09/08/23 – AULA 15

Seguindo nos exemplos:

03) $\iiint_{\Omega} \dfrac{e^{x-y+z}}{x+y-z}\, dx\,dy\,dz$, onde Ω é a região definida por:

$$0 \leq \overbrace{x-y+z}^{u} \leq 1$$
$$1 \leq \overbrace{x+y-z}^{v} \leq 2 \quad e$$
$$0 \leq \underbrace{z}_{w} \leq 1.$$

Solução:

A transformação $T(u,v,w) = (x,y,z)$ será tal que

$$\begin{cases} u = x-y+z \\ v = x+y-z \\ w = z \end{cases} \longrightarrow z = w$$

$$+\begin{cases} u = x - \cancel{y} + \cancel{w} \\ v = x + \cancel{y} - \cancel{w} \end{cases}$$

$$2x = u+v \Rightarrow x = \frac{1}{2}u + \frac{1}{2}v$$

$$y = v + z - x$$

$$y = v + w - \frac{1}{2}u - \frac{1}{2}v \Rightarrow y = -\frac{1}{2}u + \frac{1}{2}v + w$$

Disso, temos:

$$\det J(T)(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{matrix}$$

$$= \frac{1}{4} + 0 + 0 - 0 - 0 + \frac{1}{4} = \frac{1}{2} \Rightarrow \boxed{|\det J(T)(u,v,w)| = \frac{1}{2}}$$

Portanto, temos:

$$\iiint_{\Omega} \frac{e^{\overbrace{x-y+z}^{u}}}{\underbrace{x+y-z}_{v}} \, dx\,dy\,dz = \iiint_{\Omega'} \frac{e^{u}}{v} \left|\frac{1}{2}\right| . du\,dv\,dw \; ; \quad \begin{matrix} 0 \leq u \leq 1 \\ 1 \leq v \leq 2 \\ 0 \leq w \leq 1 \end{matrix}$$

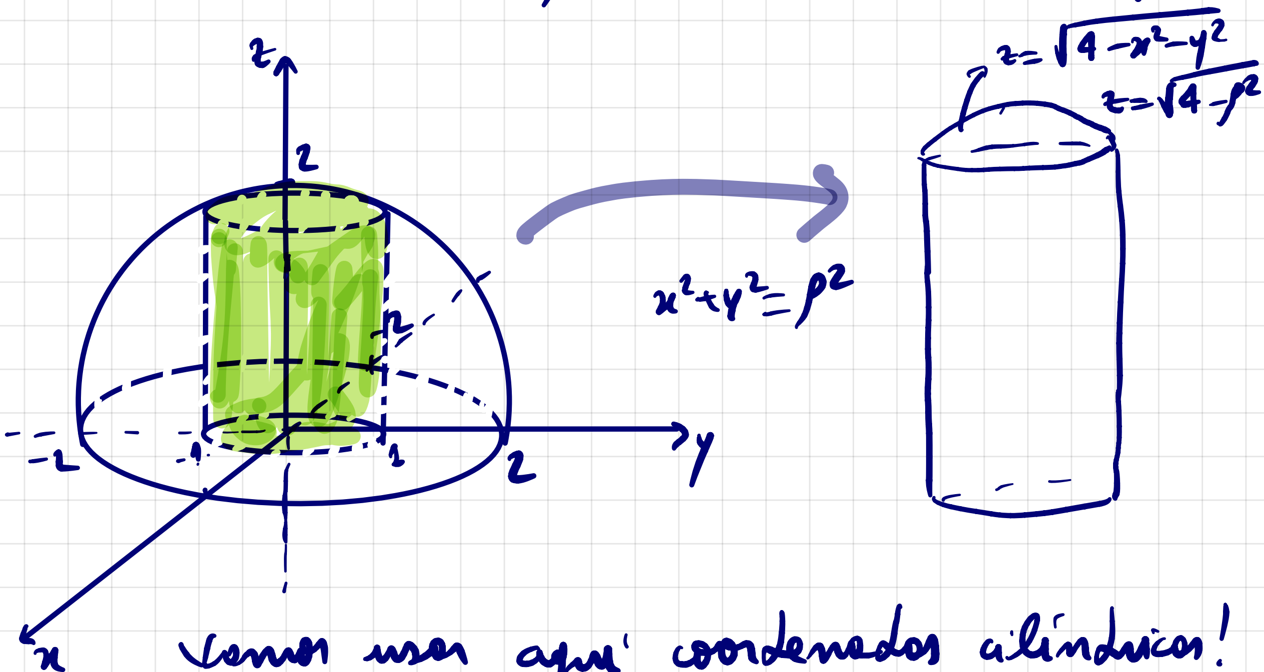
$$= \frac{1}{2} \int_{w=0}^{w=1} \int_{v=1}^{v=2} \int_{u=0}^{u=1} \frac{e^{u}}{v} \, du\,dv\,dw = \frac{1}{2} \int_{w=0}^{w=1} dw \int_{v=1}^{v=2} \frac{dv}{v} \int_{u=0}^{u=1} e^{u} du =$$

