04/08/23 — AULA 14

MUDANÇA GERAL DE COORDENADAS NO $\mathbb{R}^3$:

Do mesmo modo que fizemos no caso $\mathbb{R}^2$, mostraremos uma dedução para determinar uma mudança de variáveis para $\iiint_\Omega f(x,y,z)\,dx\,dy\,dz$, para um sistema uvw, onde esta integral torne-se "mais simples".

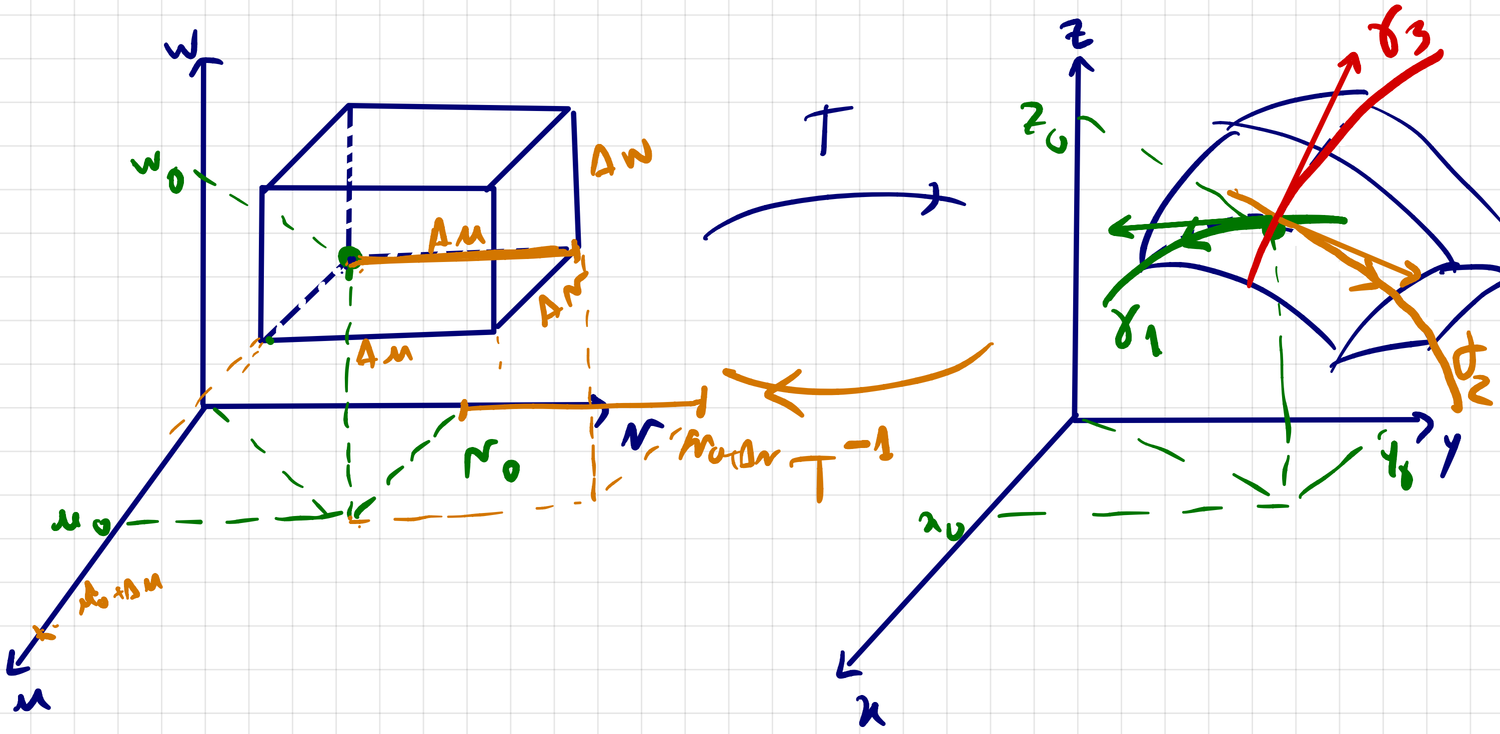
Seja $T: \Omega \subset \mathbb{R}^3 \to T(\Omega) := \Omega' \subset \mathbb{R}^3$ uma transformação da região Ω do espaço uvw para o espaço xyz, injetiva e de classe C^1, onde:

$$T(u,v,w) = (x,y,z); \text{ com}$$

$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w)$$

Seja A um paralelepípedo em $\Omega \subset \mathbb{R}^3$, de dimensões Δu, Δv e Δw. Seja (u_0, v_0, w_0) um ponto no vértice inferior esquerdo, c.f. esquema:

$$T(u_0, v_0, w_0) = (x_0, y_0, w_0)$$



Define as curvas:

$$\gamma_1 : [u_0, u_0+\Delta u] \to \mathbb{R}^3$$

$$\gamma_1(u) = \left(x(u, v_0, w_0), y(u, v_0, w_0), z(u, v_0, w_0) \right)$$

$$\gamma_2 : [v_0, v_0+\Delta v] \to \mathbb{R}^3$$

$$\gamma_2(v) = \left(x(u_0, v, w_0), y(u_0, v, w_0), z(u_0, v, w_0) \right)$$

$$\gamma_3 : [w_0, w_0+\Delta w] \to \mathbb{R}^3$$

$$\gamma_3(w) = \left(x(u_0, v_0, w), y(u_0, v_0, w), z(u_0, v_0, w) \right)$$

Os vetores tangentes em (x_0, y_0, z_0) serão dados por:

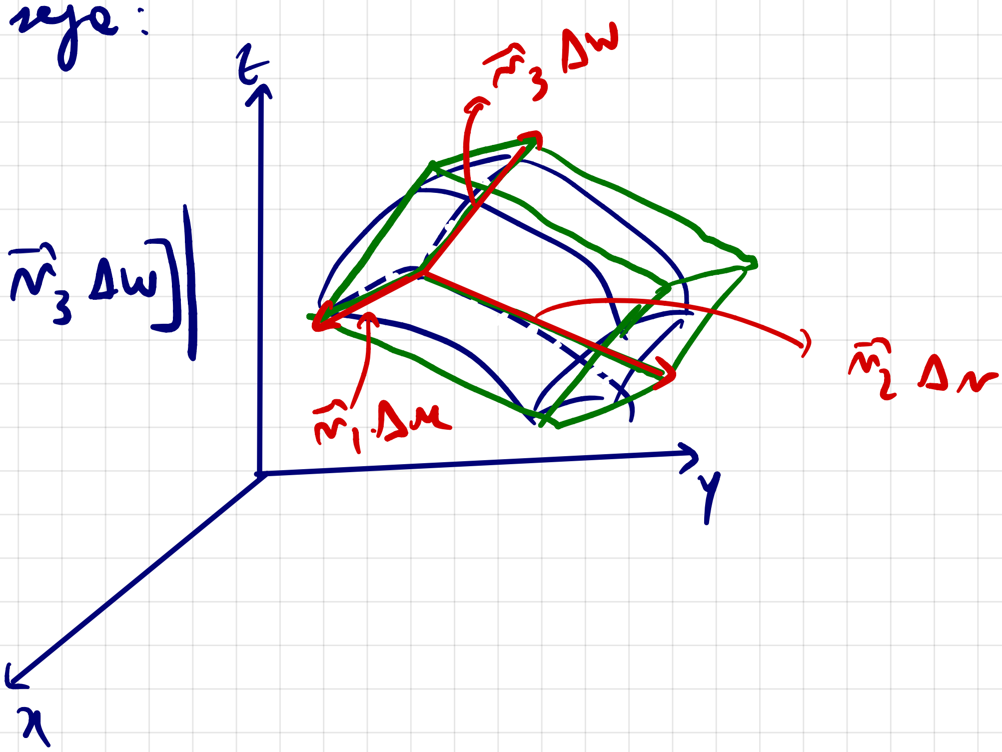
$$\vec{r}_1 = \gamma_1'(u_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0,w_0), \frac{\partial y}{\partial u}(u_0,v_0,w_0), \frac{\partial z}{\partial u}(u_0,v_0,w_0)\right)$$

$$\vec{r}_2 = \gamma_2'(v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0,w_0), \frac{\partial y}{\partial v}(u_0,v_0,w_0), \frac{\partial z}{\partial v}(u_0,v_0,w_0)\right)$$

$$\vec{r}_3 = \gamma_3'(w_0) = \left(\frac{\partial x}{\partial w}(u_0,v_0,w_0), \frac{\partial y}{\partial w}(u_0,v_0,w_0), \frac{\partial z}{\partial w}(u_0,v_0,w_0)\right)$$

