Fundação Universidade Federal de Pelotas
Departamento de Matemática e Estatística
Curso de Lic. em Matemática
Primeira Prova de Cálculo IV
Prof. Dr. Maurício Zahn

Nome: GABARITO. **Data:** 28/07/2023

Questão 01. Sejam $A \subset \mathbb{R}^m$ um bloco do $\mathbb{R}^m$, $f : A \to \mathbb{R}$ integrável, e sejam

$$m = \inf\{f(x) : x \in A\} \text{ e } M = \sup\{f(x) : x \in A\}.$$

(a) [1,5 pt] Mostre que

$$m \leq \frac{\int_A f(x)dx}{\mathrm{Vol}(A)} \leq M.$$

(b) [1,0 pt] Encontre um intervalo fechado que contenha o valor da integral $\displaystyle\int_A (x+y)e^{yz}dxdydz$, onde A é o bloco $[1,3] \times [0,2] \times [1,4]$.

Questão 02. [1,5 pt] Calcule $\displaystyle\iint_\Omega \frac{x}{x^2+y^2}dA$, onde Ω é a região

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4,\ y \geq -x \text{ e } y \geq x\}.$$

Questão 03. [3,0 pt] Calcule a integral dupla

$$\iint_\Omega \sqrt{x+y}\, \ln(x-3y)dxdy,$$

onde Ω é o quadrilátero $ABCD$ de vértices $A(\frac{7}{4}, \frac{1}{4})$, $B(\frac{9}{4}, -\frac{1}{4})$, $C(\frac{3}{2}, -\frac{1}{2})$ e $D(1,0)$.

Questão 04. [1,0 pt cada] Calcule as integrais abaixo:

(a) $\displaystyle\int_0^1 \int_y^1 \frac{\operatorname{sen} x}{x}dxdy$ (b) $\displaystyle\int_0^1 \int_0^z \int_0^y ze^{-y^2}dxdydz$ (c) $\displaystyle\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2}dxdy$

Questão 05. [1,0 pt] Calcule o volume do sólido abaixo do plano $x + 2y - z = 0$ e acima da região limitada por $y = x$ e $y = x^4$.

01) (a) $\forall x \in A$, temos

$$m \leq f(x) \leq M.$$

Integrando no bloco A, teremos:

$$\int_A m\,dx \leq \int_A f(x)\,dx \leq \int_A M.\,dx$$

$$\Rightarrow m.\underbrace{\int_A dx}_{Vol(A)} \leq \int_A f(x)\,dx \leq M.\underbrace{\int_A dx}_{Vol(A)}$$

$$\Rightarrow m.Vol(A) \leq \int_A f \leq M.Vol(A)$$

1,5

$$\Rightarrow m \leq \frac{\int_A f}{Vol(A)} \leq M.$$

(b) Pelo item (a);

$$m.Vol(A) \leq \int_A f(x,y,z)\,dx\,dy\,dz \leq M.Vol(A);$$

$$A = [1,3]\times[0,2]\times[1,4] \quad ; \quad f(x,y,z) = (x+y).e^{yz}$$

$$\Rightarrow Vol\,A = (3-1).(2-0).(4-1)$$
$$= 2.2.3 = 12.$$

Note que

$$m = \inf_A f(x,y,z) = \min_A f(x,y,z)$$
$$M = \sup_A f(x,y,z) = \max_A f(x,y,z)$$

pois f é contínua no bloco A, que é compacto do $\mathbb{R}^3$.

Como $x+y$ e e^{yz} são crescentes,

o mínimo em A vai ocorrer com $\begin{cases} x = 1 \\ y = 0 \\ z = 1 \end{cases}$ 1,0

e o máximo em A com $\begin{cases} x = 3 \\ y = 2 \\ z = 4 \end{cases}$. Assim:

$$m = (x+y) \cdot e^{yz} \Big|_{x=1;\, y=0;\, z=1} = (1+0) \cdot e^{0 \cdot 1} = 1 \, ;$$

e

$$M = (x+y) \cdot e^{yz} \Big|_{x=3;\, y=2;\, z=4} = (3+2) \cdot e^{2 \cdot 4} = 5 \cdot e^{8}$$

