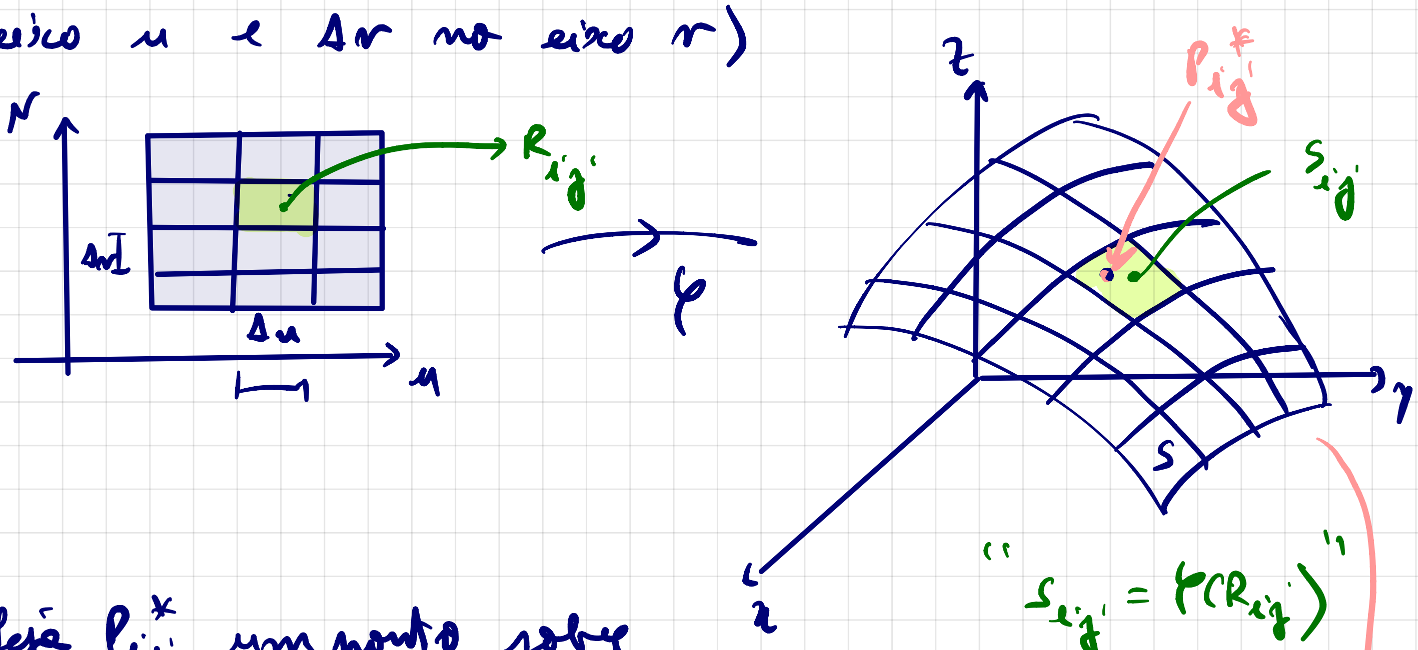
15/09/23 – AULA 25

INTEGRAIS DE SUPERFÍCIE:

Seja S a superfície do $\mathbb{R}^3$, parametrizada por $\varphi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$,

$$\varphi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

Para simplificar a introdução seja Ω um retângulo e tome P uma partição do mesmo. (regular, de comprimento de subintervalos Δu no eixo u e Δv no eixo v)



Seja P_{ij}^* um ponto sobre o retalho S_{ij}.