O volume V do paralelepípedo de lados $\vec{r}_1 \cdot \Delta u$, $\vec{r}_2 \Delta v$ e $\vec{r}_3 \Delta w$, no espaço x, y, z, será dado pelo produto misto entre eles, ou seja:

$$V = \left| \left[\vec{r}_1 \Delta u, \vec{r}_2 \Delta v, \vec{r}_3 \Delta w\right] \right|$$



onde:

$$[\vec{v}_1 \Delta u, \vec{v}_2 \Delta v, \vec{v}_3 \Delta w] =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} \Delta u & \frac{\partial y}{\partial u} \Delta u & \frac{\partial z}{\partial u} \Delta u \\ \frac{\partial x}{\partial v} \Delta v & \frac{\partial y}{\partial v} \Delta v & \frac{\partial z}{\partial v} \Delta v \\ \frac{\partial x}{\partial w} \Delta w & \frac{\partial y}{\partial w} \Delta w & \frac{\partial z}{\partial w} \Delta w \end{vmatrix} =$$

$$= \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}}_{\det J(T)(u,v,w)} \Delta u \, \Delta v \, \Delta w$$

$$= \det J(T)(u,v,w) \cdot \Delta u \, \Delta v \, \Delta w$$

Assumindo $\Delta u, \Delta v, \Delta w > 0$, então:

$$V = |\det J(T)(u,v,w)| \cdot \Delta u \, \Delta v \, \Delta w$$

Seja $f(x,y,z)$ integrável em $\Omega \subset \mathbb{R}^3$; e considere $T: \Omega \subset \mathbb{R}^3 \to T(\Omega) = \Omega' \subset \mathbb{R}^3$ transformação C^1 e injetiva;

$$T(u,v,w) = (x,y,z).$$

Então, sendo P uma partição de Ω, o volume V do sólido será aproximado pela soma de Riemann:

$$\sum_{i=1}^{n} f(x_i,y_i,z_i) \cdot V_i =$$

$$= \sum_{i=1}^{n} f(x(u_i,v_i,w_i), y(u_i,v_i,w_i), z(u_i,v_i,w_i)) \cdot$$

$$\cdot |\det J(T)(u_i,v_i,w_i)| \cdot \Delta u_i \Delta v_i \Delta w_i$$

Então:

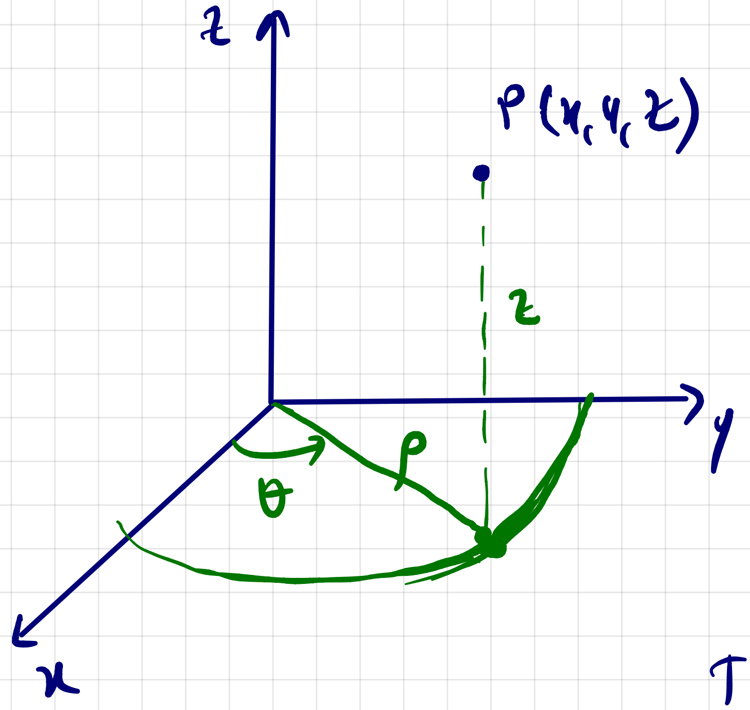
$$\underbrace{\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\, V_i}$$

$$\underbrace{= \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot |\det J(T)(u,v,w)|\,du\,dv\,dw},$$

que é a mesma fórmula do caso $\mathbb{R}^2$.