Então,

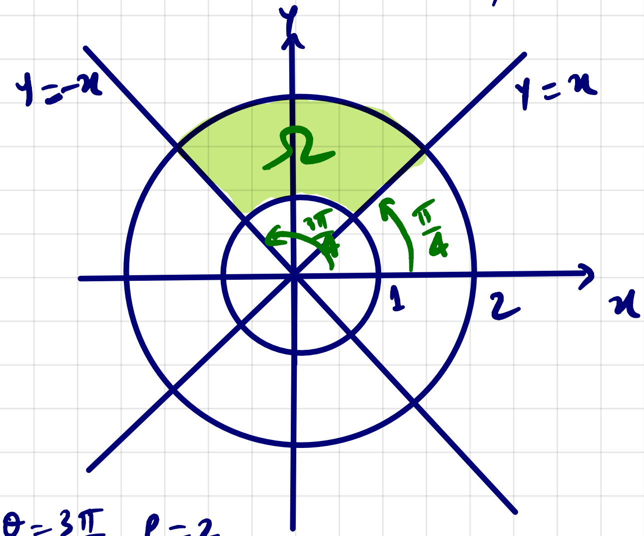
$$m \cdot \mathrm{Vol}\, A \leq \int_A (x+y) \cdot e^{yz} \, dx\, dy\, dz \leq M \cdot \mathrm{Vol}\, A$$

$$\Rightarrow \quad 1 \cdot 12 \leq \int_A (x+y) \cdot e^{yz} \, dx\, dy\, dz \leq 5 \cdot e^{8} \cdot 12$$

$$\Rightarrow \quad \boxed{12 \leq \int_A (x+y) \cdot e^{yz} \, dx\, dy\, dz \leq 60 \cdot e^{8}}$$

02) $\displaystyle\int_{\Omega} \frac{x}{x^2+y^2}\, dA$

$\Omega = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4 ;\ y \geq -x ;\ y \geq x\}$

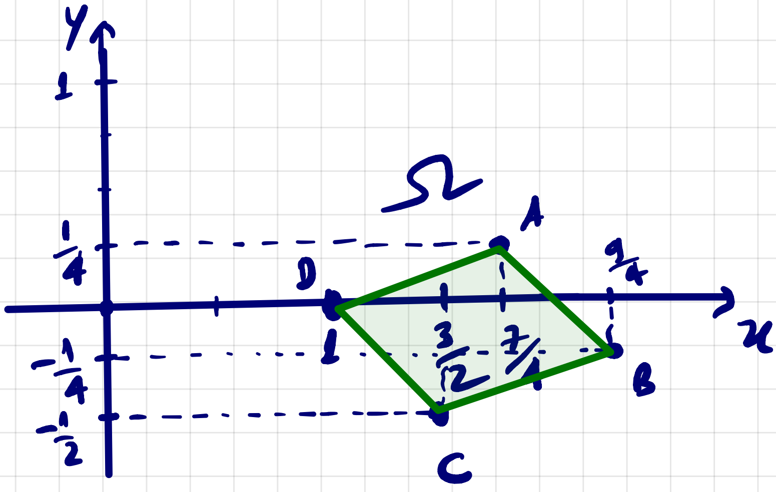


$$\int_{\Omega} \frac{x\, dA}{x^2+y^2} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{3\pi}{4}} \int_{\rho=1}^{\rho=2} \frac{\cancel{\rho}\cos\theta}{\cancel{\rho^2}} \cdot \cancel{\rho} \cdot d\rho\, d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{3\pi}{4}} \cos\theta \cdot d\theta \cdot \int_{\rho=1}^{\rho=2} d\rho = \left. \operatorname{sen}\theta \right|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \left. \rho \right|_{1}^{2} =$$

$$= \left(\operatorname{sen}\frac{3\pi}{4} - \operatorname{sen}\frac{\pi}{4}\right) \cdot (2-1) = \underbrace{\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)}_{=0} \cdot 1 = 0$$

03) $\displaystyle\iint_{\Omega} \sqrt{x+y}\cdot \ln(x-3y)\,dx\,dy$



Escreva $\begin{cases} u = x+y \rightsquigarrow x = u - y \\ v = x-3y \end{cases}$

