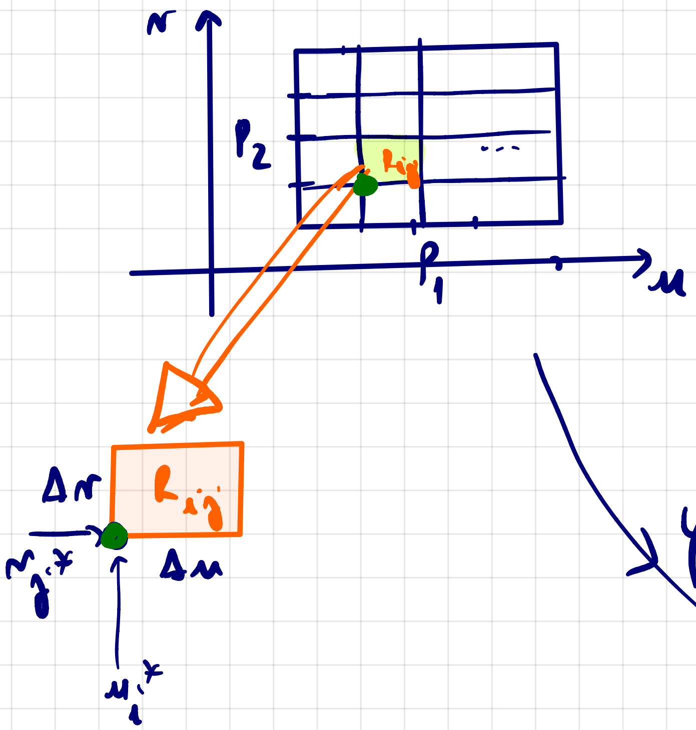
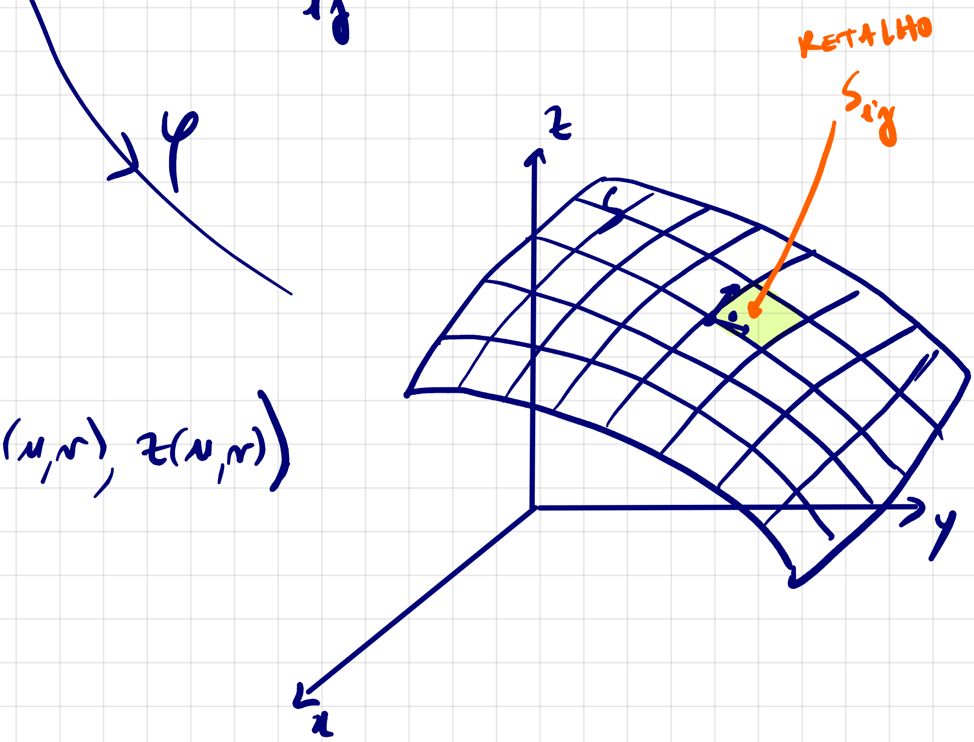
13/09/23 – AULA 24

ÁREA DA SUPERFÍCIE:

Para simplificar, suponha Ω um retângulo do $\mathbb{R}^2$. Seja $P = P_1 \times P_2$ uma partição regular de Ω, determinando sub-retângulos R_{ij} de dimensões Δu e Δv.



Esta partição divide a superfície S em RETALHOS S_{ij}



$$\varphi(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big)$$

A ideia para obter a área da superfície S será então em aproximá-la por uma soma de paralelogramos determinados pelos vetores tangentes no vértice inferior esquerdo de S_{ij}.

