{\cancel{\operatorname{sen}^2\alpha}} + \frac{\cos^2\alpha}{\operatorname{sen}^2\alpha} = \frac{1}{\operatorname{sen}^2\alpha}$$

$$\boxed{1 + \cot^2\alpha = \csc^2\alpha}$$

Valores nas extremidades do ciclo:



$\operatorname{sen} 0^\circ = 0 = \operatorname{sen} 360^\circ$

$\cos 0^\circ = 1 = \cos 360^\circ$

$\tan 0^\circ = 0 = \tan 360^\circ$

$\operatorname{sen} 90^\circ = 1$

$\cos 90^\circ = 0$

$\tan 90^\circ = \nexists$ $\left(\tan 90^\circ = \frac{\operatorname{sen} 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \nexists\right)$

$\operatorname{sen} 180^\circ = 0$

$\cos 180^\circ = -1$

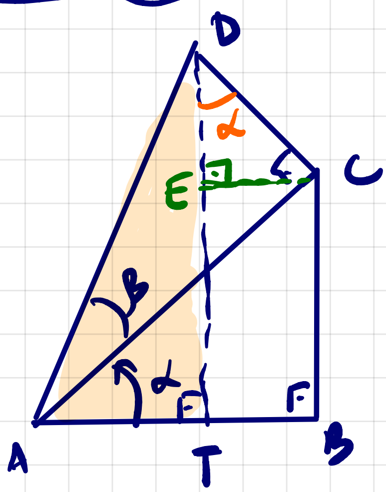
$\tan 180^\circ = 0$ $\left(\tan 180^\circ = \frac{\operatorname{sen} 180^\circ}{\cos 180^\circ} = \frac{0}{-1} = 0\right)$

$\operatorname{sen} 270° = -1$

$\cos 270° = 0$

$\tan 270° = \nexists$ $\left(\tan 270° = \dfrac{\operatorname{sen} 270°}{\cos 270°} = \dfrac{-1}{0} = \nexists \right)$

ADIÇÃO E SUBTRAÇÃO DE ARCOS:



$\operatorname{sen}(\alpha + \beta) = ?$

obs.: por simplicidade, no esquema ao lado temos $\alpha, \beta \in 1°q$; mas a fórmula a ser deduzida valerá $\forall \alpha, \beta \in \mathbb{R}$.

$$\operatorname{sen}(\alpha+\beta) = \frac{\overline{DT}}{\overline{AD}} = \frac{\overline{DE} + \overline{ET}}{\overline{AD}} = \frac{\overline{DE} + \overline{BC}}{\overline{AD}} =$$

pois $\overline{ET} = \overline{BC}$

$$= \frac{\overline{DE}}{\overline{AD}} + \frac{\overline{BC}}{\overline{AD}} = \frac{\overline{DE}}{\overline{AD}} \cdot \frac{\overline{DC}}{\overline{DC}} + \frac{\overline{BC}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{AC}}$$

$$= \frac{\overline{DE}}{\overline{DC}} \cdot \frac{\overline{DC}}{\overline{AD}} + \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}}$$

$$= \cos\alpha \cdot \operatorname{sen}\beta + \operatorname{sen}\alpha \cdot \cos\beta$$

Ou seja, acabamos de mostrar que

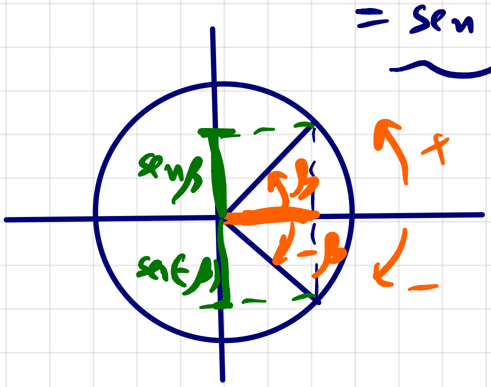
$$\boxed{\operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha \cdot \cos\beta + \operatorname{sen}\beta \cdot \cos\alpha .}$$

As demais fórmulas podem ser obtidas da acima.

- $\operatorname{sen}(\alpha-\beta) = \operatorname{sen}(\alpha+(-\beta))$

$$= \operatorname{sen}\alpha \cdot \cos(-\beta) + \operatorname{sen}(-\beta)\cdot\cos\alpha$$

$$= \operatorname{sen}\alpha \cdot \cos\beta - \operatorname{sen}\beta \cdot \cos\alpha$$



$\cos(\beta) = \cos(-\beta)$

$\operatorname{sen}(-\beta) = -\operatorname{sen}\beta$

- $\cos(\alpha+\beta) = \operatorname{sen}(90 - (\alpha+\beta))$

$$= \operatorname{sen}[(90^\circ-\alpha)-\beta] =$$

$$= \underbrace{\operatorname{sen}(90^\circ-\alpha)}_{\cos\alpha}\cdot\cos\beta - \operatorname{sen}\beta\cdot\underbrace{\cos(90^\circ-\alpha)}_{\operatorname{sen}\alpha}$$

$$= \cos\alpha\cdot\cos\beta - \operatorname{sen}\alpha\cdot\operatorname{sen}\beta$$

- $\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \operatorname{sen}\alpha\cdot\operatorname{sen}\beta$ (exercício)

