16/08/23 – AULA 17

Da aula passada:

01) Mostre que $\operatorname{rot}(\nabla F) = \vec{0}$.

solução:

$$F : \Omega \subset \mathbb{R}^3 \to \mathbb{R} \quad \text{(FUNÇÃO ESCALAR)}$$

$$F(x,y,z) = w$$

$$\Rightarrow \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

Assim:

$$\operatorname{rot}(\nabla F) = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{array}\right.$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial y}\right)\right]\vec{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\right]\vec{k}$$

$$\underbrace{\left[\frac{\partial^2 F}{\partial y \partial z} - \frac{\partial^2 F}{\partial z \partial y}\right]}_{=0}\vec{i} + \underbrace{\left[\frac{\partial^2 F}{\partial z \partial x} - \frac{\partial^2 F}{\partial x \partial z}\right]}_{=0}\vec{j} +$$

$$+ \underbrace{\left[\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial x}\right]}_{0}\vec{k} =$$

$$= 0\vec{i} + 0\vec{j} - 0\vec{k} = \vec{0}.$$

02) $\vec{F}(x,y,z) = (\overbrace{x y^2}^{F_1}, \overbrace{2x \operatorname{sen} z}^{F_2}, \overbrace{x^2 z^3}^{F_3})$.

Obter $\operatorname{div} \vec{F}$.

Solução:

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= y^2 + 0 + 3x^2 z^2$$

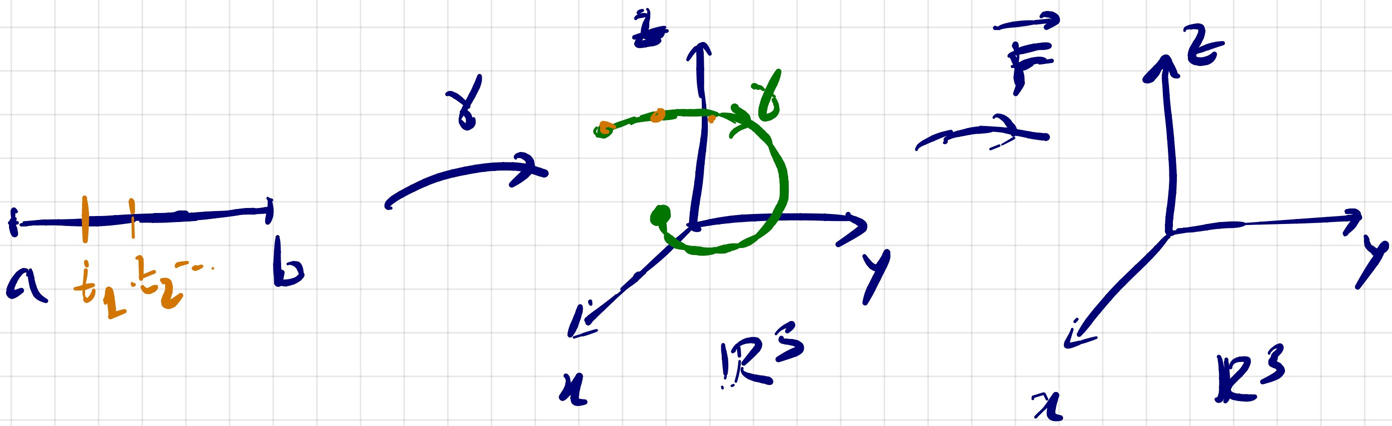
$$= y^2 + 3x^2 z^2$$

# INTEGRAIS DE LINHA:

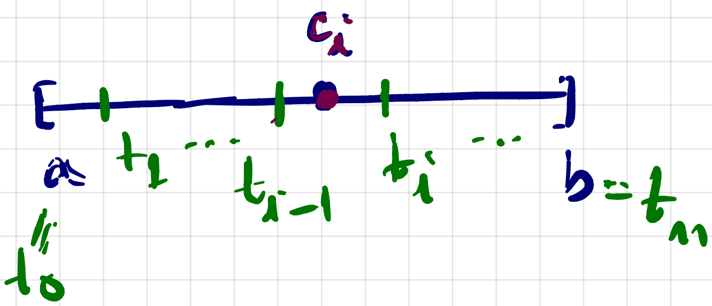
Sejam $\gamma : [a,b] \to \mathbb{R}^3$, $\gamma(t) = (x(t), y(t), z(t))$ uma curva suave (i.e., derivável, com $\gamma'(t) \neq 0$) e $\vec{F} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ um campo vetorial dado por:

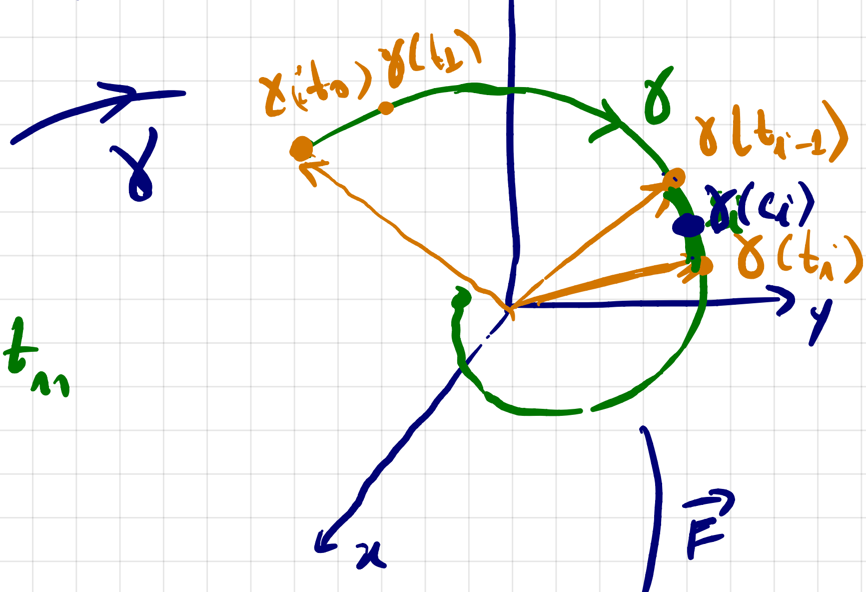
$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

definida sobre a curva $\gamma$.

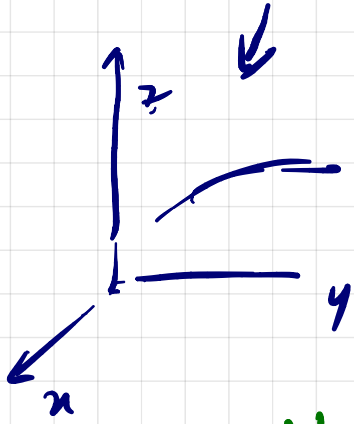



Seja $P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$ uma partição de $[a,b]$, que nos determina arcos $S_i = \overset{\frown}{\gamma(t_{i-1})\,\gamma(t_i)}$ sobre $\gamma$.



Seja $c_i \in [t_{i-1}, t_i]$ um ponto qualquer em cada subintervalo $[t_{i-1}, t_i]$




Escreva

$$S_i = \gamma(t_i) - \gamma(t_{i-1}) = \gamma'(d_i).\overbrace{(t_i - t_{i-1})}^{\Delta t_i}$$

T.V.M ; $d_i$ entre $t_{i-1}$ e $t_i$

Então, montamos a soma de Riemann:

$$\sum_{i=1}^{n} \vec{F}(\gamma(c_i)).S_i = \sum_{i=1}^{n} \vec{F}(\gamma(c_i)).\gamma'(d_i).\Delta t_i$$

Fazendo passagem do limite com $n \to \infty$, vamos obter

$$\int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t).dt := \int_\gamma \vec{F}.d\vec{r},$$

onde $\vec{r}$ é o traçado da curva $\gamma$.