$$= \frac{1}{2} \cdot w \Big|_0^1 \cdot \ln|v| \Big|_1^2 \cdot e^{u} \Big|_0^1 = \frac{1}{2} \cdot (\ln 2 - \underbrace{\ln 1}_{0}) \cdot (e^1 - e^0)$$

$$\frac{\ln 2 \cdot (e-1)}{2}$$

04) Calcule o volume do sólido abaixo da semi esfera $x^2+y^2+z^2=4$, $z \geq 0$, limitado pelo cilindro $x^2+y^2=1$ e pelo plano xy; usando integrais triplas.

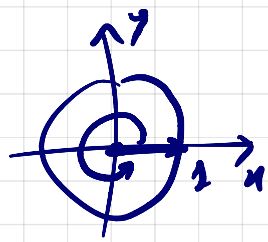
SOLUÇÃO:



Vamos usar aqui coordenadas cilíndricas!

$$P(x,y,z) = P(\rho\cos\theta, \rho \operatorname{sen}\theta, z)$$

$$V = \iiint_{\Omega} \underbrace{dV}_{//} = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=0}^{z=\sqrt{4-\rho^2}} \rho \, d\rho \, d\theta \, dz =$$



$$dx\,dy\,dz \xrightarrow[\text{COORD. CILÍNDRICAS}]{} \rho\, d\rho\, d\theta\, dz$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\rho=0}^{\rho=1} \rho \cdot \left(\int_{z=0}^{z=\sqrt{4-\rho^2}} dz\right) d\rho = \theta\Big|_0^{2\pi} \cdot \int_{\rho=0}^{\rho=1} \rho\left(z\Big|_0^{\sqrt{4-\rho^2}}\right) d\rho =$$

$$= 2\pi \cdot \int_{\rho=0}^{\rho=1} \sqrt{4-\rho^2} \cdot \rho\, d\rho = \cancel{2}\pi \cdot \left(-\frac{1}{\cancel{2}}\right) \int_0^1 (4-\rho^2)^{\frac{1}{2}} (-2\rho)\, d\rho$$

$$\int u^k du \qquad u = 4-\rho^2 \qquad du = -2\rho\, d\rho$$

$$= -\pi \cdot \frac{(4-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 = -\frac{2\pi}{3}\left[(4-1^2)^{\frac{3}{2}} - (4-0^2)^{\frac{3}{2}}\right]$$