- $\tan(\alpha+\beta) = ?$

$$\tan(\alpha+\beta) = \frac{\operatorname{sen}(\alpha+\beta)}{\cos(\alpha+\beta)} =$$

$$= \frac{\operatorname{sen}\alpha \cdot \cos\beta + \operatorname{sen}\beta \cos\alpha}{\cos\alpha\cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta} = \frac{\dfrac{\operatorname{sen}\alpha \cancel{\cos\beta}}{\cos\alpha \cancel{\cos\beta}} + \dfrac{\operatorname{sen}\beta \cancel{\cos\alpha}}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}} - \dfrac{\operatorname{sen}\alpha \cdot \operatorname{sen}\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\dfrac{\operatorname{sen}\alpha}{\cos\alpha} + \dfrac{\operatorname{sen}\beta}{\cos\beta}}{1 - \dfrac{\operatorname{sen}\alpha}{\cos\alpha} \cdot \dfrac{\operatorname{sen}\beta}{\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

- $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$ (exercício)

FÓRMULAS DA PROSTAFÉRESE (TRANSFORMAÇÃO DE SOMA EM PRODUTO)

Exercício

$$+\begin{cases} a + b = p \\ a - b = q \end{cases}$$

$$2a = p + q \Rightarrow a = \frac{p+q}{2}$$

$$\Rightarrow b = \frac{p-q}{2}$$

$$\operatorname{sen}(a+b) = \operatorname{sen} a \cdot \cos b + \operatorname{sen} b \cdot \cancel{\cos a}$$

$$+ \quad \operatorname{sen}(a-b) = \operatorname{sen} a \cdot \cos b - \operatorname{sen} \cancel{b} \cdot \cos a$$

$$\operatorname{sen}(\underbrace{a+b}_{p}) + \operatorname{sen}(\underbrace{a-b}_{q}) = 2 \cdot \operatorname{sen} \underline{a} \cdot \cos \underline{b}$$

$$\boxed{\operatorname{sen} p + \operatorname{sen} q = 2 \cdot \operatorname{sen} \frac{p+q}{2} \cdot \cos \frac{p-q}{2}}$$

$$\operatorname{sen}(a+b) = \operatorname{sen} a \cdot \cancel{\cos b} + \operatorname{sen} b \cdot \cos a$$

$$- \quad \operatorname{sen}(a-b) = \operatorname{sen} a \cdot \cancel{\cos b} - \operatorname{sen} b \cos a$$

$$\underbrace{\operatorname{sen}(a+b)}_{p} - \underbrace{\operatorname{sen}(a-b)}_{q} = +2 \operatorname{sen} b \cdot \cos a$$

$$\Rightarrow \boxed{\operatorname{sen} p - \operatorname{sen} q = 2 \cdot \operatorname{sen} \frac{p-q}{2} \cdot \cos \frac{p+q}{2}}$$

$$+ \begin{cases} \cos(a+b) = \cos a \cdot \cos b - \cancel{\operatorname{sen} a \cdot \operatorname{sen} b} \\ \cos(a-b) = \cos a \cdot \cos b + \cancel{\operatorname{sen} a \cdot \operatorname{sen} b} \end{cases}$$

$$\cos \underbrace{(a+b)}_{p} + \cos \underbrace{(a-b)}_{q} = 2 \cdot \cos a \cdot \cos b$$

$$\boxed{\cos p + \cos q = 2 \cdot \cos \frac{p+q}{2} \cdot \cos \frac{p-q}{2}}$$

$$\boxed{\cos p - \cos q = -2 \cdot \operatorname{sen} \frac{p+q}{2} \cdot \operatorname{sen} \frac{p-q}{2}}$$

EXEMPLOS:

01) Determine $\cos 75^\circ$.

SOLUÇÃO: $\cos 75^\circ = \cos(30^\circ + 45^\circ)$

$$= \cos 30^\circ \cdot \cos 45^\circ - \operatorname{sen} 30^\circ \cdot \operatorname{sen} 45^\circ$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

02) Dados $x, y \in 1^\circ q$ tais que $x + y = 120^\circ$. Sabendo que $\operatorname{sen} y = \frac{1}{3}$, determine $\cos x$.

SOLUÇÃO:

$$x + y = 120^\circ$$

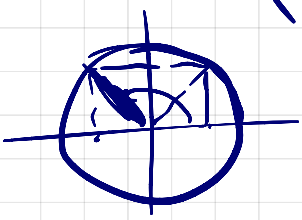
$$x = 120^\circ - y$$

$$\Rightarrow \cos x = \cos(120^\circ - y) =$$

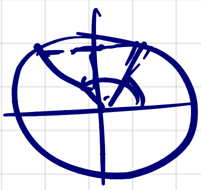
$$= \cos 120^\circ \cdot \cos y + \operatorname{sen} 120^\circ \cdot \operatorname{sen} y$$

• $\cos 120^\circ = -\cos(180^\circ - 120^\circ) = -\cos 60^\circ = -\frac{1}{2}$

Reduzir ao $1^\circ q$



• $\operatorname{sen} 120^\circ = \operatorname{sen}(180^\circ - 120^\circ) = \operatorname{sen} 60^\circ = \frac{\sqrt{3}}{2}$



Assim, temos:

$$\cos x = -\frac{1}{2} \cdot \underbrace{\cos y}_{?} + \frac{\sqrt{3}}{2} \cdot \frac{1}{3}$$

$$\operatorname{sen}^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \operatorname{sen}^2 y$$

$$\cos y = +\sqrt{1 - \operatorname{sen}^2 y} \qquad (y \in 1.^{\circ} Q)$$

$$\cos y = +\sqrt{1 - \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Portanto,

$$\cos x = -\frac{1}{2}\left(\frac{2\sqrt{2}}{3}\right) + \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = -\frac{2\sqrt{2}}{6} + \frac{\sqrt{3}}{6}$$

$$= \frac{\sqrt{3} - 2\sqrt{2}}{6}$$