Fundação Universidade Federal de Pelotas
Departamento de Matemática e Estatística
Curso de Licenciatura em Matemática
Segunda Prova de Cálculo IV
Prof. Dr. Maurício Zahn

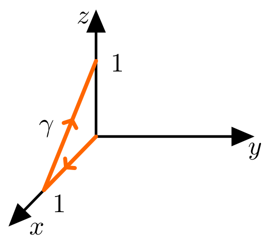
Nome: **Data:** 27/09/2023

Questão 01. [Peso 1,5] Use o Teorema de Green para calcular $\oint_\gamma \cos(x-3y)dx + \ln(x+y)dy$, onde γ é o quadrilátero $ABCD$ de vértices $A(\frac{7}{4}, \frac{1}{4})$, $B(\frac{9}{4}, -\frac{1}{4})$, $C(\frac{3}{2}, -\frac{1}{2})$ e $D(1, 0)$.

Questão 02. [Peso 1,5] Calcule o volume do sólido abaixo da semi-esfera $x^2+y^2+z^2=4$, $z \geq 0$, limitado pelo cilindro $x^2+y^2=1$ e pelo plano xy, usando integrais triplas.

Questão 03. Seja o campo vetorial $\vec{F}$ dado por $\vec{F}(x,y,z) = (e^x \cos y, e^x \operatorname{sen} y, z)$.

(a) [Peso 1,0] Calcule $\int_\gamma \vec{F} d\vec{r}$, onde γ é o caminho dado na ilustração abaixo.



(b) [Peso 1,0] Calcule a divergência e o rotacional do campo $\vec{F}$.

Questão 04. Seja $\vec{F} : \mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R}\} \to \mathbb{R}$ o campo vetorial dado por

$$\vec{F}(x,y) = \frac{2x}{y^3}\vec{i} + \frac{y^2-3x^2}{y^4}\vec{j}.$$

(a) [Peso 1,5] Mostre que $\vec{F}$ é um campo gradiente e obtenha uma função potencial φ para $\vec{F}$.

(b) [Peso 1,0] Conclua que $\int_\gamma \vec{F} d\vec{r}$ independe do caminho γ. Em seguida, calcule esta integral de linha do ponto $A(1,1)$ ao ponto $B(2,2)$.

Questão 05. [Peso 2,5] Use o Teorema de Green na forma vetorial para provar a *primeira identidade de Green*:

$$\iint_\Omega f \Delta g dA = \oint_\gamma f \nabla g\, \vec{n} ds - \iint_\Omega \nabla f \cdot \nabla g dA,$$

onde Ω e γ satisfazem as hipóteses do Teorema de Green e as derivadas parciais apropriadas de f e g existem e são contínuas em Ω.

Em seguida, supondo f harmônica em Ω, com $f(x) = 0$, $\forall x \in \gamma = \partial\Omega$, usando a primeira identidade de Green, conclua que

$$\iint_\Omega ||\nabla f||^2 dA = 0.$$

Questão 06. [Peso 1,0] Calcule a área formada pelo parabolóide $z = x^2+y^2$ abaixo do plano $z = \sqrt{2}$.

Questão 07. [Peso 1,0] Calcule a integral de superfície $\iint_S (x^2+y^2+z^2)dS$, onde S é o cone dado pela parametrização

$$\varphi(u,v) = (u\cos v, u \operatorname{sen} v, u), \quad 0 \leq u \leq 1 \ \text{ e } \ 0 \leq v \leq \frac{\pi}{2}.$$

GABARITO:

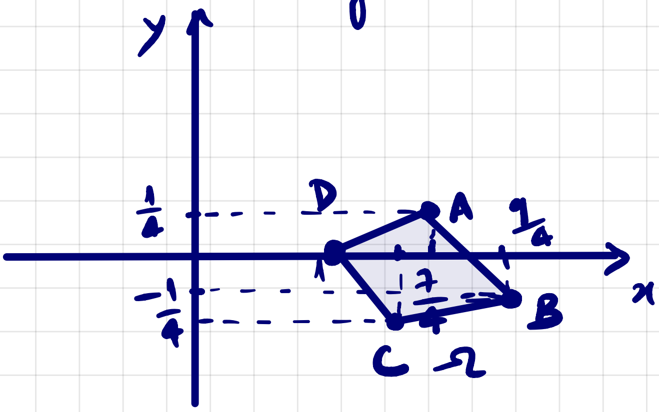
$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$P = \cos(x-3y) \Rightarrow \frac{\partial P}{\partial y} = -\operatorname{sen}(x-3y)\cdot(-3)$$

$$= +3\operatorname{sen}(x-3y)$$

$$Q = \ln(x+y) \Rightarrow \frac{\partial Q}{\partial x} = \frac{1}{x+y}$$

$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{1}{x+y} - 3\cdot\operatorname{sen}(x-3y)\right) dA$$



$$\xrightarrow{T}$$

$$T: \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$$

$T(x,y) = (u,v)$; onde

$$\begin{cases} u = x - 3y \\ v = x + y \end{cases}$$

$$u - v = -3y - y$$

$$u - v = -4y$$

$$\boxed{y = -\frac{1}{4}u + \frac{1}{4}v}$$

$$x = v - y$$

$$x = v - \left(-\frac{1}{4}u + \frac{1}{4}v\right)$$

$$\boxed{x = \frac{1}{4}u + \frac{3}{4}v}$$

obtenção da região Ω':

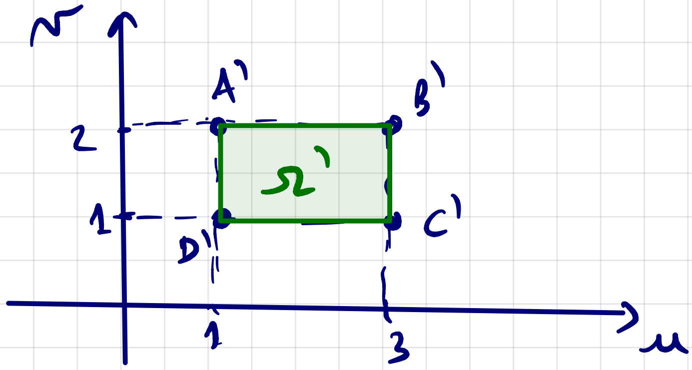
$$(x,y) \xrightarrow{T^{-1}} (u,v) = (x-3y,\ x+y)$$

$$A\left(\frac{7}{4}, \frac{1}{4}\right) \xrightarrow{T^{-1}} \left(\frac{7}{4} - \frac{3}{4},\ \frac{7}{4} + \frac{1}{4}\right) = (1,2) = A'$$

$$B\left(\frac{9}{4}, \frac{-1}{4}\right) \xrightarrow{T^{-1}} \left(\frac{9}{4} + \frac{3}{4},\ \frac{9}{4} - \frac{1}{4}\right) = (3,2) = B'$$

$$C\left(\frac{3}{2}, -\frac{1}{2}\right) \xrightarrow{T^{-1}} \left(\frac{3}{2} + \frac{3}{2},\ \frac{3}{2} - \frac{1}{2}\right) = (3,1) = C'$$

$$D(1,0) \xrightarrow{T^{-1}} (1-0,\ 1+0) = (1,1) = D'$$



Além disso:

$$\det\left(J(T)(u,v)\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$$

Disso, temos que:

$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{1}{x+y} - 3\,\mathrm{sen}(x-3y)\right) dA =$$

$$= \iint_{\Omega'} \left(\frac{1}{v} - 3 \cdot \mathrm{sen}\, u\right) \cdot \underbrace{\left|\det J(T)(u,v)\right|}_{\frac{1}{4}} \cdot du\,dv =$$

$$= \int_{v=1}^{v=2} \int_{u=1}^{u=3} \left(\frac{1}{v} - 3\,\mathrm{sen}\, u\right) \cdot \frac{1}{4} \cdot du\,dv =$$

$$\frac{1}{4}\int_{r=1}^{r=2}\left(\frac{1}{r}\cdot u+3\cdot\cos u\right)\Bigg|_{u=1}^{u=3} dr =$$

$$\frac{1}{4}\cdot\left[\int_1^2\left(\frac{3}{r}+3\cos 3-\frac{1}{r}-3\cos 1\right)dr\right]$$