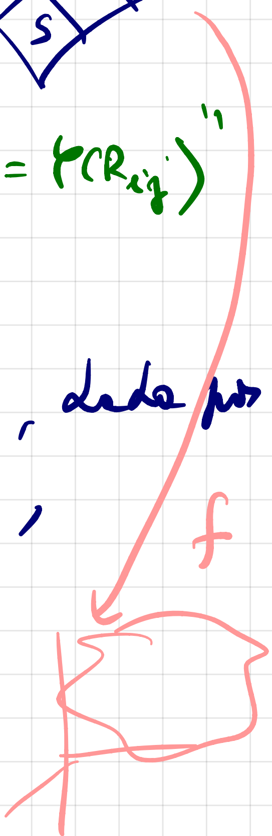
A área do retalho S_{ij} será ΔS_{ij}, dada por

$$\Delta S_{ij} \cong \|\varphi_u \times \varphi_v\| \cdot \Delta u \cdot \Delta v,$$

onde

$$\varphi_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\varphi_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$



Calcule $f(P_{ij}^*)$, construímos a soma de Riemann:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) . \Delta S_{ij}.$$

$[f : D \subset \mathbb{R}^3 \to \mathbb{R}]$

$S \subset D.$

Tomando o limite com $m, n \to \infty$, vamos obter:

$$\boxed{\iint_S f(x,y,z)\, dS := \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) . \Delta S_{ij}.}$$

INTEGRAL DE SUPERFÍCIE

Sendo $\Delta S_{ij} = \| \varphi_u \times \varphi_v \| . \Delta u \Delta v$; então:

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_\Omega f(\varphi(u,v)) . \| \varphi_u \times \varphi_v \| \, du\, dv}$$

Quando $z = g(x,y)$, então usamos a parametrização

$$\varphi(x,y) = (x, y, g(x,y))$$

Disso:

$$\varphi_x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial g}{\partial x} \right) = \left(1, 0, \frac{\partial g}{\partial x} \right)$$

$$\varphi_y = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial g}{\partial y} \right) = \left(0, 1, \frac{\partial g}{\partial y} \right)$$

Logo: $\varphi_x \times \varphi_y = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ 1 & 0 \\ 0 & 1 \end{matrix}$

$$= 0\vec{i} + 0\vec{j} + \vec{k} - 0\vec{k} - \frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j}$$

$$= \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

$$\Longrightarrow \|\varphi_x \times \varphi_y\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}.$$

Portanto,

$$\iint_S f(x,y,z)\, dS = \iint_\Omega \underbrace{f(\varphi(x,y))}_{f(x,y,g(x,y))} \cdot \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, \underbrace{dA}_{dxdy}$$

Ex.: Calcule $\iint_S y\, dS$, onde S é a superfície

$z = x + y^2$; $0 \leq x \leq 1$ e $0 \leq y \leq 2$.

Solução: $f(x,y,z) = y$

$$\varphi(x,y) = (x, y, g(x,y)) = (x, y, x+y^2)$$

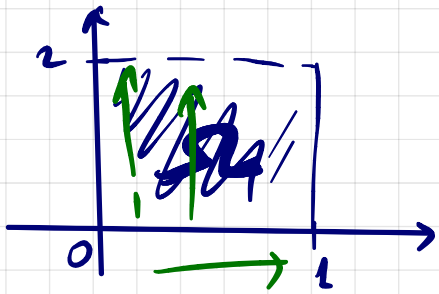
$$\Longrightarrow \boxed{f(\varphi(x,y)) = y}$$

$$g(x,y) = x + y^2$$

$$\frac{\partial g}{\partial x} = 1 \quad ; \quad \frac{\partial g}{\partial y} = 2y$$

Portanto:

$$\iint_S y \cdot dS = \iint_\Omega f(\varphi(x,y)) \cdot \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \cdot dA$$



$$= \iint_\Omega y \cdot \sqrt{1 + 1 + (2y)^2}\, dy\, dx$$

$$= \frac{1}{8} \int_{x=0}^{x=1} \left(\int_{y=0}^{y=2} 8 \cdot y \cdot (2 + 4y^2)^{\frac{1}{2}} dy \right) dx$$

$$\int v^k dv = \frac{v^{k+1}}{k+1} + C$$

$$v = 2 + 4y^2$$

$$dv = 8y\, dy$$

$$= \frac{1}{8} \int_{x=0}^{x=1} dx \cdot \left. \frac{(2+4y^2)^{\frac{3}{2}}}{\frac{3}{2}} \right|_{y=0}^{y=2}$$

