13/09/23. AULA 23

No final da aula passada vimos o Teor. da
REGRA DA CADEIA:

$f: \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^p$, f diferenciável em $a \in \Omega$, g diferenciável em $b = g(a)$, então, $g\circ f$ é diferenciável em a, e

$$(g\circ f)'(a) = g'(f(a)).f'(a)$$

Em termos de diferenciais: $\quad g\circ f: \mathbb{R}^m \to \mathbb{R}^p$

$$d_a(g\circ f) = d_{f(a)}\, g \cdot d_a f \quad \to \text{transformações lineares.}$$

$$[\ \cdot\ ]_{\underline{\underline{p\times m}}} = [\ ]\cdot[\ ]_{\underline{\underline{p\times n}}\ \ n\times \underline{\underline{m}}} \quad \to \text{MATRIZES DAS TRANSF. LINEARES.}$$

Esta regra, na prática, é aplicada em apenas "uma variável desejada", ou seja, olhando apenas um elemento de posição ij da matriz do $d_a(g\circ f)$ dada acima.

Ou seja, teremos o seguinte teorema:

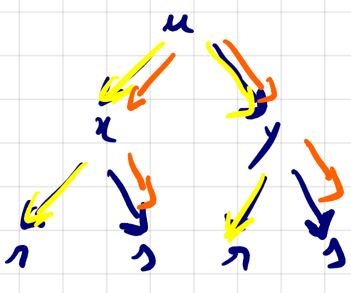
TEOREMA (REGRA DA CADEIA)

Seja $u = f(x,y)$ uma função $\mathbb{R}^2 \to \mathbb{R}$ diferenciável, e $x = x(r,s)$, $y = y(r,s)$ funções de r e de s, tais que $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial r}$ e $\frac{\partial y}{\partial s}$ existam. Então:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r},$$

e

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$



DEMONSTR.: Temos que $u = f(x,y)$ é diferenciável, por hipótese. Disso, pela def. de diferenciabilidade via incremento, podemos escrever:

$$\boxed{\Delta u = \Delta f(x,y) = \frac{\partial u}{\partial x}\cdot \Delta x + \frac{\partial u}{\partial y}\cdot \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \Delta y,} \quad (*)$$

com $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = 0$ e $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2 = 0$.

Seja $\Delta r \neq 0$ um incremento para a variável r (e, neste caso, s fica fixo).

Observe que, sendo $x = x(r,s)$ e $y = y(r,s)$, então

$$\Delta x = x(r+\Delta r, s) - x(r,s) \quad \text{e}$$

$$\Delta y = y(r+\Delta r, s) - y(r,s).$$

Dividindo $(*)$ por $\Delta r \neq 0$, obtemos:

$$\frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial x}\cdot\frac{\Delta x}{\Delta r} + \frac{\partial u}{\partial y}\cdot\frac{\Delta y}{\Delta r} + \varepsilon_1 \cdot \frac{\Delta x}{\Delta r} + \varepsilon_2 \cdot \frac{\Delta y}{\Delta r}$$

Fazendo a passagem ao limite com $\Delta r \to 0$, obtemos:

$$\underbrace{\lim_{\Delta r\to 0}\frac{\Delta u}{\Delta r}}_{\frac{\partial u}{\partial r}} = \frac{\partial u}{\partial x}\cdot \lim_{\Delta r\to 0}\frac{\Delta x}{\Delta r} + \frac{\partial u}{\partial y}\cdot\lim_{\Delta r\to 0}\frac{\Delta y}{\Delta r} +$$

$$+ \lim_{\Delta r\to 0}\Big(\underbrace{\varepsilon_1\cdot\frac{\Delta x}{\Delta r}}_{\to 0} + \underbrace{\varepsilon_2\cdot\frac{\Delta y}{\Delta r}}_{\to 0}\Big)$$

pois $\varepsilon_1, \varepsilon_2 \to 0$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \overbrace{\lim_{\Delta r \to 0} \frac{x(r+\Delta r, s) - x(r,s)}{\Delta r}}^{\frac{\partial x}{\partial r}} + 0 + 0$$

$$+ \frac{\partial u}{\partial y} \cdot \underbrace{\lim_{\Delta r \to 0} \frac{y(r+\Delta r, s) - y(r,s)}{\Delta r}}_{\frac{\partial y}{\partial r}} + 0$$