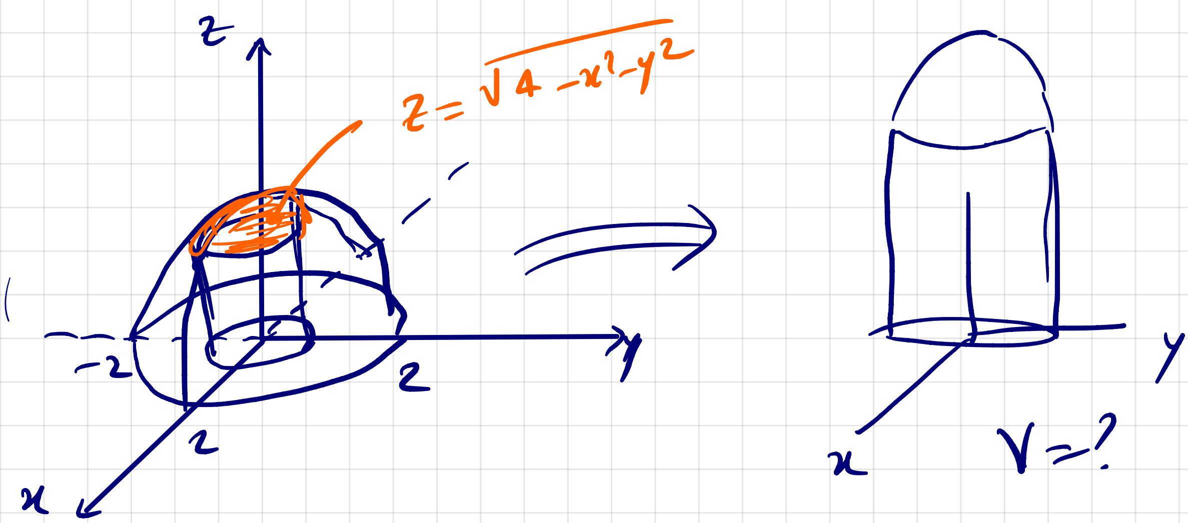
$$\frac{1}{4}\left[\int_1^2\frac{2}{r}\,dr+3(\cos 3-\cos 1)\cdot\int_1^2 dr\right]$$

$$=\frac{1}{4}\cdot 2\cdot\ln|r|\Big|_1^2+\frac{1}{4}\left(3(\cos 3-\cos 1)\cdot r\Big|_1^2\right)$$

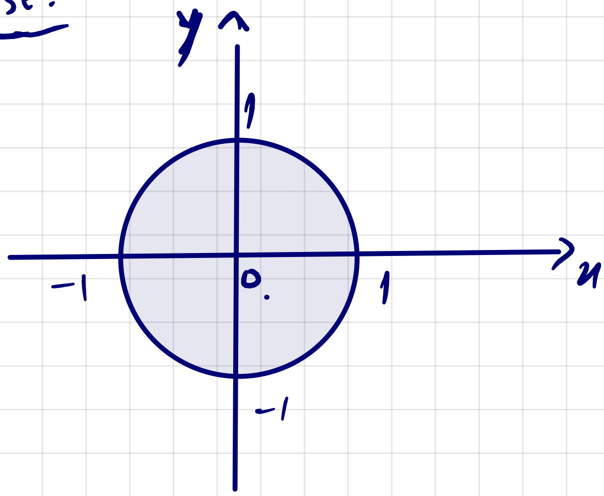
$$=\frac{1}{2}\Big[\ln(2)-\underbrace{\ln 1}_{=0}\Big]+\frac{3}{4}\cdot(\cos 3-\cos 1)\cdot(2-1)$$

$$\frac{1}{2}\ln 2+\frac{3}{4}(\cos 3-\cos 1)$$

02)



BASE:



Vamos usar coordenadas cilíndricas:

$$x = \rho\cos\theta$$
$$y = \rho\,\mathrm{sen}\,\theta$$
$$z = z$$
$$x^2+y^2=\rho^2$$
$$z=\sqrt{4-x^2-y^2}=\sqrt{4-\rho^2}$$

$$V = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=0}^{z=\sqrt{4-\rho^2}} 1\cdot\rho\cdot d\rho\cdot d\theta\, dz = \int_{\theta=0}^{\theta=2\pi} d\theta\cdot\int_{\rho=0}^{\rho=1}\rho\left(\int_{z=0}^{\rho=\sqrt{4-\rho^2}} dz\right)d\rho =$$

$$= \theta\Big|_0^{2\pi}\cdot\int_{\rho=0}^{\rho=1}\rho\cdot z\Big|_{z=0}^{z=\sqrt{4-\rho^2}} d\rho = 2\pi\cdot\int_{\rho=0}^{\rho=1}\rho\cdot\left(\sqrt{4-\rho^2}-0\right)d\rho$$

$$= 2\pi\cdot\left(-\frac{1}{2}\right)\int_0^1 (4-\rho^2)^{\frac{1}{2}}\cdot(-2\rho)\,d\rho = -\pi\cdot\frac{(4-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 =$$

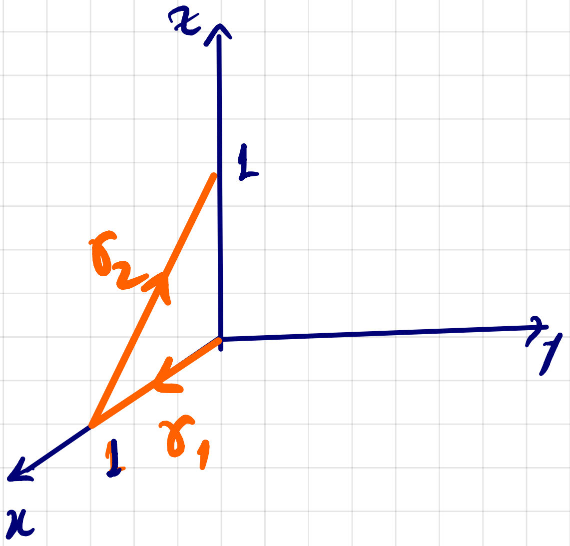
$$= -\pi\cdot\frac{2}{3}\left[(4-1)^{\frac{3}{2}}-(4-0)^{\frac{3}{2}}\right] =$$

$$= -\pi\cdot\frac{2}{3}\cdot\left[3^{\frac{3}{2}}-4^{\frac{3}{2}}\right] = -\frac{2\pi}{3}\left(\sqrt{3^3}-\sqrt{4^3}\right)$$

$$= -\frac{2\pi}{3}\left(3\sqrt{3}-4\cdot 2\right) = +\frac{2\pi}{3}\left(8-3\sqrt{3}\right)$$ unidades de volume

03) (a)

$$\oint_{\gamma} \vec{F}\, d\vec{r} = ?$$



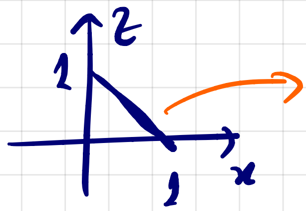
$$\oint_{\gamma} \vec{F}\, d\vec{r} = \oint_{\gamma_1} \vec{F}\cdot d\vec{r} + \oint_{\gamma_2} \vec{F}\cdot d\vec{r}$$

Sejam as parametrizações para γ_1 e γ_2:

γ_1: (no eixo Ox):

$$\gamma_1 : \begin{cases} x = t & \Rightarrow dx = dt \\ y = 0 & \Rightarrow dy = 0 \\ z = 0 & \Rightarrow dz = 0 \end{cases} \quad ; \ \gamma_1 : [0,1] \to \mathbb{R}^3$$

γ_2: (no plano xz : reta :



eq. da reta:

$z = -x + 1$

$$\gamma_2 : \begin{cases} x = t \Rightarrow dx = dt \\ y = 0 \Rightarrow dy = 0 \\ z = -t + 1 \Rightarrow dz = dt \end{cases} \quad ; \ \gamma_2 : [0,1] \to \mathbb{R}^3$$

$$\vec{F}(x,y,z) = (\underbrace{e^x \cos y}_{P}, \underbrace{e^x \operatorname{sen} y}_{Q}, \underbrace{z}_{R})$$

1º:

$$\oint_{\gamma_1} \vec{F}\, d\vec{r} = \int_0^1 P\,dx + Q\,dy + R\,dz =$$

$$= \int_0^1 (e^t \cdot \underbrace{\cos 0}_{1} \cdot dt + e^t \cdot \operatorname{sen} 0 \cdot 0 + 0 \cdot 0) = \int_0^1 e^t \cdot dt = e^t \Big|_0^1$$

$$= e^1 - e^0 = \underline{e - 1}$$

2º: $\oint_{\gamma_2} \vec{F}\, d\vec{r} = \int_0^1 P\,dx + Q\,dy + R\,dz =$

$$\int_0^1 (e^t \cdot \underbrace{\cos 0}_{=1} \cdot dt + e^t \cdot \operatorname{sen} 0 \cdot 0 + (-t+1)\,dt) =$$

$$= \int_0^1 e^t dt + \int_0^1 (1-t)\,dt = e^t \Big|_0^1 - \frac{(1-t)^2}{2}\Big|_0^1$$

$$= e^1 - e^0 - \frac{1}{2}\left[\underbrace{(1-1)^2}_{=0} - (1-0)^2\right] = e - 1 - \frac{1}{2}(-1)$$

$$= e - 1 + \frac{1}{2} = \underline{e - \frac{1}{2}}$$

Portanto, obtemos:

$$\oint_{\gamma} \vec{F}\, d\vec{r} = \oint_{\gamma_1} \vec{F}\, d\vec{r} + \oint_{\gamma_2} \vec{F} \cdot d\vec{r} = e - 1 + e - \frac{1}{2} = \underline{2e - \frac{3}{2}}$$

(b) $\vec{F} = (P, Q, R) = (e^x \cos y, e^x \operatorname{sen} y, z)$

$$\underline{\operatorname{div} \vec{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(e^x \operatorname{sen} y) + \frac{\partial}{\partial z}(z)$$

$$= e^x \cdot \cos y + e^x \cdot \cos y + 1 = \underline{2 \cdot e^x \cos y + 1.}$$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y & e^x \cdot \text{sen}\, y & z \end{vmatrix}\begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^x\cos y & e^x\cdot\text{sen}\,y \end{matrix}$$

$- \quad - \quad - \quad -1 \quad + \quad +$

$$= 0\cdot\vec{i} + 0\vec{j} + e^x\text{sen}\,y\cdot\vec{k} - (-e^x\cdot\text{sen}\,y)\vec{k} - 0\vec{i} + 0\vec{j}$$

$$= (0, 0, 2e^x\cdot\text{sen}\,y)$$

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04) (a) $\vec{F} = (P, Q) = \left(\frac{2x}{y^3}, \frac{y^2-3x^2}{y^4}\right)$.

Note que

- $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy^{-3}) = -6xy^{-4} = -\frac{6x}{y^4}$;
- $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^4}\cdot(y^2-3x^2)\right) = 0 - \frac{1}{y^4}\cdot 6x = -\frac{6x}{y^4}$

Como $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, segue que $\vec{F}$ é um campo gradiente.

Seja $\varphi(x,y)$ tal que $\nabla\varphi = \vec{F}$.