EXEMPLOS:

01) SISTEMA DE COORDENADAS CILÍNDRICAS:



Um ponto $P(x,y,z)$ terá as coordenadas cilíndricas dadas por

$$T: \begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = z \end{cases}$$

$$P(\rho\cos\theta, \rho\operatorname{sen}\theta, z)$$

$$J(T)(\rho,\theta,z) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho\operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \det J(T)(\rho,\theta,z) = \left| \begin{matrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho\operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{matrix} \right| \begin{matrix} \cos\theta & \operatorname{sen}\theta \\ -\rho\operatorname{sen}\theta & \rho\cos\theta \\ 0 & 0 \end{matrix}$$

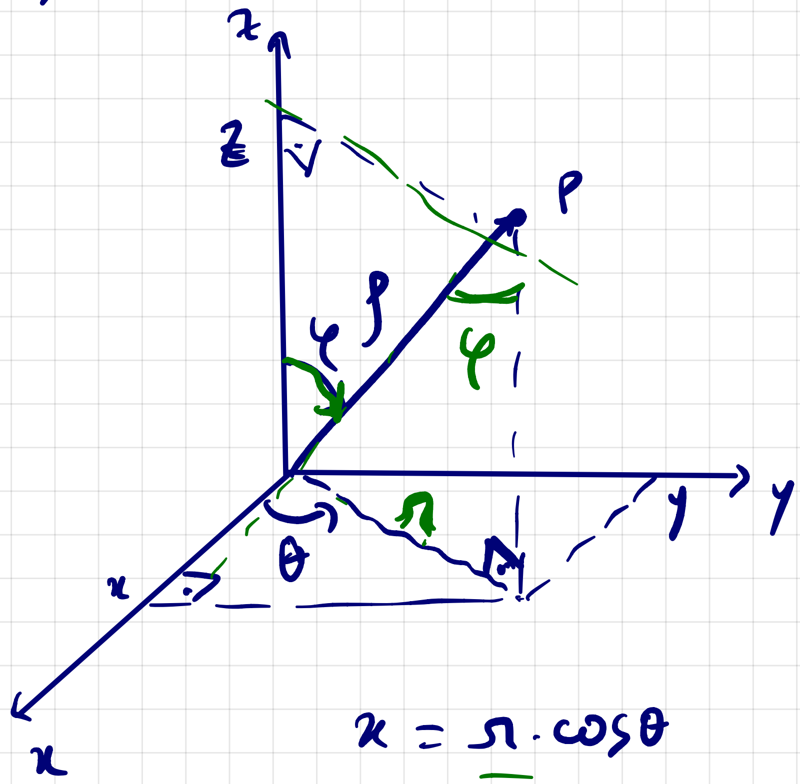
$$= \rho\cos^2\theta + \rho\operatorname{sen}^2\theta = \rho\,(\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1}) = \rho$$

Então:

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega'} f(\rho\cos\theta, \rho\operatorname{sen}\theta, z)\cdot\rho\cdot d\rho\,d\theta\,dz$$

MUDANÇA USANDO
COORDENADAS CILÍNDRICAS

02) SISTEMA DE COORDENADAS ESFÉRICAS:



$P(x, y, z) \rightarrow (\rho, \varphi, \theta)$

$0 \leq \varphi \leq 180^0$

$x = r \cdot \cos\theta$

$\text{sen}\,\varphi = \dfrac{r}{\rho}$

$\Downarrow$

$r = \rho \cdot \text{sen}\,\varphi$

$$\boxed{x = \rho \cdot \text{sen}\,\varphi \cdot \cos\theta}$$

$y = r \cdot \text{sen}\,\theta$

$$\boxed{y = \rho\, \text{sen}\,\varphi \cdot \text{sen}\,\theta}$$

$\cos\varphi = \dfrac{z}{\rho} \Rightarrow \boxed{z = \rho \cos\varphi}$

$P(x, y, z) = P(\rho\, \text{sen}\,\varphi \cos\theta, \rho\, \text{sen}\,\varphi\, \text{sen}\,\theta, \rho \cos\varphi)$