Para isto sejam (u_i^*, v_j^*) o vértice inferior esquerdo de R_{ij}.

Denote por

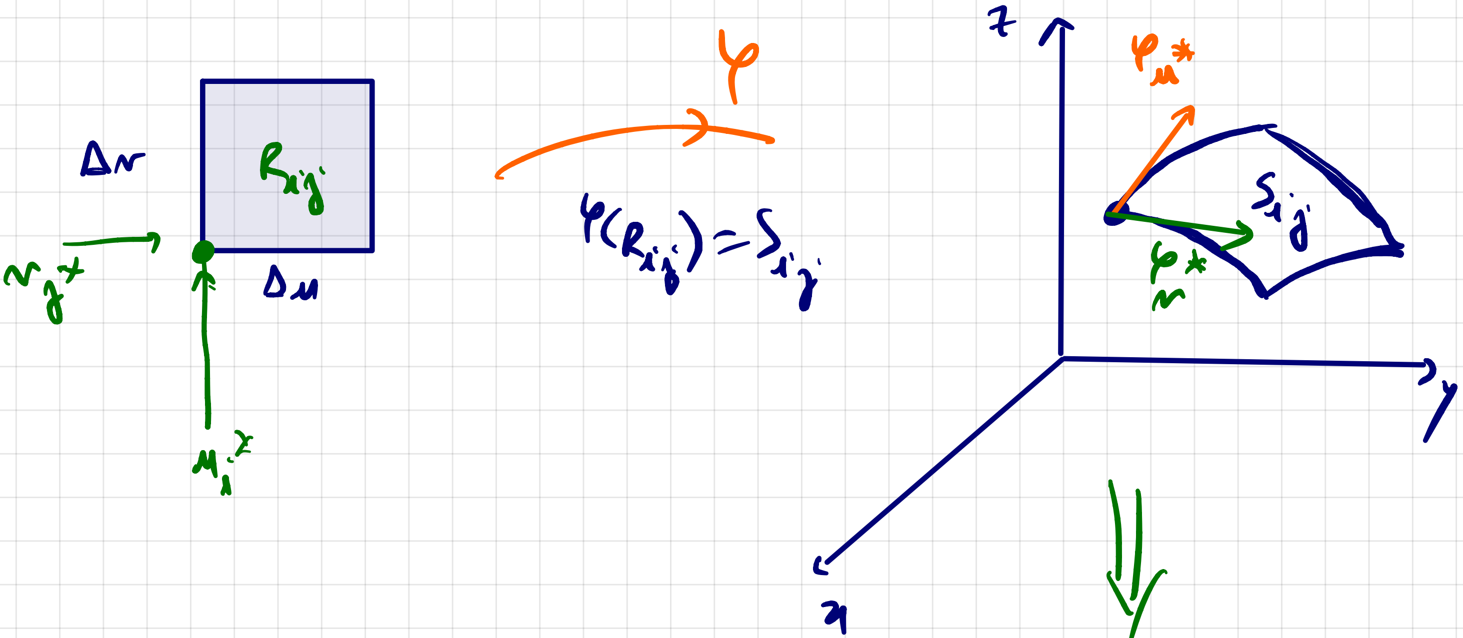
$$\varphi_u^* = \varphi_u(u_i^*, v_j^*) \quad e$$

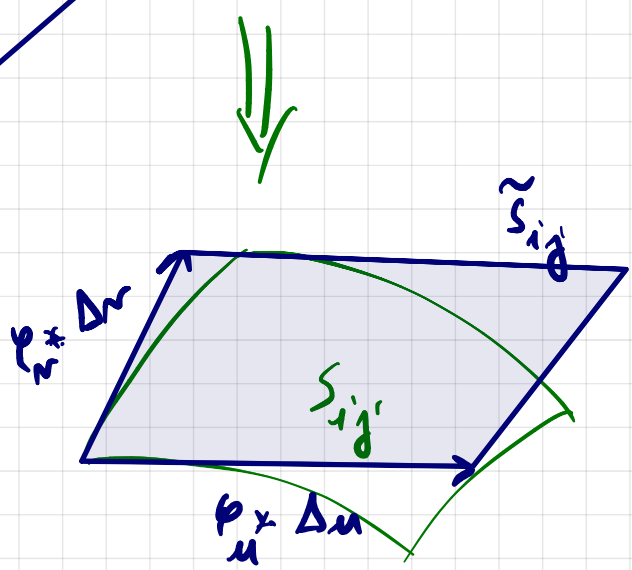
$$\varphi_v^* = \varphi_v(u_i^*, v_j^*),$$