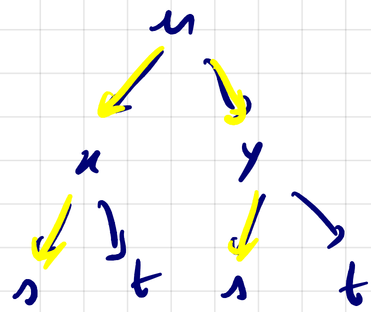
$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}}$$

Analogamente se prova a outra igualdade. $\square$

Ex.: Seja $u = x^2 \operatorname{sen}(xy)$; $x = s \cdot t^2$; $y = s^2 t^3$.

Obtenha $\frac{\partial u}{\partial s}$ e $\frac{\partial u}{\partial t}$.

Solução:



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s};$$ onde:

$u = x^2 \operatorname{sen} xy$

$$\frac{\partial u}{\partial x} = x^2 \cdot \cos xy \cdot y + 2x \cdot \operatorname{sen} xy$$

$$\frac{\partial u}{\partial y} = x^2 \cdot \cos xy \cdot x$$

$$x = st^2 \to \boxed{\frac{\partial x}{\partial s} = t^2}$$
$$\to \frac{\partial x}{\partial t} = 2st$$

$$y = s^2t^3 \to \boxed{\frac{\partial y}{\partial s} = 2st^3}$$
$$\to \frac{\partial y}{\partial t} = 3s^2t^2$$

Assim;

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial s} = (x^2y \cdot \cos xy + 2x \operatorname{sen} xy) \cdot t^2 + x^3 \cos xy \cdot 2st^3$$

com: $\boxed{x = st^2 \;;\; y = s^2t^3}$

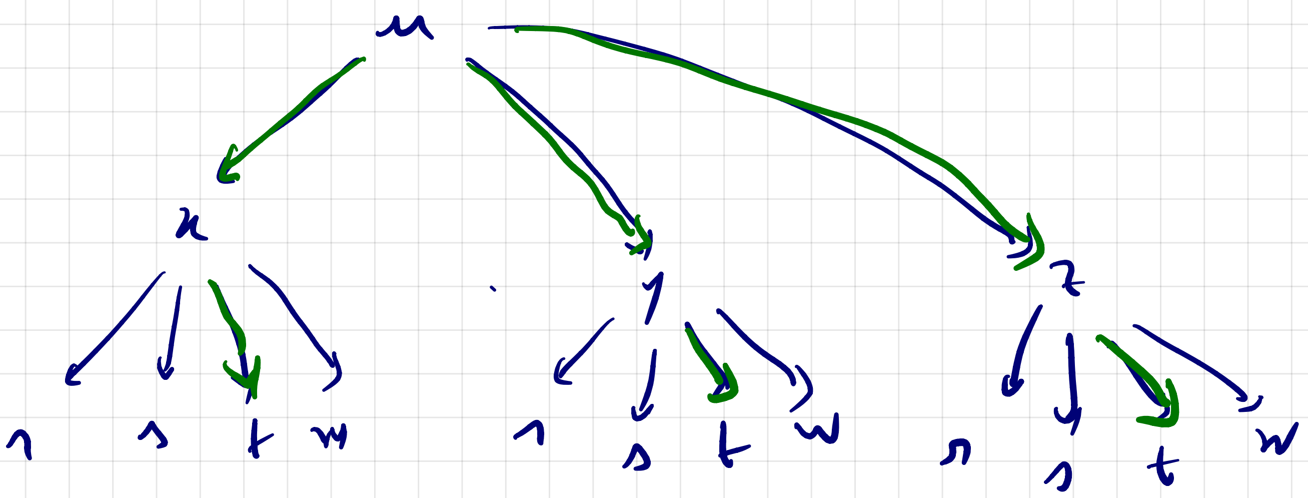
$$\Rightarrow \frac{\partial u}{\partial s} = \left[(st^2)^2 \cdot s^2t^3 \cdot \cos(st^2s^2t^3) + 2st^2 \cdot \operatorname{sen}(st^2s^2t^3)\right] t^2 +$$
$$+ (st^2)^3 \cdot \cos(st^2s^2t^3) \cdot 2st^3$$