A integral

$$\int_\gamma \vec{F}.d\vec{r} = \int_a^b \vec{F}(\gamma(t)).\gamma'(t)dt$$

Chama-se INTEGRAL DE LINHA de $\vec{F}$ ao longo da curva $\gamma$.

Podemos escrever esta integral de linha do seguinte modo:

$$\int_\gamma \vec{F}.d\vec{r} = \int_a^b \vec{F}(\gamma(t)).\gamma'(t)dt =$$

$$= \int_a^b \left[ P(x(t),y(t),z(t)), Q(x(t),y(t),z(t)), R(x(t),y(t),z(t)) \right.$$
$$\left. .(x'(t), y'(t), z'(t)) \right].dt$$

Como $(x'(t), y'(t), z'(t))dt =$

$(x'(t)dt, y'(t).dt, z'(t)dt)$

$= (dx, dy, dz)$, e então

$$\int_\gamma \vec{F}.d\vec{r} =$$

$$= \int_a^b (P(x,y,z), Q(x,y,z), R(x,y,z)).(dx, dy, dz)$$

PRODUTO ESCALAR

$$= \int_a^b P \cdot dx + Q\,dy + R\,dz$$

Vejamos alguns exemplos:

01) Seja $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$.

Calcule a integral de linha de $\vec{F}$ ao longo da hélice

$$\gamma(t) = (\underbrace{\cos t}_{x}, \underbrace{\operatorname{sen} t}_{y}, \underbrace{t}_{z}); \quad t \in [0, 2\pi].$$

Solução:

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t)\,dt \quad ; \quad \text{onde:}$$

$$\vec{F}(\gamma(t)) = \cos t\,\vec{i} + \operatorname{sen} t\,\vec{j} + t\,\vec{k} \; ;$$

$$\gamma'(t) = (-\operatorname{sen} t, \cos t, 1)$$

Assim:

$$\int_\gamma \vec{F}\,d\vec{r} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t)\,dt =$$

$$= \int_0^{2\pi} (\cos t, \operatorname{sen} t, t) \cdot (-\operatorname{sen} t, \cos t, 1)\,dt$$

PRODUTO ESCALAR

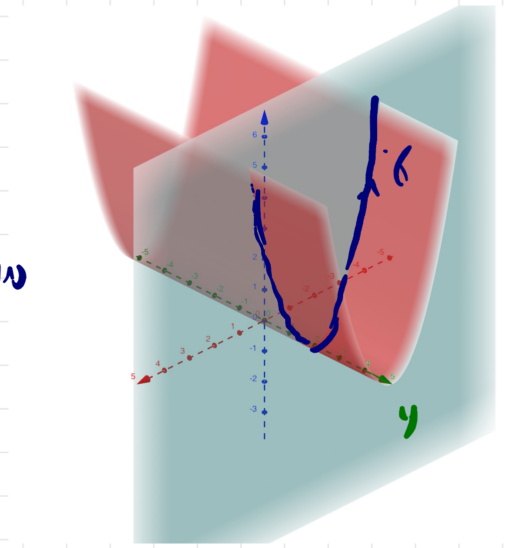
$$= \int_0^{2\pi} (-\text{sen}\, t \cancel{\cos t} + \text{sen}\, t \cancel{\cos t} + t)\, dt$$

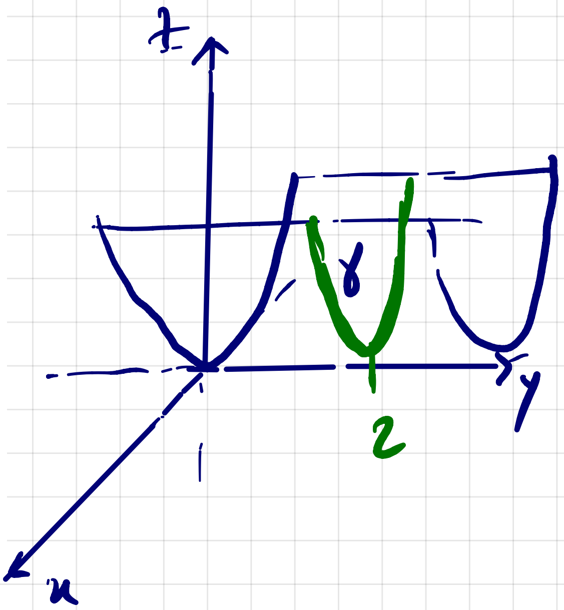
$$= \int_0^{2\pi} t\, dt = \frac{t^2}{2}\Big|_0^{2\pi} = \frac{4\pi^2}{2} - 0 = 2\pi^2$$