onde

$$\varphi_u(u_i^*, v_j^*) = \left(\frac{\partial x}{\partial u}(u_i^*, v_j^*), \frac{\partial y}{\partial u}(u_i^*, v_j^*), \frac{\partial z}{\partial u}(u_i^*, v_j^*) \right)$$

e $\varphi_v(u_i^*, v_j^*) = \left(\frac{\partial x}{\partial v}(u_i^*, v_j^*), \frac{\partial y}{\partial v}(u_i^*, v_j^*), \frac{\partial z}{\partial v}(u_i^*, v_j^*) \right)$





A área de S_{ij} é aproximadamente a área do paralelogramo $\tilde{S}_{ij}$

Assim: $\tilde{S}_{ij} = \| \varphi_u^* \cdot \Delta u \times \varphi_v^* \cdot \Delta v \| =$

$$= \| \varphi_u^* \times \varphi_v^* \| . \Delta u . \Delta v$$

Assim, a área $A(S)$ da superfície fica aproximada por:

$$A(S) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{S}_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} \| \varphi_u^* \times \varphi_v^* \| . \Delta u \Delta v$$

Fazendo a passagem ao limite com $(m,n) \to (0,0)$ vamos obter o limite da soma de Riemann, que fornecerá a área $A(S)$ da superfície dada. Ou seja, obtemos o seguinte conceito:

Def: Se S é uma superfície do $\mathbb{R}^3$ suave, parametrizada por $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$;

$$\varphi(u,v) = (x(u,v),\ y(u,v),\ z(u,v));$$

então, a área de S na região $\Omega \subset \mathbb{R}^2$ será dada por:

$$A(S) = \iint_{\Omega} \| \varphi_u \times \varphi_v \| . \underbrace{du\, dv}_{dA}$$

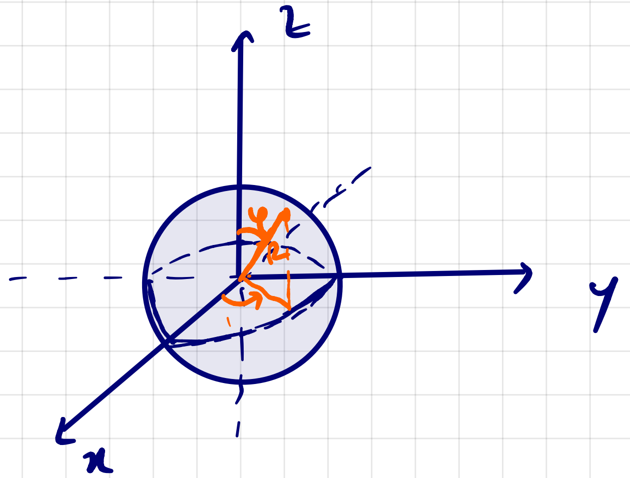
[Lembre que $\varphi_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$ e

$$\varphi_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$]

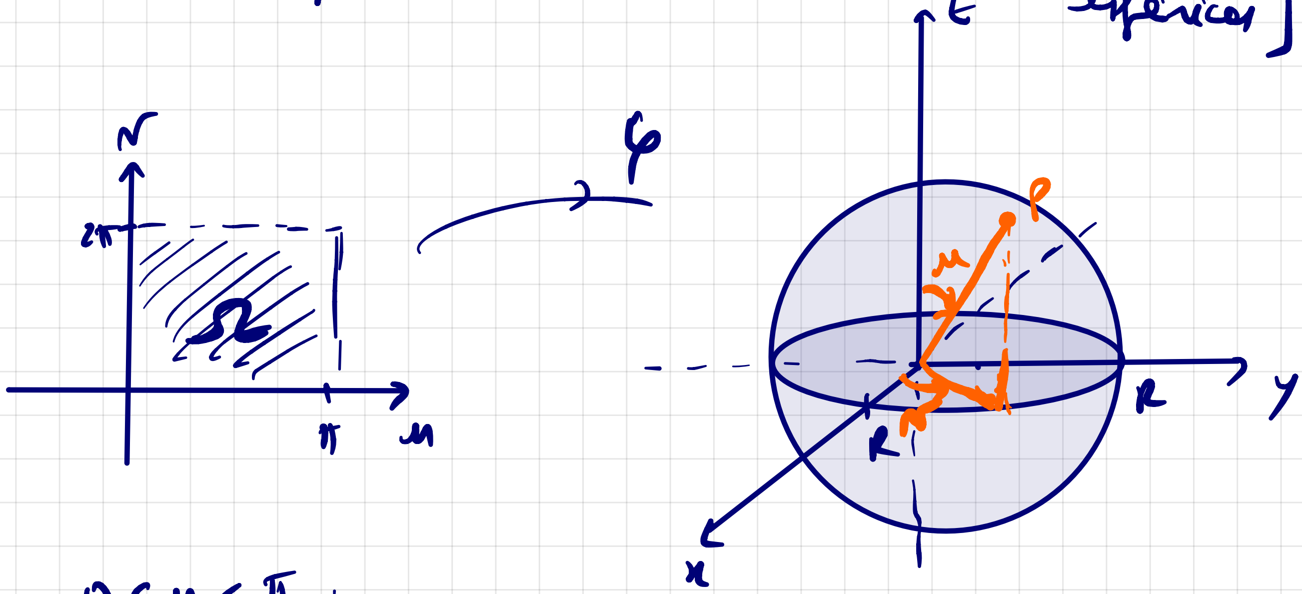
Ex.: Obtenha a fórmula da área da superfície esférica de raio $R>0$.