SIST. COORD. ESFÉRICAS

$$J(T) = ?$$

$$T(x,y,z) = \begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

$$J(T)(\rho,\varphi,\theta) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{sen}\varphi \cos\theta & \operatorname{sen}\varphi \operatorname{sen}\theta & \cos\varphi \\ \rho \cos\varphi \cos\theta & \rho \cos\varphi \operatorname{sen}\theta & -\rho \operatorname{sen}\varphi \\ -\rho \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \operatorname{sen}\varphi \cos\theta & 0 \end{bmatrix}$$

$$\det J(T)(\rho,\varphi,\theta) =$$

$$= \left| \begin{array}{ccc|cc} \operatorname{sen}\varphi \cos\theta & \operatorname{sen}\varphi \operatorname{sen}\theta & \cos\varphi & \operatorname{sen}\varphi \cos\theta & \operatorname{sen}\varphi \operatorname{sen}\theta \\ \rho \cos\varphi \cos\theta & \rho \cos\varphi \operatorname{sen}\theta & -\rho \operatorname{sen}\varphi & \rho \cos\varphi \cos\theta & \rho \cos\varphi \operatorname{sen}\theta \\ -\rho \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \operatorname{sen}\varphi \cos\theta & 0 & -\rho \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \operatorname{sen}\varphi \cos\theta \end{array} \right.$$

$$= 0 + \rho^2 \operatorname{sen}^3\varphi \operatorname{sen}^2\theta + \rho^2 \cos^2\varphi \operatorname{sen}\varphi \cos^2\theta$$

$$+ \rho^2 \cos^2\varphi \operatorname{sen}\varphi \operatorname{sen}^2\theta + \rho^2 \operatorname{sen}^3\varphi \cos^2\theta - 0$$

$$= \rho^2 \operatorname{sen}\varphi \Big[\operatorname{sen}^2\varphi \operatorname{sen}^2\theta + \cos^2\varphi . \cos^2\theta +$$

$$+ \cos^2\varphi . \operatorname{sen}^2\theta + \operatorname{sen}^2\varphi . \cos^2\theta \Big]$$

$$= \rho^2 \operatorname{sen}\varphi \underbrace{\Big[\operatorname{sen}^2\varphi \big(\underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{1}\big) + \cos^2\varphi \big(\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{1}\big) \Big]}_{1}$$

$$= \rho^2 \operatorname{sen}\varphi \qquad \Longrightarrow |\det J| = \rho^2 \operatorname{sen}\varphi$$

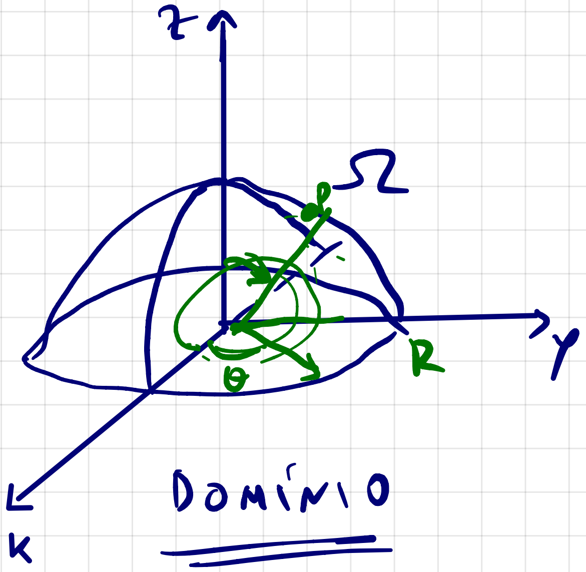
Então, a mudança do sistema retangular ao esférico será:

$$\iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz =$$

$$= \iiint_{\Omega'} f(\rho \operatorname{sen}\varphi \cos\theta, \rho \operatorname{sen}\varphi \operatorname{sen}\theta, \rho \cos\varphi)\, \rho^2 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$$

Ex.: Deduzir a fórmula do volume V de uma esfera, usando coordenadas esféricas.

SOLUÇÃO:



DOMÍNIO

Por simetria:

$$V_{esfera} = 2 \cdot \iiint_{\Omega} dV$$

$$\Omega' \quad \boxed{\begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq R \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{array}}$$

$$V_{esfera} = 2 \cdot \iiint_{\Omega'} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta =$$

COORD. ESFÉRICAS

$$= 2 \cdot \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= 2 \cdot \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi \cdot \int_{\rho=0}^{\rho=R} \rho^2 \, d\rho$$

$$2\cdot(2\pi)\,(-\cos\varphi)\Big|_0^{\frac{\pi}{2}}\cdot\frac{\rho^3}{3}\Big|_0^R =$$

$$4\pi\cdot\underbrace{\left(-\cos\frac{\pi}{2}+\cos 0\right)}_{=1}\cdot\frac{R^3}{3} = \frac{4\pi R^3}{3}$$