$$\frac{\partial u}{\partial s} = s^4t^9 \cdot \cos(s^3t^5) + 2st^4 \cdot \operatorname{sen}(s^3t^5) +$$
$$+ 2s^4t^8 \cdot \cos(s^3t^5)$$

$$\frac{\partial u}{\partial t} = \ldots \quad (\text{exercício})$$

Obs.: Esta regra pode ser ampliada para mais variáveis. Por exemplo:

$$u = u(x, y, z) \, ; \qquad \begin{aligned} x &= (r, s, t, w) \\ y &= (r, s, t, w) \\ z &= (r, s, t, w) \end{aligned}$$



Por exemplo:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t}$$

Obs.: Conforme visto anteriormente, em geral:

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

$$d_a(g \circ f) = d_{f(a)} g \cdot d_a f \qquad [\text{MATRICIAL}]$$

Por exemplo tome $g(x, y) = u$

$u : \mathbb{R}^2 \to \mathbb{R}$. $\quad [d_a u]_{1 \times 2}$ $\qquad$ $g : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad [d_{f(a)} g]_{2 \times 2}$ $\qquad$ $\left.\begin{aligned} x &= x(r, s) \\ y &= y(r, s) \end{aligned}\right\} = f$

Então $\quad u = g(x, y) = g(x(r, s), y(r, s)) = (g \circ f)\underbrace{(r, s)}_{a}$

Disso:

$$d_a u = d_a\, g\circ f = d_{f(a)}\, g \cdot d_a f .$$

PRODUTO DE MATRIZES.

Logo; lembrando que $a = (r, s)$

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \\ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \end{cases}$$

, as duas fórmulas do Teorema.

LISTA 09 |

10 - a) $u = \ln(xy) + y^2$; $x = e^t$; $y = e^{-t}$

Obter $\frac{\partial u}{\partial t}$.

SOLUÇÃO:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Obs.: como x e y dependem apenas de t, podemos escrever:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Temos:

$$\frac{\partial u}{\partial x} = \frac{\cancel{y}}{x\cancel{y}} + 0 = \frac{1}{x} \quad ; \quad \frac{\partial x}{\partial t} = e^{t} \quad ; \quad \frac{\partial y}{\partial t} = -e^{-t}$$

$$(\ln v)' = \frac{v'}{v}$$

$$\frac{\partial u}{\partial y} = \frac{\cancel{x}}{xy} + 2y = \frac{1}{y} + 2y$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial t} = \frac{1}{x} \cdot e^{t} + \left(\frac{1}{y} + 2y\right) \cdot (-e^{-t})$$

$$= \frac{1}{\cancel{e^{t}}} \cdot \cancel{e^{t}} + \left(\frac{1}{e^{-t}} + 2 \cdot e^{-t}\right) \cdot (-e^{-t})$$

$$= 1 - (e^{t} + 2 \cdot e^{-t}) \cdot e^{-t}$$

$$= 1 - (\underbrace{e^{t} \cdot e^{-t}}_{1} - 2 \cdot e^{-t} \cdot e^{-t})$$

$$= \cancel{1} - \cancel{1} + 2 \cdot e^{-2t} = 2e^{-2t}$$