$v = u - y - 3y$

$v = u - 4y$

$\Rightarrow \boxed{y = \frac{1}{4}u - \frac{1}{4}v}$

Assim:

$x = u - y$

$x = u - \left(\frac{1}{4}u - \frac{1}{4}v\right)$

$\boxed{x = \frac{3}{4}u + \frac{1}{4}v}$

E disso, segue que

$$T(u,v) = (x,y) = \left(\frac{3}{4}u + \frac{1}{4}v,\ \frac{1}{4}u - \frac{1}{4}v\right)$$

$$\det J(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{3}{16} - \frac{1}{16} = -\frac{1}{4}$$

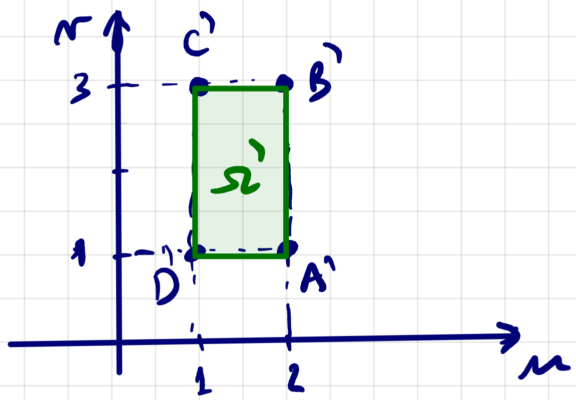
$\quad\ x \quad y \qquad\qquad\qquad u = x+y\ ;\ v = x-3y$

$A\left(\frac{7}{4}, \frac{1}{4}\right) \rightsquigarrow A'\left(\frac{7}{4}+\frac{1}{4}, \frac{7}{4}-\frac{3}{4}\right) = A'(2,1)$

$B\left(\frac{9}{4}, -\frac{1}{4}\right) \rightsquigarrow B'\left(\frac{9}{4}-\frac{1}{4}, \frac{9}{4}+\frac{3}{4}\right) = B'(2,3)$

$C\left(\frac{3}{2}, -\frac{1}{2}\right) \rightsquigarrow C'\left(\frac{3}{2}-\frac{1}{2}, \frac{3}{2}+\frac{3}{2}\right) = C'(1,3)$

$D(1,0) \rightsquigarrow D'(1+0, 1-0) = D'(1,1)$



Logo, teremos, pela mudança de variáveis:

$$\iint_{\Omega} \sqrt{x+y}\,\ln(x-3y)\,dx\,dy = \iint_{\Omega'} \sqrt{u}\cdot \ln v \cdot \underbrace{|\det J(T)(u,v)|}_{=\left|-\frac{1}{4}\right|}\,du\,dv$$

$$= \int_{v=1}^{v=3}\int_{u=1}^{u=2} u^{\frac{1}{2}}\cdot \ln v \cdot \frac{1}{4}\cdot du\,dv =$$

$$= \frac{1}{4}\int_{v=1}^{v=3} \ln v \cdot dv \cdot \int_{u=1}^{u=2} u^{\frac{1}{2}}\cdot du \;=$$

$$\int u^{\frac{1}{2}}\,du = \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2u\sqrt{u}}{3} + C$$

$$\int \ln v\, dv = \int U\cdot dV = U\cdot V - \int V\cdot dU$$

$$\begin{cases} U = \ln v \Rightarrow dU = \dfrac{dv}{v} \\ dV = dv \Rightarrow V = v \end{cases}$$

$$\Rightarrow \int \ln v\, dv = v\cdot \ln v - \int \cancel{v}\cdot \frac{dv}{\cancel{v}}$$

$$= v\cdot \ln v - v + C$$

$$= \frac{1}{4} \cdot \left(r \cdot \ln r - r\right)\Big|_{r=1}^{r=3} \cdot \left(\frac{2u\sqrt{u}}{3}\right)\Big|_{u=1}^{u=2}$$

