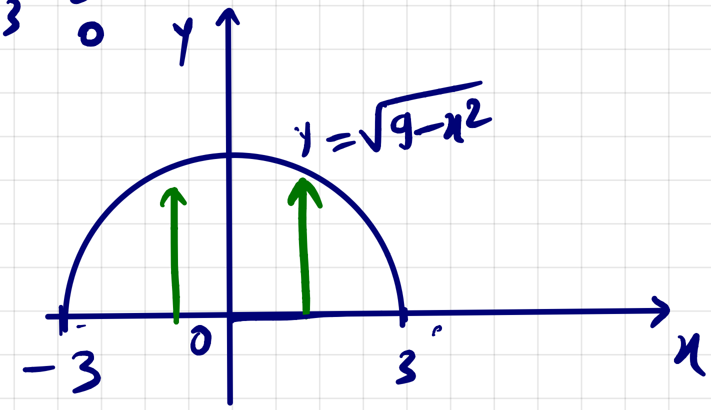
$\boxed{12/07/23 \text{ - AULA 09}}$

Da aula passada:

03) $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \operatorname{sen}(x^2+y^2)\,dy\,dx.$$



$$y=+\sqrt{9-x^2}$$

$$y^2=9-x^2$$

$$x^2+y^2=9$$

$$\rho^2 = x^2+y^2$$

Assim, teremos:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \operatorname{sen}(x^2+y^2)\,dy\,dx. = \int_{\theta=0}^{\theta=\pi}\left(\frac{1}{2}\int_{\rho=0}^{\rho=3} \operatorname{sen}\rho^2\, 2\rho\, d\rho\right) d\theta =$$

$$\int \operatorname{sen} u\, du$$

$$u=\rho^2 \Rightarrow du = 2\rho\, d\rho$$

$$\int_{\theta=0}^{\theta=\pi} d\theta \cdot \frac{1}{2}\left(-\cos\rho^2\right)\Big|_{\rho=0}^{\rho=3} =$$

$$\theta\Big|_0^{\pi} \cdot \frac{1}{2}\left(-\cos\rho^2\right)\Big|_0^{3} = (\pi-0)\cdot\frac{1}{2}\left(-\cos(3)^3+\underbrace{\cos 0}_{1}\right)$$

$$= \pi\cdot\left(-\cos 27+1\right) = \pi\cdot\left(1-\cos 27\right)$$

MUDANÇA GERAL DE VARIÁVEIS NO $\mathbb{R}^2$.

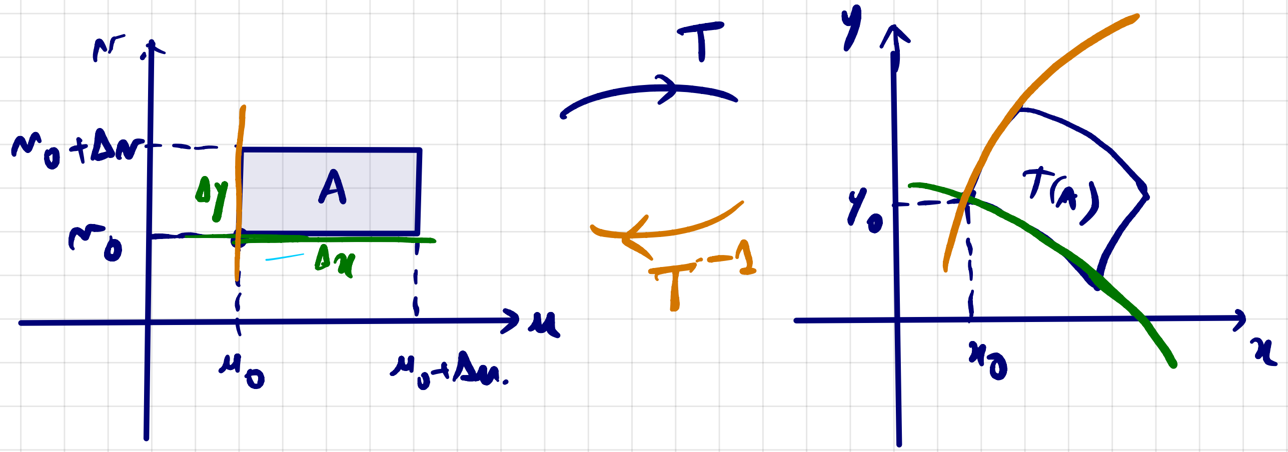
Dada $\iint_{\Omega} f(x,y)\,dx\,dy$, vamos determinar um procedimento geral de mudança de variáveis com o intuito de transformar a integral dada numa integral "mais simples". De fato, já exploramos na seção anterior com o estudo de coordenadas polares. No entanto, ali só podíamos lidar com integrais onde a região de integração envolvesse arcos de circunferência.