Então; $\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = \left(\frac{2x}{y^3}, \frac{y^2-3x^2}{y^4}\right)$ $(*)$

$$\frac{\partial\varphi}{\partial x} = \frac{2x}{y^3} \Rightarrow \varphi(x,y) = \int \frac{2x}{y^3}\,dx$$

$$\Rightarrow \varphi(x,y) = \frac{\cancel{2}}{y^3} \cdot \frac{x^2}{\cancel{2}} + g(y)$$

$$\Rightarrow \varphi(x,y) = \frac{x^2}{y^3} + g(y) = x^2 y^{-3} + g(y)$$

Derivando em y, vem:

$$\frac{y^2 - 3x^2}{y^4} = \frac{\partial \varphi}{\partial y} = -3x^2 y^{-4} + g'(y)$$

por (*)

$$\Rightarrow \frac{y^2}{y^4} - \cancel{\frac{3x^2}{y^4}} = -\cancel{\frac{3x^2}{y^4}} + g'(y)$$

$$\Rightarrow g'(y) = y^{-2}$$

$$\Rightarrow g(y) = \int y^{-2} \cdot dy = \frac{y^{-1}}{-1} + C$$

$$\Rightarrow g(y) = -\frac{1}{y} + C$$

Portanto, obtemos

$$\boxed{\varphi(x,y) = \frac{x^2}{y^3} - \frac{1}{y} + C}$$ – FUNÇÃO POTENCIAL.

(b) Como $\vec{F}$ é um campo vetorial contínuo em

$\Omega = \mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R}\}$ e é um campo gradiente (i.e., conservativo), segue por teor. que $\vec{F}$ independe do caminho.

Além disso, sendo $A(1,1)$ e $B(2,2)$:

$$\int_\gamma \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A) = \varphi(2,2) - \varphi(1,1).$$

$\varphi(x,y) = \frac{x^2}{y^3} - \frac{1}{y} + C$, pelo item (a)

$$= \frac{2^2}{2^3} - \frac{1}{2} + \cancel{C} - \left(\underbrace{\frac{1^2}{1^3} - \frac{1}{1}}_{=0} + \cancel{C} \right)$$

$$= \frac{4}{8} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$$ //

05) Pelo T. de divergência:

$$\oint_\gamma \vec{F} \cdot \vec{n} \, ds = \iint_\Omega \operatorname{div} \vec{F} \, dA.$$

Como $\vec{F} = f \cdot \nabla g = f \cdot \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = \left(f \cdot \frac{\partial g}{\partial x}, f \cdot \frac{\partial g}{\partial y} \right)$

Logo, $\operatorname{div} \vec{F} = \operatorname{div} \left(f \cdot \frac{\partial g}{\partial x}, f \cdot \frac{\partial g}{\partial y} \right) =$

$$= \frac{\partial}{\partial x}\left(f \cdot \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y}\left(f \cdot \frac{\partial g}{\partial y} \right) =$$

$$= f\cdot \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + f\cdot\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}$$

$$= f\cdot\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + f\cdot\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} =$$

$$= f\cdot\underbrace{\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right)}_{\Delta g} + \underbrace{\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}}_{\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\cdot\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right)}$$

$$= f\cdot\Delta g + \nabla f\cdot\nabla g$$

$$\Rightarrow \operatorname{div} F = f\cdot\Delta g + \nabla f\cdot\nabla g$$

Assim, do T. da divergência:

$$\oint_\gamma \vec{F}\cdot\vec{n}\cdot ds = \iint_\Omega \operatorname{div}\vec{F}\cdot dA \quad ; \quad \vec{F} = f\cdot\nabla g$$

$$\oint_\gamma f\cdot\nabla g\cdot\vec{n}\, ds = \iint_\Omega (f\cdot\Delta g + \nabla f\cdot\nabla g)\, dA$$

$$\oint_\gamma f\cdot\nabla g\,\vec{n}\cdot ds = \iint_\Omega f\cdot\Delta g\, dA + \iint_\Omega \nabla f\cdot\nabla g\cdot dA$$

$$\Rightarrow \iint_\Omega f\cdot\Delta g\cdot dA = \oint_\gamma f\cdot\nabla g\cdot\vec{n}\, ds - \iint_\Omega \nabla f\cdot\nabla g\cdot dA \qquad (*)$$

Em seguida, suponto $\Delta f = 0$ (i.e., f harmônica) e $f \equiv 0$ em $\partial\Omega$, por $(*)$, vem; tomando $f = g$:

$$\iint_\Omega f \cdot \underbrace{\Delta f}_{0} \cdot dA = \oint_\gamma \underbrace{f}_{0 \text{ em } \gamma} \cdot \nabla f \cdot \vec{n}\, ds - \iint_\Omega \nabla f \cdot \nabla f \cdot dA$$