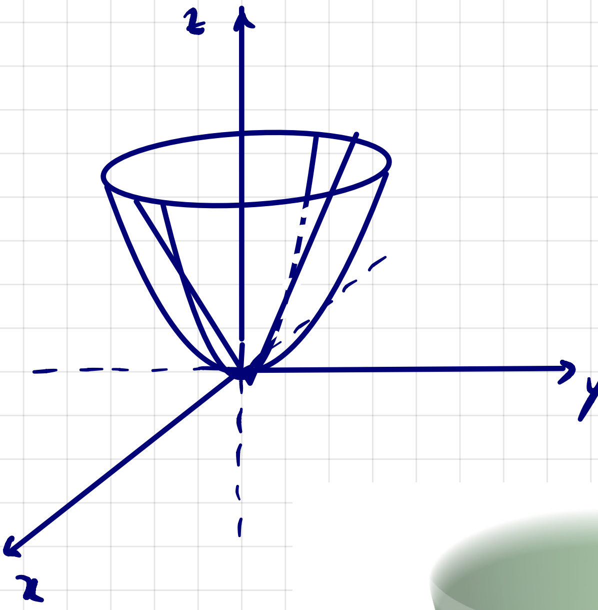
$$= -\frac{2\pi}{3}\cdot\left[3\sqrt{3} - 8\right] = \frac{(8-3\sqrt{3})\cdot 2\pi}{3}$$

//

05) Calcule $\displaystyle\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, onde:

$$\Omega = \{(x,y,z)\in \mathbb{R}^3 : x^2+y^2 \le z \le \sqrt{x^2+y^2}\}$$

Solução:

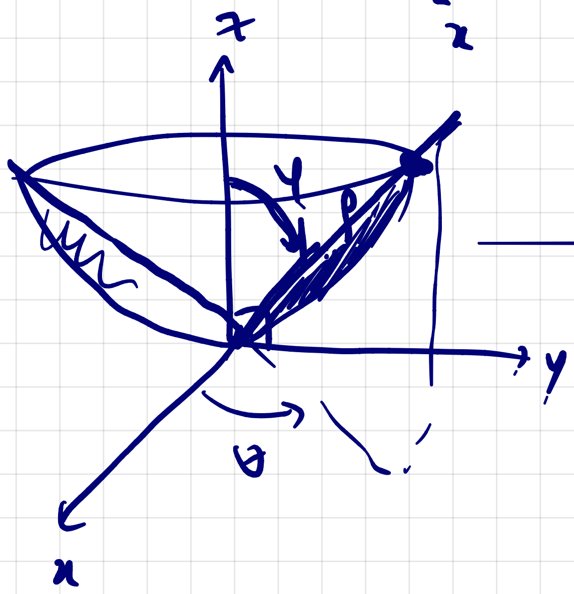


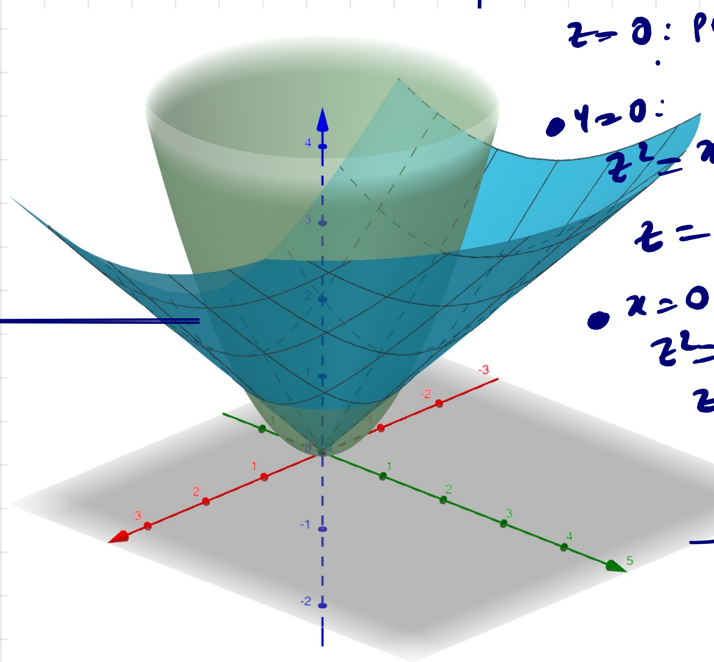
- $z = x^2+y^2$ é um paraboloide.
- $z = \sqrt{x^2+y^2}$