Seja $T : \Omega \subset \mathbb{R}^2 \to T(\Omega) \subset \mathbb{R}^2$ uma transformação do plano uv para o plano xy, injetiva e de classe C^1 (T é de classe C^1 se as derivadas parciais forem contínuas).

Com isso, $\exists\, T^{-1} : T(\Omega) := \Omega' \to \Omega$ transformação inversa.

$$T : \Omega \to T(\Omega)$$
$$T(u,v) = (x,y) \quad ; \text{ onde}$$
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Seja A um retângulo de dimensões Δu e Δv no plano uv. Seja (u_0,v_0) coordenada do ponto no canto inferior esquerdo de A.



$$T(u_0, v_0) = (x_0, y_0)$$

Sejam $\gamma_1 : [u_0, u_0+\Delta u] \longrightarrow \mathbb{R}^2$

$$\gamma_1(u) = T(u, v_0) = (x(u,v_0),\ y(u,v_0))$$

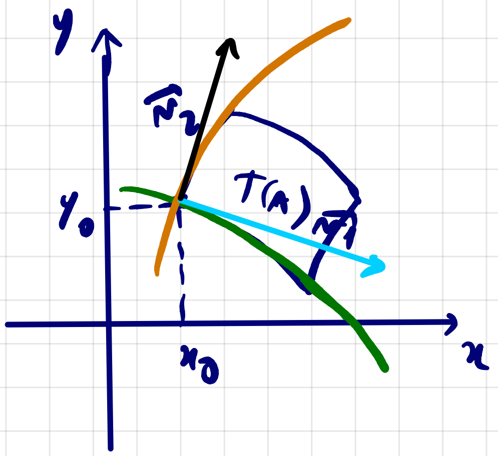
e

$$\gamma_2 : [v_0, v_0+\Delta v] \longrightarrow \mathbb{R}^2$$

$$\gamma_2(v) = T(u_0, v) = (x(u_0,v),\ y(u_0,v))$$

parametrizações das curvas que suportam os lados de $T(A)$, passando por (x_0, y_0).

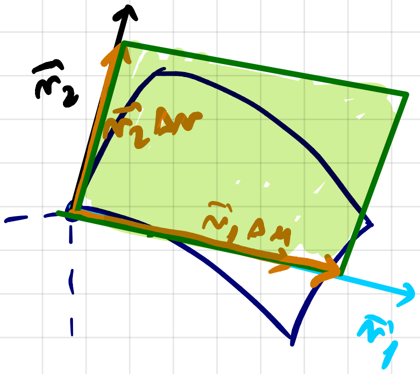
Os vetores tangentes à γ_1 e γ_2, passando por (x_0, y_0) serão:



$$\vec{v_1} = \gamma_1'(u_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0),\ \frac{\partial y}{\partial u}(u_0,v_0)\right)$$

$$\vec{v_2} = \gamma_2'(v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0),\ \frac{\partial y}{\partial v}(u_0,v_0)\right)$$

A área do paralelogramo determinado pelos vetores $\vec{v_1} \cdot \Delta u$ e $\vec{v_2}\, \Delta v$ será dada por:



$$A = \| \vec{r}_1 \Delta u \times \vec{r}_2 \Delta v \| = \| \vec{r}_1 \times \vec{r}_2 \| . \Delta u . \Delta v \; ;$$

TOMAMOS $\Delta u, \Delta v > 0$

$$\| \alpha \vec{u} \| = |\alpha| . \| \vec{u} \|$$

onde:

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} \cdot \vec{k} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \cdot \vec{k} - 0\vec{j} - 0\vec{i}$$

$$= \left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \right) \vec{k}$$

$$\Longrightarrow \| \vec{r}_1 \times \vec{r}_2 \| = \left| \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \right|$$

Logo:

$$A = \left| \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \right| . \Delta u . \Delta v$$