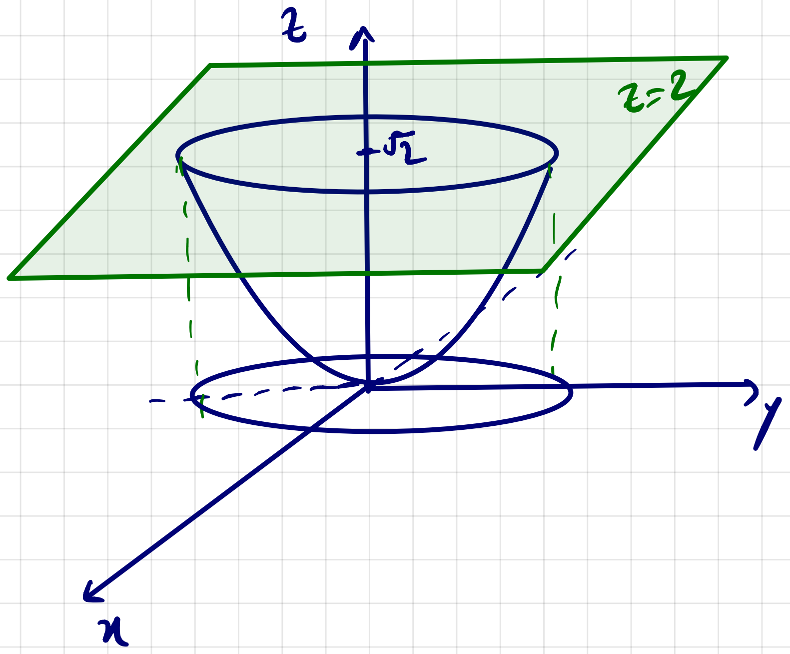
$$0 = 0 - \iint_\Omega \nabla f \cdot \nabla f \cdot dA$$

; e como $\nabla f \cdot \nabla f = \|\nabla f\|^2$,

vem:

$$\boxed{\iint_\Omega \|\nabla f\|^2 dA = 0}$$

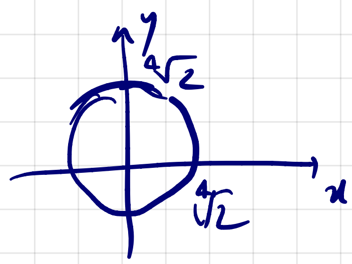
06)



$z = \sqrt{2}$

$\Updownarrow$

$x^2 + y^2 = \sqrt{2}$



Seja $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ a parametrização da superfície

$$\varphi(x,y) = (x,\ y,\ x^2+y^2),$$

onde $\Omega = \{(x,y) \in \mathbb{R}^2 :\ x^2+y^2 \leq \sqrt{2}\}$.

Então,

$$\varphi_x = \left(\frac{\partial}{\partial x}(x),\ \frac{\partial}{\partial x}(y),\ \frac{\partial}{\partial x}(x^2+y^2)\right) = (1,\ 0,\ 2x)$$

$$\varphi_y = \left(\frac{\partial}{\partial y}(x),\ \frac{\partial}{\partial y}(y),\ \frac{\partial}{\partial y}(x^2+y^2)\right) = (0,\ 1,\ 2y)$$

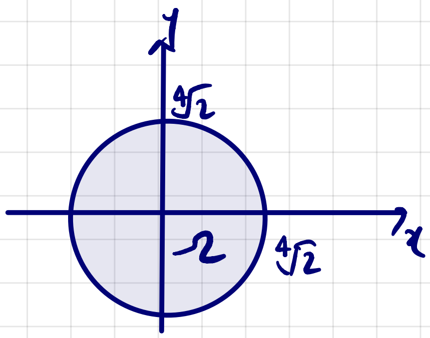
$$\varphi_x \times \varphi_y = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 0 & 2x & 1 & 0 \\ 0 & 1 & 2y & 0 & 1 \end{array}\right.$$

$$= 0\cdot\vec{i} + 0\vec{j} + 1\vec{k} - 0\vec{k} - 2x\cdot\vec{i} - 2y\vec{j}$$

$$= (-2x,\ -2y,\ 1)$$

$\Rightarrow \|\varphi_x \times \varphi_y\| = \sqrt{4x^2+4y^2+1}$. Assim, obtemos:

$$A(S) = \iint_{\Omega} \|\varphi_x \times \varphi_y\|\, dA = \iint_{\Omega} \sqrt{4(x^2+y^2)+1}\cdot dA =$$



