12/09/23 - AULA 23

EXERCÍCIOS DAS LISTAS:

LISTA 04.

11) (a) $\vec{F}(x,y,z) = (\underbrace{e^x \cdot \cos y}_{F_1}, \underbrace{e^x \cdot \operatorname{sen} y}_{F_2}, \underbrace{z}_{F_3})$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \text{ ; onde:}$$

$$\frac{\partial F_1}{\partial x} = e^x \cdot \cos y$$

$$\frac{\partial F_2}{\partial y} = e^x \cdot \cos y$$

$$\frac{\partial F_3}{\partial z} = 1$$

$$\Rightarrow \operatorname{div} \vec{F} = e^x \cos y + e^x \cos y + 1$$

$$= 2 \cdot e^x \cdot \cos y + 1$$

$$\operatorname{rot} \vec{F} = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y & e^x \operatorname{sen} y & z \end{matrix} \right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^x \cos y & e^x \operatorname{sen} y \end{matrix}$$

$-$ $-$ $-$ $+$ $+$ $+$

$$= \underbrace{\frac{\partial}{\partial y}(z)}_{=0}\vec{i} + \underbrace{\frac{\partial}{\partial z}(e^x \cos y)}_{=0}\vec{j} + \frac{\partial}{\partial x}(e^x \operatorname{sen} y)\vec{k} -$$

$$- \frac{\partial}{\partial y}(e^x \cos y)\vec{k} - \underbrace{\frac{\partial}{\partial z}(e^x \operatorname{sen} y)\vec{i}}_{=0} - \underbrace{\frac{\partial}{\partial x}(z)}_{=0}\vec{j}$$

$$= 0\vec{i} + 0\vec{j} + e^x \operatorname{sen} y\,\vec{k} + e^x \operatorname{sen} y\,\vec{k} - 0\vec{i} - 0\vec{j}$$

$$= 0\vec{i} + 0\vec{j} + 2 e^x \operatorname{sen} y\,\vec{k}$$

$$\Rightarrow \boxed{\operatorname{rot}\vec{F} = (0, 0, 2e^x \operatorname{sen} y)}$$

EXTRA: mostre que

$$\operatorname{div}(\nabla f) = \Delta f,$$

onde $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (tome $n = 2$ para simplificar)

$\uparrow$ ABERTO.

obs! $z = f(x, y)$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \quad (\text{Do cálculo II})$$

$$z = f(x,y) \Rightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Então:

$$\operatorname{div}(\nabla f) = \operatorname{div}\left(\underbrace{\frac{\partial f}{\partial x}}_{F_1}, \underbrace{\frac{\partial f}{\partial y}}_{F_2}\right) =$$

$$= \frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} := \Delta f$$

LISTA 06 |

08) $g: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ $\qquad g = g(x,y)$

Ω aberta de $\mathbb{R}^2$ e γ curva fechada suave e simples em Ω. (na verdade, $\gamma = \partial\Omega$)

Mostrar: (Pelo T. da Divergência)

$$\oint_\gamma g \cdot \frac{\partial g}{\partial \vec{n}} \cdot ds = \iint_\Omega (g \cdot \Delta g + \|\nabla g\|^2) \cdot dA$$

obs: $\frac{\partial g}{\partial \vec{n}} := \nabla g \cdot \vec{n}$ (DERIVADA NORMAL)

$\downarrow$

derivada direcional.

T. Divergência:

$$\oint_{\gamma} \vec{F} \cdot \vec{n} \cdot ds = \iint_{\Omega} \operatorname{div} \vec{F} \cdot dA$$

Tome $\vec{F} = g \cdot \nabla g$. Neste caso,

$$\vec{F} \cdot \vec{n} = (g \cdot \nabla g) \cdot \vec{n} = g \cdot (\nabla g \cdot \vec{n}) = g \cdot \frac{\partial g}{\partial \vec{n}}$$

ASSOCIATIVIDADE

Então:

$$\operatorname{div} \vec{F} = \operatorname{div}(g \cdot \nabla g) =$$

$$= \operatorname{div}\left(g \cdot \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\right)$$

$$= \operatorname{div}\Big(\underbrace{g \cdot \frac{\partial g}{\partial x}}_{F_1}, \underbrace{g \cdot \frac{\partial g}{\partial y}}_{F_2}\Big) =$$

$$= \frac{\partial}{\partial x}\left(g \cdot \frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(g \cdot \frac{\partial g}{\partial y}\right) =$$

$(u \cdot v)' = u \cdot v' + u' \cdot v$

$$= g \cdot \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) + \frac{\partial g}{\partial x} \cdot \frac{\partial g}{\partial x} + g \cdot \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) + \frac{\partial g}{\partial y} \cdot \frac{\partial g}{\partial y}$$

$$= g \cdot \frac{\partial^2 g}{\partial x^2} + \left(\frac{\partial g}{\partial x}\right)^2 + g \cdot \frac{\partial^2 g}{\partial y^2} + \left(\frac{\partial g}{\partial y}\right)^2$$

$$= \underbrace{g \cdot \left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right]}_{\Delta g} + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2$$

$\hookrightarrow \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$

$\Rightarrow \|\nabla g\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$

$$= g \cdot \Delta g + \|\nabla g\|^2$$

Portanto, mostramos que:

$$\operatorname{div} \vec{F} = \operatorname{div}(g \cdot \nabla g) = g \cdot \Delta g + \|\nabla g\|^2$$

Assim, c.f o T. da Divergência, obtemos:

$$\oint_\gamma \underbrace{\vec{F} \cdot \vec{n}}_{= g \cdot \frac{\partial g}{\partial \vec{n}}} \, ds = \iint_\Omega \underbrace{\operatorname{div} \vec{F}}_{= g \cdot \Delta g + \|\nabla g\|^2} \, dA$$

$$\Rightarrow \boxed{\oint_\gamma g \cdot \frac{\partial g}{\partial \vec{n}} \, ds = \iint_\Omega \left(g \cdot \Delta g + \|\nabla g\|^2\right) dA}$$

LISTA 06

07) $u, v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. $\gamma = \partial\Omega$

Mostre que

$$\oint_\gamma uv\,dx + uv\,dy = \iint_\Omega \left[v\cdot\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + u\cdot\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)\right]\cdot dA$$

$$\boxed{\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$ (T. DE GREEN)

Tome $P = u\cdot v$

$Q = u\cdot v$

De fato: $\dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial x}(u\cdot v) = u\cdot\dfrac{\partial v}{\partial x} + v\cdot\dfrac{\partial u}{\partial x}$

e $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}(u\cdot v) = u\cdot\dfrac{\partial v}{\partial y} + v\cdot\dfrac{\partial u}{\partial y}$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = v\cdot\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right] + u\cdot\left[\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right]$$

Portanto, aplicando o T. de Green para $P = u\cdot v$ e $Q = u\cdot v$, segue a igualdade desejada.