Solução:

$\varphi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ parametrização para a esfera.



↳ para isto usaremos o sist. de coordenadas esféricas]



$0 \leq u \leq \pi$;

$0 \leq v \leq 2\pi$

SIST. COORD. ESFÉRICAS
$$\begin{cases} x = R \cdot \cos v \cdot \operatorname{sen} u \\ y = R \cdot \operatorname{sen} v \cdot \operatorname{sen} u \\ z = R \cdot \cos u \end{cases}$$
— fornece a parametrização.

$$\varphi(u,v) = (\overbrace{R \cos v \cdot \operatorname{sen} u}^{x}, \overbrace{R \operatorname{sen} v \cdot \operatorname{sen} u}^{y}, \overbrace{R \cdot \cos u}^{z})$$

$$A(S) = \iint_{\Omega} \| \varphi_u \times \varphi_v \| \cdot dA \quad ; \quad \text{onde:}$$

$$\bullet\ \varphi_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$= \left(R\cos v \cdot \cos u,\ R\cdot \operatorname{sen} v \cdot \cos u,\ -R\cdot \operatorname{sen} u\right)$$

$$\bullet\ \varphi_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

$$= \left(-R\cdot \operatorname{sen} v \cdot \operatorname{sen} u,\ R\cdot\cos v\cdot \operatorname{sen} u,\ 0\right)$$

$$\Rightarrow \varphi_u \times \varphi_v = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos v\cos u & R\operatorname{sen} v\cos u & -R\operatorname{sen} u \\ -R\operatorname{sen} v\operatorname{sen} u & R\cos v \operatorname{sen} u & 0 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ R\cos v\cos u & R\operatorname{sen} v\cos u \\ -R\operatorname{sen} v\operatorname{sen} u & R\cos v\operatorname{sen} u \end{matrix}$$

$$\vdots$$

$$\varphi_u \times \varphi_v = \left(R^2\operatorname{sen}^2 u\cos v,\ R^2\operatorname{sen}^2 u\cdot\operatorname{sen} v,\ R^2\operatorname{sen} u\cos u\right)$$

$$\Rightarrow \|\varphi_u \times \varphi_v\| = \cdots = R^2 \operatorname{sen} u$$

$$\Rightarrow \underbrace{A(S)} = \iint_\Omega \|\varphi_u\times\varphi_v\|\cdot dA = \int_{v=0}^{v=2\pi}\int_{u=0}^{u=\pi} R^2\cdot\operatorname{sen} u\, du\, dv$$

$$= R^2 \int_{v=0}^{v=2\pi} (-\cos u)\Big|_{u=0}^{u=\pi} dv = R^2 \cdot (-\cos u)\Big|_0^\pi \cdot v\Big|_0^{2\pi}$$

$$= R^2\cdot(-\cos\pi + \cos 0)\cdot(2\pi - 0)$$

$$= R^2 (2)\cdot 2\pi = \underbrace{4\pi R^2}$$

ÁREA DE UMA SUPERFÍCIE DADA POR $z = f(x,y)$

Seja $z = f(x,y)$: $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$.