$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt[4]{2}} \sqrt{4\rho^2+1}\cdot \rho\, d\rho\, d\theta =$$

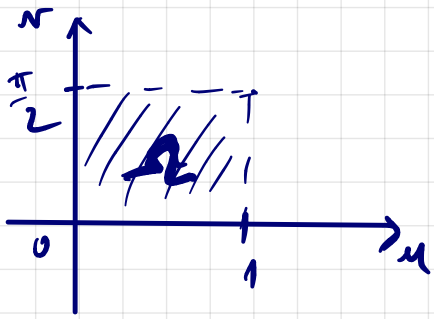
$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \frac{1}{8} \int_{\rho=0}^{\rho=\sqrt[4]{2}} (1+4\rho^2)^{\frac{1}{2}}\, 8\rho\, d\rho = \frac{2\pi}{8} \cdot \left.\frac{(1+4\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^{\sqrt[4]{2}}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} \cdot \left[\left(1 + 4\sqrt[4]{2}\right)^{\frac{3}{2}} - 1 \right]$$

$$= \frac{\pi}{6} \left(\left(1 + 4\sqrt[4]{2}\right)^{\frac{3}{2}} - 1 \right) \text{ unidades de área.}$$

07) $\displaystyle\iint_S (x^2 + y^2 + z^2)\, dS =$

$$= \iint_\Omega \left[(u\cos v)^2 + (u \operatorname{sen} v)^2 + u^2 \right] \cdot \| \varphi_u \times \varphi_v \| \, du\, dv$$



$$\varphi_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$= (\cos v, \operatorname{sen} v, 1) \; ;$$

$$\varphi_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

$$= (-u \operatorname{sen} v, u\cos v, 0)$$

$$\Rightarrow \varphi_u \times \varphi_v = \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \cos v & \operatorname{sen} v & 1 & \cos v & \operatorname{sen} v \\ -u \operatorname{sen} v & u\cos v & 0 & -u\operatorname{sen} v & u\cos v \end{array} \right.$$

$$- \qquad - \qquad - \qquad + \qquad + \qquad +$$

$$= 0.\vec{i} - u \operatorname{sen} v\, \vec{j} + u\cos^2 v\, \vec{k} + u \operatorname{sen}^2 v\, \vec{k} - u\cos v\, \vec{i} - 0\vec{j}$$

$$= (-u\cos v, -u \operatorname{sen} v, u)$$

$$\Rightarrow \|\varphi_u \times \varphi_v\| = \sqrt{u^2\cos^2 v + u^2 \operatorname{sen}^2 v + u^2}$$

$$= \sqrt{u^2 \underbrace{(\cos^2 v + \operatorname{sen}^2 v)}_{=1} + u^2} = u\sqrt{2}$$

Assim:

$$\iint_S (x^2+y^2+z^2)\, dS = \iint_\Omega (u^2\cos^2 v + u^2\operatorname{sen}^2 v + u^2) \cdot u\sqrt{2}\, du\, dv$$

$$= \iint_\Omega 2u^2 \cdot u\sqrt{2}\, du\, dv = 2\sqrt{2}\int_{v=0}^{v=\frac{\pi}{2}} dv \int_{u=0}^{u=1} u^3\, du$$

$$= \cancel{2}\sqrt{2} \cdot \frac{\pi}{\cancel{2}} \cdot \frac{u^4}{4}\Big|_0^1 = \sqrt{2}\pi \cdot \frac{1}{4} = \frac{\sqrt{2}\,\pi}{4}$$