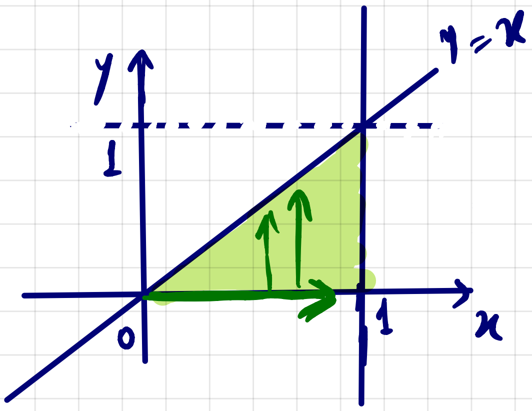
$$= \frac{1}{4} \cdot \left(3\ln 3 - 3 - [1 \cdot \underbrace{\ln 1}_{0} - 1]\right) \cdot \left(\frac{4\sqrt{2}}{3} - \frac{2}{3}\right)$$

$$= \frac{1}{\cancel{4}} \cdot \left(3\ln 3 - 3 + 1\right) \cdot \left(\frac{\cancel{2}(2\sqrt{2}-1)}{3}\right) = \frac{(3\ln 3 - 2)(2\sqrt{2}-1)}{6}$$

---

04)

a) $$\int_{y=0}^{y=1} \int_{x=y}^{x=1} \frac{\operatorname{sen} x}{x}\, dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\operatorname{sen} x}{x}\, dy\, dx =$$



$$= \int_{x=0}^{x=1} \frac{\operatorname{sen} x}{x} \left(\int_{y=0}^{y=x} dy\right) dx =$$

$$= \int_{x=0}^{x=1} \frac{\operatorname{sen} x}{x} \cdot y\Big|_{y=0}^{y=x} dx =$$

$$= \int_0^1 \frac{\operatorname{sen} x}{\cancel{x}} \cdot \cancel{x}\, dx = \int_0^1 \operatorname{sen} x\, dx =$$

$$= (-\cos x)\Big|_0^1 = -\cos 1 + \underbrace{\cos 0}_{=1} = 1 - \cos 1$$

b) $\displaystyle\int_{z=0}^{z=1}\int_{y=0}^{y=z}\left(\int_{x=0}^{x=y} z\cdot e^{-y^2}dx\right)dy\,dz = \int_{z=0}^{z=1} z\int_{y=0}^{y=z} e^{-y^2}\left(\int_{x=0}^{x=y}dx\right)dy\,dz =$

$\displaystyle= \int_{z=0}^{z=1} z \int_{y=0}^{y=z} e^{-y^2}\cdot x\Big|_0^y dy\,dz = -\frac{1}{2}\int_{z=0}^{z=1} z\left(\int_{y=0}^{y=z} e^{-y^2}\cdot(-2y)\,dy\right)dz =$

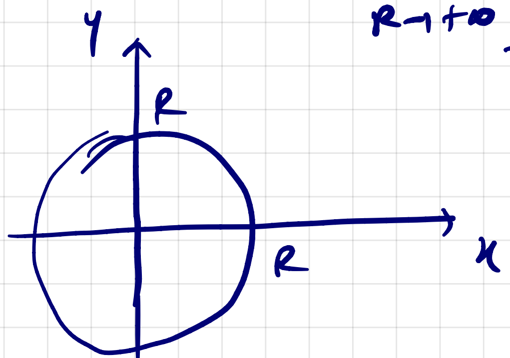
$\displaystyle= -\frac{1}{2}\int_{z=0}^{z=1} z\cdot e^{-y^2}\Big|_{y=0}^{y=z} dz = -\frac{1}{2}\int_{z=0}^{z=1} z\cdot\left(e^{-z^2}-e^{0}\right)dz =$

$\displaystyle= -\frac{1}{2}\int_0^1 z\cdot\left(e^{-z^2}-1\right)dz = -\frac{1}{2}\left(-\frac{1}{2}\right)\int_0^1 e^{-z^2}\cdot(-2z)\,dz + \frac{1}{2}\int_0^1 z\,dz$

$\displaystyle= \frac{1}{4}\, e^{-z^2}\Big|_0^1 + \frac{1}{2}\,\frac{z^2}{2}\Big|_0^1 = \frac{1}{4}\cdot\left(e^{-1}-e^{0}\right)+\frac{1}{4}\left(1^2-0^2\right)$