  $z^2 = x^2+y^2$

  ↳ CONE

  $z=0$: PONTO ORIGEM
- $y=0$: $z^2 = x^2$, $z = \pm x$
- $x=0$: $z^2 = y^2$, $z = \pm y$





— PELO GEOGEBRA —

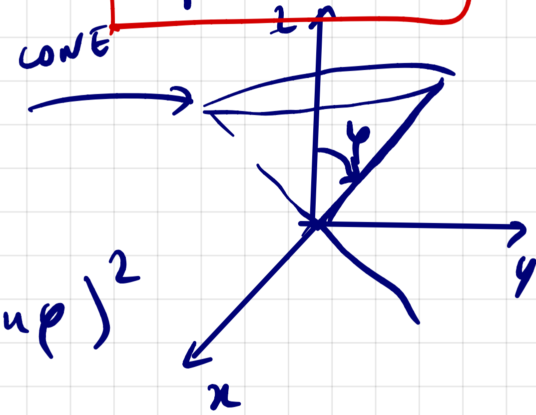
Vamos usar o sist. de coordenadas esféricas!

$$\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$$

$$x = \rho \cos\theta \operatorname{sen}\varphi$$

$$y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi$$

$$z = \rho \cos\varphi \quad , \quad \text{onde} \quad \boxed{0 \leq \theta \leq 2\pi} \quad \boxed{\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}}$$



$z = x^2+y^2$ fornece:

$$\rho \cos\varphi = (\rho\cos\theta \operatorname{sen}\varphi)^2 + (\rho \operatorname{sen}\theta \operatorname{sen}\varphi)^2$$

$$\rho \cos\varphi = \rho^2 \cos^2\theta \operatorname{sen}^2\varphi + \rho^2 \operatorname{sen}^2\theta \operatorname{sen}^2\varphi$$

$$\rho \cos\varphi = \rho^2 \operatorname{sen}^2\varphi \, \underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1}$$

$$\cancel{\rho} \cos\varphi = \rho^{\cancel{2}} \operatorname{sen}^2\varphi$$

$$\rho = \frac{\cos\varphi}{\operatorname{sen}^2\varphi}$$

Logo;

$$\boxed{0 \leq \rho \leq \frac{\cos\varphi}{\operatorname{sen}^2\varphi}}$$

Assim, teremos:

$$\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$$

$$= \iiint_{\Omega'} \sqrt{(\rho\cos\theta\,\text{sen}\,\varphi)^2 + (\rho\,\text{sen}\,\theta\,\text{sen}\,\varphi)^2 + (\rho\cdot\cos\varphi)^2}\cdot \underbrace{\rho^2 \text{sen}\,\varphi\, d\rho\, d\varphi\, d\theta}_{dx\,dy\,dz}$$

$$= \iiint_{\Omega'} \sqrt{\underbrace{\rho^2\cos^2\theta\,\text{sen}^2\varphi + \rho^2\text{sen}^2\theta\cdot\text{sen}^2\varphi}_{} + \rho^2\cos^2\varphi}\cdot \rho^2\text{sen}\,\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \iiint_{\Omega'} \sqrt{\rho^2\cdot\underbrace{\left[\text{sen}^2\varphi\,(\underbrace{\text{sen}^2\theta + \cos^2\theta}_{}) + \cos^2\varphi\right]}_{1}}\ \rho^2\cdot\text{sen}\,\varphi\, d\rho\, d\varphi\, d\theta$$

1

$$= \iiint_{\Omega'} \rho^3\cdot\text{sen}\,\varphi\, d\rho\, d\varphi\, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=\frac{\cos\varphi}{\text{sen}^2\varphi}} \rho^3\,\text{sen}\,\varphi\, d\rho\cdot d\varphi\, d\theta = \ ---$$

et cetera

[EXERCÍCIO]

ENTREGAR NA SEXTA ; LISTA 03 , n.º 14 ou n.º 16.