02) Calcule $\int_\gamma (2x\, dx + yz\, dy + 3z\, dz)$ ao longo da parábola $z = x^2$, $y = 2$, do ponto $A(0,2,0)$ ao ponto $B(2,2,4)$.

Solução:

Obs.: o desenho ao lado é opcional




$$\gamma(t) = (x(t), y(t), z(t))$$

$$\gamma : \begin{cases} x = t \\ y = 2 \\ z = t^2 \end{cases} \quad z = x^2$$

Queremos a integral de $A(0,2,0)$ a $B(2,2,4)$

para A: $\left.\begin{array}{l} x = 0 = t \\ y = 2 = 2 \\ z = 0 = t^2 \end{array}\right\}$ $t = 0$

para B: $\left.\begin{array}{l} x = 2 = t \\ y = 2 = 2 \\ z = 4 = t^2 \end{array}\right\}$ $t = 2$

$t \in [0,2]$

Assim: $x = t \rightsquigarrow dx = dt$

$y = 2 \rightsquigarrow dy = 0$

$z = t^2 \rightsquigarrow dz = 2t\,dt$

Assim, teremos:

$$\int_\gamma (2x\,dx + yz\,dy + 3z\,dz) =$$

$$= \int_0^2 2 \cdot t \cdot dt + \underbrace{2 \cdot t^2 \cdot 0}_{0} + 3 \cdot t^2 \cdot 2t\,dt$$

$$= \int_0^2 (2t + 6t^3)\,dt = \left(t^2 + \frac{3}{2}t^4\right)\Big|_0^2 =$$

$$4 + \frac{3}{\cancel{2}} \cdot \overset{8}{\cancel{16}} - 0 = \underline{\underline{28}}$$

03) $\vec{F}(x,y) = (\overset{P}{x^2y}, \overset{Q}{xy})$.

Calcule $\int_\gamma \vec{F}\, d\vec{r}$, onde $\gamma$ é a parábola

$x = y^2$ do ponto $(0,0)$ ao ponto $(4,2)$.

<u>Solução:</u> parametrização para $\gamma$:

$$\gamma: \begin{cases} x = t^2 \rightsquigarrow dx = 2t\,dt \\ y = t \rightsquigarrow dy = dt \end{cases}$$

Além disso,

$$A(0,0) \Longrightarrow \begin{cases} x = 0 = t^2 \\ y = 0 = t \end{cases} \Rightarrow t = 0$$

$$B(4,2) \Rightarrow \begin{cases} x = 4 = t^2 \\ y = 2 = t \end{cases} \Rightarrow t = 2$$

Logo, $\gamma$ está definida no intervalo $[0,2]$.

Assim, temos:

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^2 P\,dx + Q\,dy =$$

$$= \int_0^2 x^2y\,dx + xy\,dy$$

$$= \int_0^2 (t^2)^2 \cdot t \cdot 2t\,dt + t^2 \cdot t \cdot dt$$

$$= \int_0^2 (2t^6 + t^3)\,dt = \frac{2t^7}{7} + \frac{t^4}{4} \Big|_0^2 = \ldots$$

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EXERCÍCIO PARA ENTREGAR NA SEXTA: (LISTA 04)

- CAROLINE : 8a + 13b
- CÁTIA : 8b + 13c
- GEILSON : 8a + 13a
- GUSTAVO : 8c + 13c
- WILIAN : 8b + 13b