24/09/23 — AULA 27

LISTA 07.

02) $\varphi(u,v) = (\cos u, v, \operatorname{sen} u)$

$u^2 + 4v^2 \leq 1$.

$\dfrac{u^2}{1} + \dfrac{v^2}{\frac{1}{4}} \leq 1$ (região interna de uma elipse)

área de superfície S?

$$\boxed{A(S) = \iint_{\Omega} \|\varphi_u \times \varphi_v\| \, du\,dv}$$; onde

$$\varphi(u,v) = (\underbrace{x(u,v)}_{\cos u}, \underbrace{y(u,v)}_{v}, \underbrace{z(u,v)}_{\operatorname{sen} u})$$

$$\Rightarrow \varphi_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$= (-\operatorname{sen} u, 0, \cos u)$$

$$\Rightarrow \varphi_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

$$= (0, 1, 0)$$

Disso, temos:

$$\varphi_u \times \varphi_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\operatorname{sen} u & 0 & \cos u \\ 0 & 1 & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -\operatorname{sen} u & 0 \\ 0 & 1 \end{matrix}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$\varphi_u \times \varphi_v = 0\vec{i} + 0\vec{j} - \operatorname{sen} u \cdot \vec{k} - 0\vec{k} - \cos u\, \vec{i} + 0\vec{j}$$

$$= (-\cos u, 0, -\operatorname{sen} u)$$

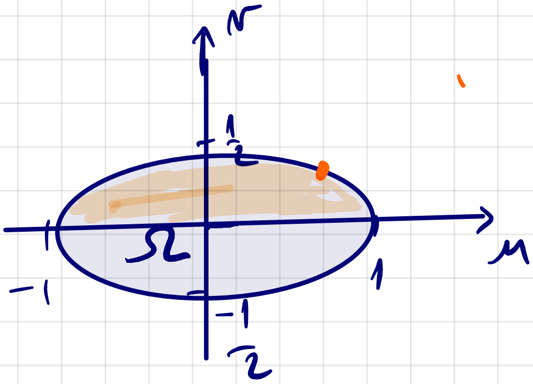
Então: $\|\varphi_u \times \varphi_v\| = \sqrt{(-\cos u)^2 + 0^2 + (-\operatorname{sen} u)^2}$

$$= \sqrt{\cos^2 u + \operatorname{sen}^2 u} = 1.$$

Logo, temos:

$$A(S) = \iint_\Omega \|\varphi_u \times \varphi_v\| \cdot du\,dv =$$

$$\iint_\Omega 1 \cdot du\,dv = 2 \cdot \int_{u=-1}^{u=1} \left(\int_{v=0}^{v=\frac{1}{2}\sqrt{1-u^2}} 1 \cdot dv \right) du =$$

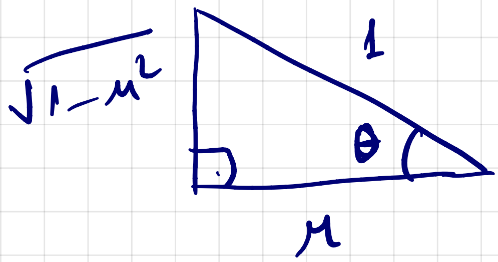


$$u^2 + 4v^2 = 1$$

$$\Leftrightarrow v = \frac{1}{2}\sqrt{1-u^2}$$

$$2\cdot\int_{u=-1}^{u=1} r\Big|_{r=0}^{r=\frac{1}{2}\sqrt{1-u^2}}\cdot du =$$

$$\cancel{2}\cdot\int_{u=-1}^{u=1}\frac{1}{\cancel{2}}\cdot\sqrt{1-u^2}\cdot du = \int_{-1}^{1}\sqrt{1-u^2}\,du =$$



$$\operatorname{sen}\theta = \frac{\sqrt{1-u^2}}{1}$$

$$\sqrt{1-u^2} = \operatorname{sen}\theta$$

$$\cos\theta = \frac{u}{1} \Rightarrow u = \cos\theta$$

$$du = -\operatorname{sen}\theta\,d\theta$$

$$\int\sqrt{1-u^2}\,du = \int \operatorname{sen}\theta\cdot(-\operatorname{sen}\theta\,d\theta) =$$

$$-\int \operatorname{sen}^2\theta\,d\theta = -\int\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= -\frac{1}{2}\int d\theta + \frac{1}{2}\cdot\frac{1}{2}\int \cos 2\theta\; 2\,d\theta$$

$$= -\frac{\theta}{2} + \frac{1}{4}(\operatorname{sen}2\theta) + C =$$

$u = \cos\theta$

$\theta = \arccos(u)$

$$= -\frac{1}{2}\operatorname{arc}(\cos u) + \frac{1}{4}\cdot 2\cdot\sqrt{1-u^2}\cdot u + C$$

$$\boxed{\operatorname{sen}2\theta = 2\cdot\operatorname{sen}\theta\cdot\cos\theta}$$