$$= \frac{1}{8} \cdot x \Big|_0^1 \cdot \frac{2}{3} \left[(2+16)^{\frac{3}{2}} - 2^{\frac{3}{2}} \right] =$$

$$\frac{1}{\cancel{8}_{4}} \cdot \frac{\cancel{2}}{3} \cdot \left[\sqrt{18^3} - \sqrt{2^3} \right]$$

$$= \frac{1}{12} \cdot \left[\sqrt{18^2 \cdot 9 \cdot 2} - \sqrt{2^2 \cdot 2} \right]$$

$$= \frac{1}{12} \cdot \left[18 \cdot 3 \cdot \sqrt{2} - 2\sqrt{2} \right] = \frac{52\sqrt{2}}{12} = \frac{13\sqrt{2}}{3}$$

EXERCÍCIO: (ENTREGAR ATÉ SEGUNDA, DIA 18/09, POR E-MAIL):

Calcule a integral de superfície $\iint_S (x+y+z)\,dS$, onde S é o paralelogramo com equações paramétricas $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \leq u \leq 2$ e $0 \leq v \leq 1$.

$\boxed{\text{FIM DO CURSO}}$!

LISTA DE CONTEÚDO PARA ESTUDO POSTERIOR.

→ SUPERFÍCIES ORIENTADAS

→ TEOR. DE STOKES E DA DIVERGÊNCIA NO $\mathbb{R}^3$.

DICA: LIVRO DO STWEART, VOL II.

RESOLUÇÃO DE EXERCÍCIOS DAS LISTAS:

LISTA 06

11) b. $\varphi(u,v) = (u, v, v^2 - u^2)$

$x = u$

$y = v$

$z = v^2 - u^2$

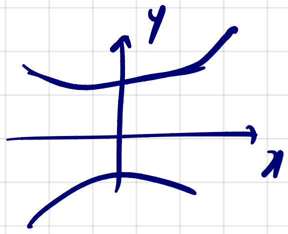
$\boxed{z = y^2 - x^2}$

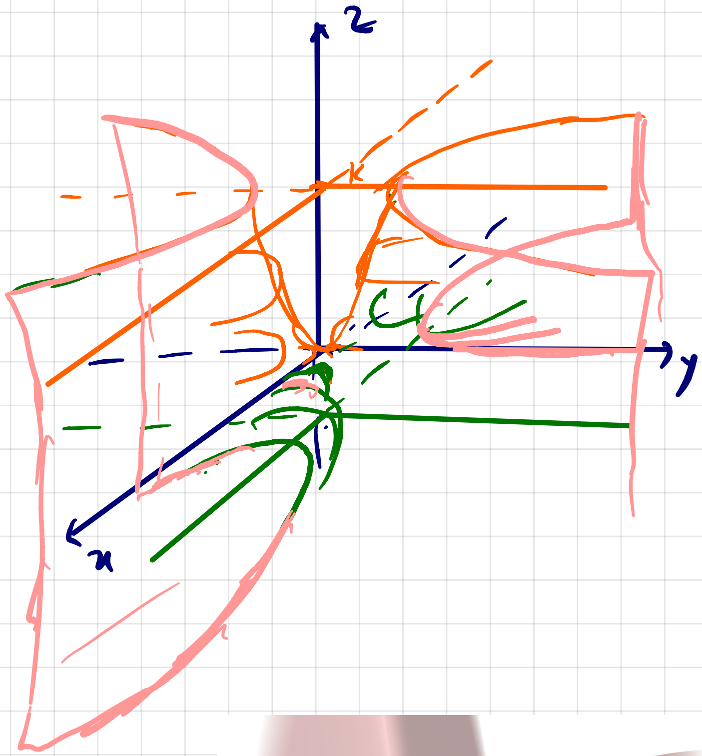
$z = k.$

$y^2 - x^2 = k$

1º: $k > 0$:

$\frac{y^2}{k} - \frac{x^2}{k} = 1.$



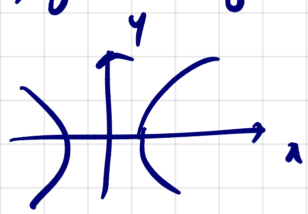


2º: $k < 0$: $(-k > 0)$

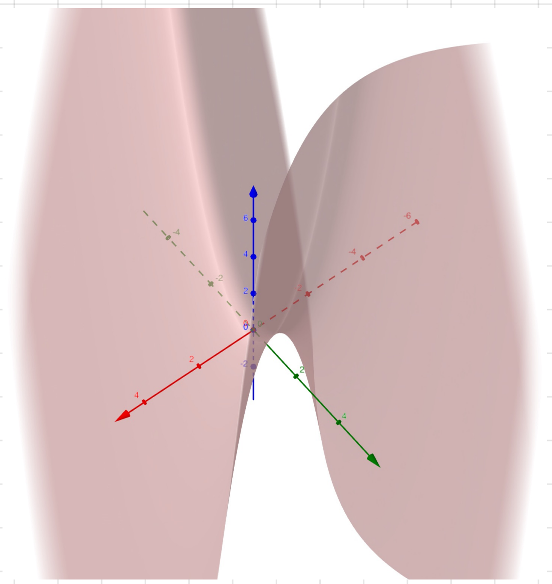
$y^2 - x^2 = k$

$-y^2 + x^2 = -k$

$\frac{x^2}{\underbrace{-k}_{>0}} - \frac{y^2}{\underbrace{-k}_{>0}} = 1$



HIPÉRBOLES.



(c) $\varphi(u,v) = \left(v\cdot\cos u,\ v\cdot \operatorname{sen} u,\ \frac{1}{v^2}\right);$

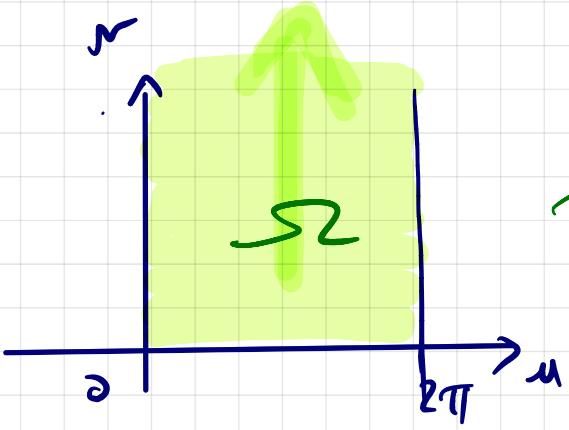
$0 < u \leq 2\pi \ ; \ v > 0.$

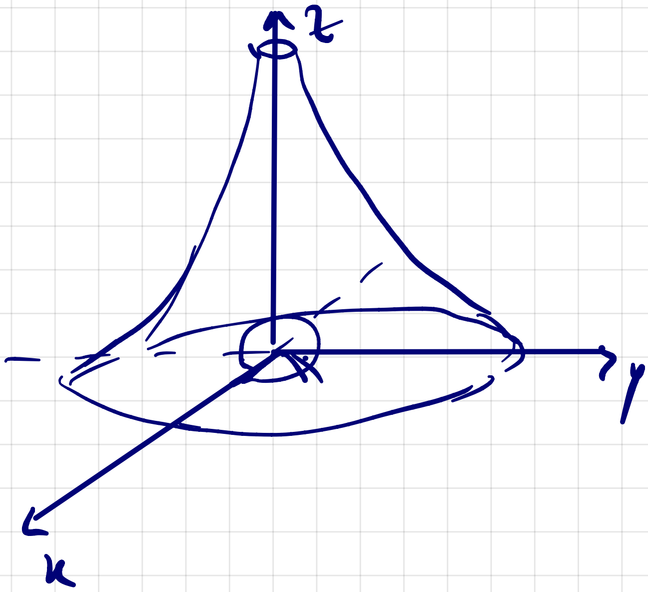
$$\left.\begin{aligned} x &= v\cos u \\ y &= v \operatorname{sen} u \end{aligned}\right\} \qquad \begin{aligned} x^2 &= v^2\cos^2 u \\ + \ y^2 &= v^2 \operatorname{sen}^2 u \end{aligned}$$