Então uma parametrização para f é simplesmente

$$\varphi(x,y) = (x, y, f(x,y))$$

Neste caso:
$$\varphi_x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial f}{\partial x} \right)$$

$$\Rightarrow \varphi_x = \left(1, 0, \frac{\partial f}{\partial x} \right) \quad ;$$

e
$$\varphi_y = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial f}{\partial y} \right)$$

$$= \left(0, 1, \frac{\partial f}{\partial y} \right).$$

Logo:
$$A(S) = \iint_\Omega \| \varphi_x \times \varphi_y \| \cdot dx\,dy \quad , \quad \text{onde}$$

$$\varphi_x \times \varphi_y = \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 0 & \frac{\partial f}{\partial x} & 1 & 0 \\ 0 & 1 & \frac{\partial f}{\partial y} & 0 & 1 \end{array} \right.$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

$$\varphi_x \times \varphi_y = 0\vec{i} + 0\vec{j} + \vec{k} - 0\vec{k} - \frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j}$$

$$= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

$$\Rightarrow \quad \|\varphi_x \times \varphi_y\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

Ou seja, obtemos a fórmula:

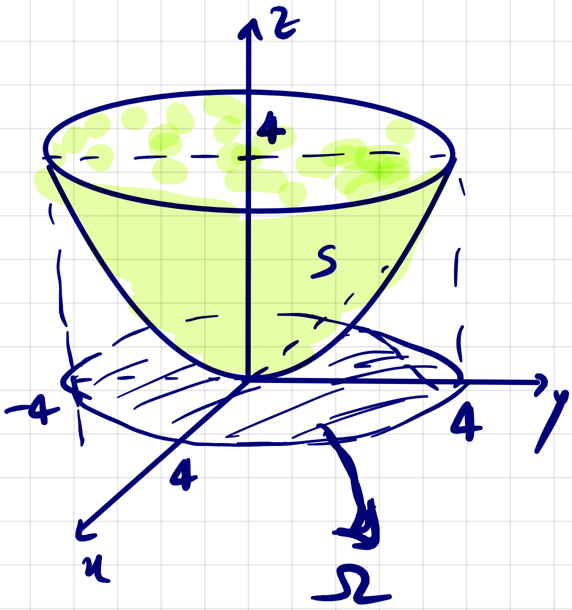
$$\underset{\sim}{A(S)} = \iint_{\Omega} \|\varphi_x \times \varphi_y\| \cdot dx\, dy =$$

$$\underbrace{= \iint_{\Omega} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \cdot dx\, dy}$$

↳ FÓRMULA PARA CÁLCULO DA ÁREA DE UMA SUPERFÍCIE $z = f(x,y)$ em $\Omega \subset \mathbb{R}^2$

Ex. 1 $z = x^2 + y^2$ (paraboloide).

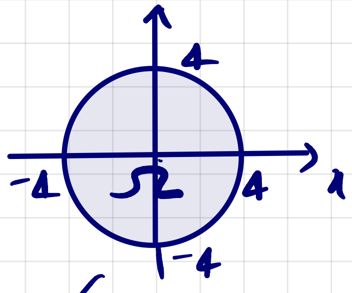
Achar sua área abaixo do plano $z = 4$.



SOLUÇÃO:

$$A(S) = \iint_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \cdot dA$$

$$\frac{\partial z}{\partial x} = 2x \quad ; \quad \frac{\partial z}{\partial y} = 2y$$



$$A(S) = \iint_{\Omega} \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_{\Omega} \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy =$$

COORD. POLARES

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=4} \sqrt{1 + 4\cdot\rho^2\cos^2\theta + 4\cdot\rho^2 \operatorname{sen}^2\theta} \cdot \rho \, d\rho \, d\theta =$$

$x = \rho\cos\theta$
$y = \rho \operatorname{sen}\theta$
$dx\,dy = \rho\, d\rho\, d\theta$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=4} \sqrt{1 + 4\rho^2 \underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1}} \cdot \rho \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=4} \sqrt{1 + 4\rho^2} \cdot \rho \, d\rho \, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \frac{1}{8} \int_{\rho=0}^{\rho=4} (1+4\rho^2)^{\frac{1}{2}} \cdot 8\rho \, d\rho =$$

$$\int u^k du$$

$$u = 1 + 4\rho^2 \Rightarrow du = 8\rho\, d\rho$$

$$\theta \Big|_0^{2\pi} \cdot \frac{1}{8} \cdot \frac{(1+4\rho^2)^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_0^4 =$$

$$= (2\pi - 0) \cdot \frac{1}{8} \cdot \frac{2}{3} \cdot \left[(1 + 4\cdot 16)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}} \right]$$

$$= \frac{2\pi}{8} \cdot \frac{2}{3} \cdot \left[\sqrt[3]{65^2} - 1 \right] = \frac{\pi}{6} \cdot \left(\sqrt[3]{4225} - 1 \right)$$

unidades de volume