$$= -\frac{1}{2}\arccos(u) + \frac{1}{2}u\cdot\sqrt{1-u^2} + C$$

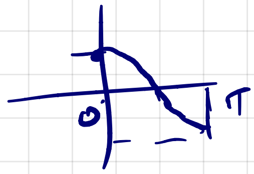
Assim, temos:

$$\int_{-1}^{1}\sqrt{1-u^2}\,du = \left(-\frac{1}{2}\arccos(u)+\frac{1}{2}u\cdot\sqrt{1-u^2}\right)\Bigg|_{-1}^{1}$$

$$= -\frac{1}{2}\arccos(1) + \underbrace{\frac{1}{2}\cdot 1\cdot\sqrt{1-1^2}}_{=0} + \frac{1}{2}\arccos(-1) - \underbrace{\frac{1}{2}(-1)\cdot\sqrt{1-1}}_{=0}$$

$$= -\frac{1}{2}\cdot\arccos(1) + \frac{1}{2}\arccos(-1)$$

$$= -\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot(\pi) = \frac{\pi}{2}$$



Resposta: $\boxed{A(S) = \frac{\pi}{2}}$.

LISTA 07

04) a área de $z = \sqrt{x^2+y^2}$ com

$x^2+y^2 \leq 2x$; $x \geq 1$; $y \geq 0$.

Tome $\varphi(x,y) = (x, y, \sqrt{x^2+y^2})$ $\longrightarrow$ $g(x,y) = \sqrt{x^2+y^2}$

Então

$$A(S) = \iint_{\Omega} \|\varphi_x \times \varphi_y\| \, dA =$$

$$= \iint_{\Omega} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \cdot dA$$

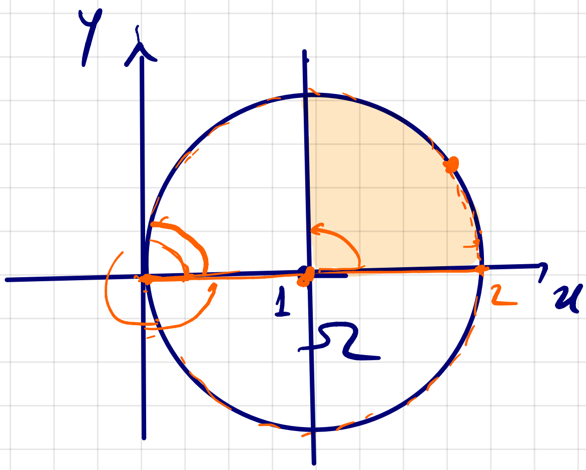
Região Ω:

$$x^2+y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 - 1 + y^2 = 0$$

$(x-1)^2 + y^2 = 1$ — circunferência de raio 1, centrada em $(1, 0)$



$0 \leq \theta \leq \frac{\pi}{2}$

$x = \rho \cos\theta$

$y = \rho \operatorname{sen}\theta$.

Obs.:

$\rho^2 = x^2 + y^2$

$x^2 + y^2 \leq 2x$

$\rho^{\cancel{2}} \leq 2\cancel{\rho}\cos\theta$

$\rho \leq 2\cos\theta$

$0 \leq \rho \leq 2\cos\theta$

Assim, teremos:

$$= \iint_{\Omega} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dA =$$

$$g(x,y) = (x^2+y^2)^{\frac{1}{2}} \Rightarrow \begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{1}{\cancel{2}} \cdot (x^2+y^2)^{-\frac{1}{2}} \cancel{2}x = \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{\partial g}{\partial y} = \dfrac{1}{\cancel{2}} \cdot (x^2+y^2)^{-\frac{1}{2}} \cancel{2}y = \dfrac{y}{\sqrt{x^2+y^2}} \end{cases}$$

$$= \iint_{\Omega} \sqrt{1 + \underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{=1}} \, dA =$$

$$= \iint_{\Omega} \sqrt{2} \, dA = \sqrt{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2\cos\theta} \rho \, d\rho \, d\theta =$$

coor. polares

$$= \sqrt{2} \cdot \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{\rho^2}{2} \Bigg|_{\rho=0}^{\rho=2\cos\theta} d\theta = \sqrt{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{\cancel{4}\cos^2\theta}{\cancel{2}} \cdot d\theta =$$

$$= 2\sqrt{2} \cdot \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \cancel{2}\sqrt{2} \cdot \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{\cancel{2}} \, d\theta =$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \sqrt{2} \cdot \int_0^{\frac{\pi}{2}} d\theta + \frac{\sqrt{2}}{2} \cdot \int_0^{\frac{\pi}{2}} \cos 2\theta \, (2 \, d\theta)$$

$$= \sqrt{2}\, \theta \Big|_0^{\frac{\pi}{2}} + \frac{\sqrt{2}}{2} \cdot \operatorname{sen} 2\theta \Big|_0^{\frac{\pi}{2}} =$$

$$= \sqrt{2} \cdot \frac{\pi}{2} + \frac{\sqrt{2}}{2} \cdot \Big(\underbrace{\operatorname{sen} \frac{\cancel{2}\pi}{\cancel{2}}}_{0} - \underbrace{\operatorname{sen} 0}_{0} \Big) = \frac{\sqrt{2}\,\pi}{2} + 0 = \frac{\sqrt{2}}{2}\pi$$

unidades de área

LISTA 07

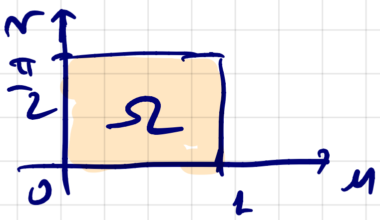
05) $\iint_S xyz \, dS$, onde S é o cone dado por:

$f(x,y,z)$

$$\begin{cases} x = u \cos v \\ y = u \cdot \operatorname{sen} v \\ z = u \end{cases}$$

, onde:

$0 \leq u \leq 1$ e $0 \leq v \leq \frac{\pi}{2}$.



$$\iint_S f(x,y,z) \, dS = \iint_\Omega f(\varphi(u,v)) \cdot \|\varphi_u \times \varphi_v\| \, du \, dv$$

onde:

$$\underbrace{f(\varphi(u,v))} = (x \cdot y \cdot z)\Big|_{\substack{x = u\cdot\cos v \\ y = u\cdot \operatorname{sen} v \\ z = u}} =$$

$$= u\cos v \cdot u \cdot \operatorname{sen} v \cdot u = \underbrace{u^3 \cdot \operatorname{sen} v \cdot \cos v}$$

$$\varphi_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad ; \quad \varphi_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

$$= (\cos v, \operatorname{sen} v, 1) \quad ; \quad \varphi_v = (-u\operatorname{sen} v, u\cdot\cos v, 0)$$

$$\Rightarrow \varphi_u \times \varphi_v = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \cos v & \operatorname{sen} v & 1 & \cos v & \operatorname{sen} v \\ -u\operatorname{sen} v & u\cos v & 0 & -u\operatorname{sen} v & u\cos v \end{array}\right.$$

$$= 0\cdot\vec{i} - u\cdot\operatorname{sen} v\,\vec{j} + u\cdot\cos^2 v\,\vec{k} + u\cdot\operatorname{sen}^2 v\,\vec{k} - u\cos v\,\vec{i} - 0\,\vec{j}$$

$$= (-u\cos v, -u\operatorname{sen} v, u\cdot\cos^2 v + u\operatorname{sen}^2 v)$$

$$= (-u\cos v, -u\operatorname{sen} v, u\,\underbrace{(\cos^2 v + \operatorname{sen}^2 v)}_{=1})$$

$$\Rightarrow \varphi_u \times \varphi_v = (-u\cos v, -u\operatorname{sen} v, u)$$

$$\Rightarrow \|\varphi_u \times \varphi_v\| = \sqrt{u^2\cos^2 v + u^2\operatorname{sen}^2 v + u^2}$$

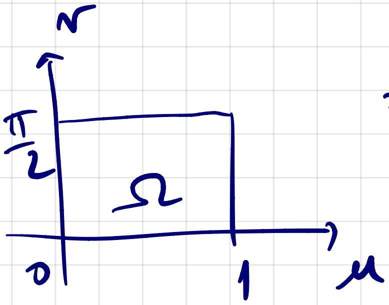
$$= \sqrt{u^2\underbrace{(\cos^2 v + \operatorname{sen}^2 v)}_{=1} + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$$

Portanto;

$$\iint_S xyz\, dS = \iint_\Omega f(\varphi(u,v)) \cdot \|\varphi_u \times \varphi_v\|\, dA$$

$$= \iint_\Omega u^3 \cdot \operatorname{sen} v \cdot \cos v \cdot \sqrt{2}\,u \cdot dA =$$

$$= \sqrt{2} \iint_\Omega u^4 \cdot \operatorname{sen} v \cdot \cos v \; du\, dv$$



$$= \sqrt{2} \cdot \int_{v=0}^{v=\frac{\pi}{2}} \left(\int_{u=0}^{u=1} u^4 \cdot \operatorname{sen} v \cos v \; du \right) dv$$

$$= \sqrt{2} \int_{v=0}^{v=\frac{\pi}{2}} \frac{u^5}{5} \Big|_0^1 \cdot \operatorname{sen} v \cos v \; dv$$

$$= \frac{\sqrt{2}}{5} \cdot \int_0^{\frac{\pi}{2}} (\underbrace{\operatorname{sen} v}_{w})^1 \underbrace{\cos v\, dv}_{dw} = \frac{\sqrt{2}}{5} \cdot \frac{\operatorname{sen}^2 v}{2} \Big|_0^{\frac{\pi}{2}} =$$

$$\int w^k\, dw = \frac{w^{k+1}}{k+1} + C$$

$$w = \operatorname{sen} v \Rightarrow dw = \cos v\, dv$$

$$= \frac{\sqrt{2}}{5} \cdot \frac{1}{2} \cdot \left(\underbrace{\operatorname{sen}^2 \frac{\pi}{2}}_{1} - \underbrace{\operatorname{sen}^2 0}_{0} \right) = \frac{\sqrt{2}}{10}$$