$$z = \frac{1}{v^2} \qquad x^2+y^2 = v^2 \underbrace{(\cos^2 u + \operatorname{sen}^2 u)}_{=1}$$

$$x^2+y^2 = v^2$$

$$\Rightarrow z = \frac{1}{v^2} = \frac{1}{x^2+y^2}$$





$z > 0$ (pois $v > 0$)

$x = v\cos u$

$y = v \operatorname{sen} u$

LISTA 06 |

03) $\vec{r} = (x, y, z) = (F_1, F_2, F_3)$

a) $\nabla \cdot \vec{r} = 3$:

$$\nabla \cdot \vec{r} = div(\vec{r})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

b) $\nabla \cdot (\|\vec{r}\| . \vec{r}) = 4 . \|\vec{r}\|$

$\nabla . (\|\vec{r}\| . \vec{r}) = div(\|\vec{r}\| . \vec{r})$; onde:

$$\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \|\vec{r}\| . \vec{r} = \sqrt{x^2+y^2+z^2} . (x, y, z)$$

$$= (\underbrace{x (x^2+y^2+z^2)^{\frac{1}{2}}}_{f_1}, \underbrace{y . (x^2+y^2+z^2)^{\frac{1}{2}}}_{f_2}, \underbrace{z . (x^2+y^2+z^2)^{\frac{1}{2}}}_{f_3})$$

$$\Rightarrow \operatorname{div}(\|\vec{r}\|\cdot\vec{r}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= x\cdot\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot\cancel{2}x + 1\cdot(x^2+y^2+z^2)^{\frac{1}{2}} +$$

$$+ y\cdot\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot\cancel{2}y + 1\cdot(x^2+y^2+z^2)^{\frac{1}{2}} +$$

$$+ z\cdot\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot\cancel{2}z + 1\cdot(x^2+y^2+z^2)^{\frac{1}{2}}$$

$$\frac{x^2}{\underbrace{\sqrt{x^2+y^2+z^2}}_{=\|\vec{r}\|}} + \underbrace{\sqrt{x^2+y^2+z^2}}_{\|\vec{r}\|} + \frac{y^2}{\underbrace{\sqrt{x^2+y^2+z^2}}_{\|\vec{r}\|}} +$$

$$+ \underbrace{\sqrt{x^2+y^2+z^2}}_{\|\vec{r}\|} + \frac{z^2}{\underbrace{\sqrt{x^2+y^2+z^2}}_{\|\vec{r}\|}} + \underbrace{\sqrt{x^2+y^2+z^2}}_{=\|\vec{r}\|}$$

$$= \frac{x^2}{\|\vec{r}\|} + \|\vec{r}\| + \frac{y^2}{\|\vec{r}\|} + \|\vec{r}\| + \frac{z^2}{\|\vec{r}\|} + \|\vec{r}\|$$

$$= \frac{1}{\|\vec{r}\|}\underbrace{(x^2+y^2+z^2)}_{=\|\vec{r}\|^2} + 3\cdot\|\vec{r}\|$$

$$= \frac{\|\vec{r}\|^{\cancel{2}}}{\cancel{\|\vec{r}\|}} + 3\cdot\|\vec{r}\| = 4\cdot\|\vec{r}\|.$$

Então: $\nabla\cdot(\|\vec{r}\|\cdot\vec{r}) = \operatorname{div}(\|\vec{r}\|\cdot\vec{r}) = 4\cdot\|\vec{r}\|.$