$\displaystyle= \frac{1}{4}\cdot\left(\frac{1}{e}\cancel{-1}\cancel{+1}\right) = \frac{1}{4e}$ //

c) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}dx\,dy = \lim_{R\to+\infty}\int_{-R}^{R}\int_{-R}^{R} e^{-x^2-y^2}dx\,dy$



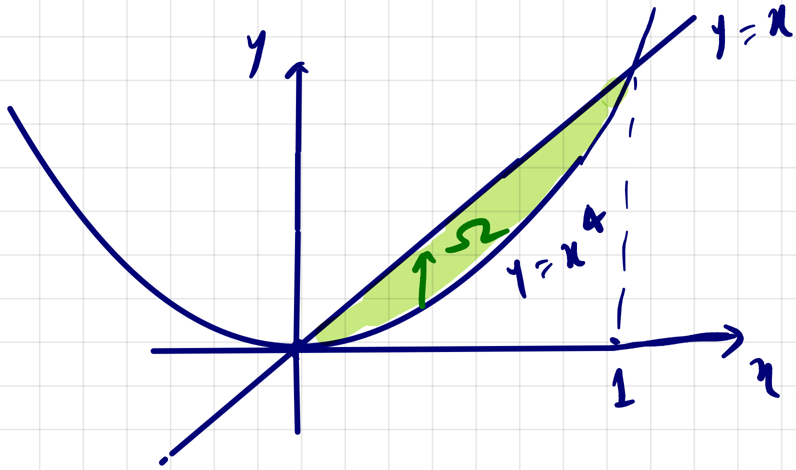
$$= \lim_{R\to+\infty} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=R} e^{-\rho^2} \cdot \rho \cdot d\rho\, d\theta = \lim_{R\to+\infty} \int_{\theta=0}^{\theta=2\pi} -\frac{1}{2} \int_{\rho=0}^{\rho=R} e^{-\rho^2} (-2\rho)\, d\rho\, d\theta =$$

$$= -\frac{1}{2} \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \lim_{R\to+\infty} e^{-\rho^2} \Big|_{\rho=0}^{\rho=R} = -\frac{1}{2}\, \theta \Big|_0^{2\pi} \cdot \lim_{R\to+\infty} \left(e^{-R^2} - e^0 \right)$$

$$= -\frac{1}{2} (2\pi - 0) \cdot \lim_{R\to+\infty} \left(\underbrace{\frac{1}{e^{R^2}}}_{\to 0} - 1 \right) = -\pi \cdot (-1) = \pi$$

---

05)



$x = x^4 \Leftrightarrow x = 0$
$x = 1$

Sendo $z = f(x,y) = x + 2y$, teremos

$$V = \int_{x=0}^{x=1} \int_{y=x^4}^{y=x} f(x,y)\, dy\, dx = \int_{x=0}^{x=1} \int_{y=x^4}^{y=x} (x + 2y)\, dy\, dx$$

$$= \int_{x=0}^{x=1} (xy + y^2) \Big|_{y=x^4}^{y=x} dx = \int_0^1 (x^2 + x^2) - [x \cdot x^4 + (x^4)^2] dx$$

$$= \int_0^1 (2x^2 - x^5 - x^8) dx = \left(2\frac{x^3}{3} - \frac{x^6}{6} - \frac{x^9}{9}\right)\Bigg|_0^1 =$$

$$= \frac{2}{3} - \frac{1}{6} - \frac{1}{9} - 0 = \frac{12 - 3 - 2}{18} = \frac{7}{18}$$

Obs.: Também poderíamos ter iniciado esta resolução do seguinte modo:

$$V = \iiint_{\Omega} dV = \int_{x=0}^{x=1} \int_{y=x^4}^{y=x} \int_{z=0}^{z=x+2y} dz\, dy\, dx =$$

$$= \int_{x=0}^{x=1} \int_{y=x^4}^{y=x} z \Big|_0^{x+2y} dy\, dx =$$

$$\int_{x=0}^{x=1} \int_{y=x^4}^{y=x} (x+2y) dy\, dx = \cdots$$ segue igual à primeira parte.