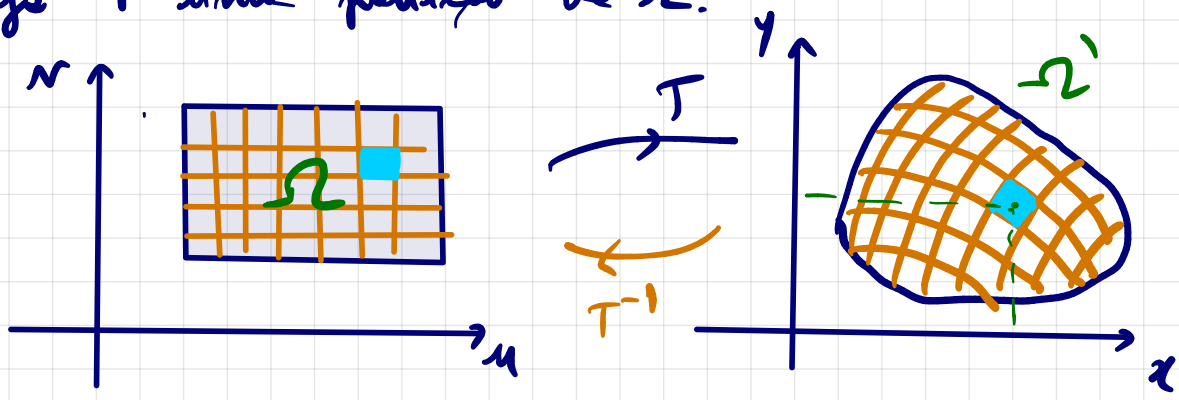
$$= \left| \; \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{} \; \right| . \Delta u . \Delta v = |\det J(T)(u,v)| . \Delta u \, \Delta v$$

$:= J(T)_{(u,v)}$ — MATRIZ JACOBIANA DA $T(u,v)$

Seja $z = f(x,y)$ definida em $\Omega \subset \mathbb{R}^2$ e considere $T(u,v) = (x,y)$ uma transf. injetiva, C^1, tal que

$$x = x(u,v)$$
$$y = y(u,v)$$

Seja P uma partição de Ω.



A área de cada bloco (retângulo da partição), mediante T, será aproximada pela área do paralelogramo, c.f. deduzido acima. Ou seja; $\forall (x_i, y_i) \in T(\Omega) := \Omega'$

$$\sum_{i=1}^{n} f(x_i, y_i)\, A_i = \sum_{i=1}^{n} f(x(u_i,v_i), y(u_i,v_i)) \cdot |\det(J(T)(u_i,v_i))| \cdot \Delta u_i \Delta v_i ,$$

uma soma de Riemann.

Ou seja, obtemos que, passando o limite com $n \to \infty$ (i.e; $\|P\| \to 0$):

$$\boxed{\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega'} f(x(u,v), y(u,v))\, |\det J(T)(u,v)| \cdot du\,dv}$$

(FÓRMULA DA MUDANÇA DE VARIÁVEL)

Obs.: Sem "saber", isso já era feito no CÁLCULO II:

EX.: $\displaystyle\int_0^{\pi} \operatorname{sen} 2x \, dx$

$T: \mathbb{R} \to \mathbb{R}$

$T(x) = 2x \overset{=}{} u \qquad J(T)(u) = f'(x) = 2 \qquad \frac{\partial u}{\partial x}$

$x = 0 \Rightarrow T(x) = 0$

$x = \pi \rightarrow T(x) = 2\pi$

$$\Omega = [0, \pi] \xrightarrow{T} \Omega' = [0, 2\pi]$$

$$\Rightarrow \int_0^{\pi} \operatorname{sen} 2x \, dx = \int_0^{2\pi} \operatorname{sen} u \, . \, |\det \underbrace{J(T)(u)}_{2}| \, . \, du =$$

$$= \int_0^{2\pi} \operatorname{sen} u \, . \, 2 \, du = 2 \int_0^{2\pi} \operatorname{sen} u \, . \, du = 2 \, . \, (-\cos u)\Big|_0^{2\pi}$$

$$= 2 \, . \, (-\underbrace{\cos 2\pi} + \cos 0) = 2 \, . \, (+1 + 1) = 4$$

EXEMPLOS:

01) A mudança em coordenadas polares:

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad T: (\rho, \theta) \rightarrow (x, y)$$

$$\det J(T)(\rho, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} =$$

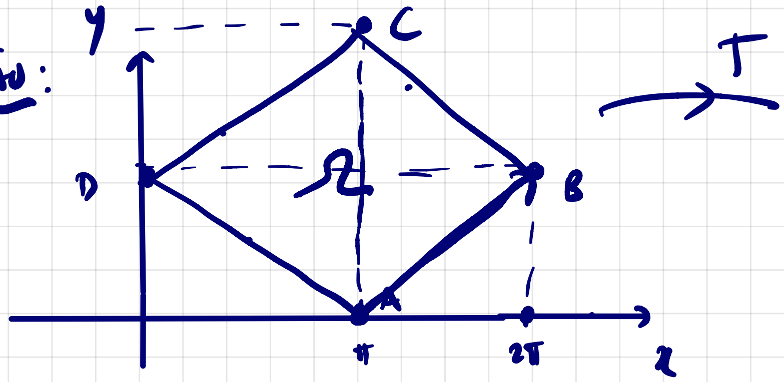
$$= \begin{vmatrix} \cos\theta & -\rho \operatorname{sen}\theta \\ \operatorname{sen}\theta & \rho\cos\theta \end{vmatrix} = \rho \, . \, \cos^2\theta - (-\rho \operatorname{sen}^2\theta) =$$

$$= \rho(\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1}) = \rho$$

Assim: $\displaystyle\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega'} f(\rho\cos\theta, \rho\operatorname{sen}\theta)\cdot \underline{\underline{\rho}}\cdot d\rho\, d\theta$

02) Calcule $\displaystyle\iint_{\Omega} (x-y)^2 \cdot \operatorname{sen}(x+y)\,dx\,dy$, onde Ω é o paralelogramo de vértices $A(\pi,0)$; $B(2\pi,\pi)$; $C(\pi,2\pi)$ e $D(0,\pi)$.

Solução:



Escreva $\begin{cases} u = x - y \\ v = x + y \end{cases}$ $+$

$u + v = 2x \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$

$\rightsquigarrow y = v - x = v - \frac{1}{2}u - \frac{1}{2}v$

$$\boxed{y = -\frac{1}{2}u + \frac{1}{2}v}$$

$$T(u,v) = (x,y) = \left(\frac{1}{2}u + \frac{1}{2}v, -\frac{1}{2}u + \frac{1}{2}v\right)$$

Disso:

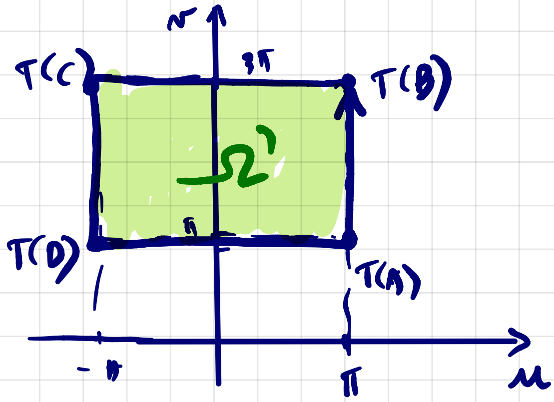
$A(\pi, 0) \Rightarrow T(A) = (\pi - 0, \pi + 0) = (\pi, \pi)$

$(x-y, x+y)$

$B(2\pi, \pi) \Rightarrow T(B) = (2\pi - \pi, 2\pi + \pi) = (\pi, 3\pi)$

$C(\pi, 2\pi) \to T(C) = (-\pi, 3\pi)$

$D(0, \pi) \Rightarrow T(D) = (-\pi, \pi)$



$$\det J(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

Assim:

$$\iint_{\Omega} (\underbrace{x-y}_{u})^2 \cdot \operatorname{sen}(\underbrace{x+y}_{v})\, dx\, dy = \iint_{\Omega'} u^2 \cdot \operatorname{sen} v \cdot |\underbrace{\det J(T)(u,v)}_{\frac{1}{2}}| \cdot du\, dv$$

$$= \int_{v=\pi}^{v=3\pi} \int_{u=-\pi}^{u=\pi} u^2 \operatorname{sen} v \cdot \frac{1}{2} \cdot du\, dv = \frac{1}{2} \int_{v=\pi}^{v=3\pi} \operatorname{sen} v\, dv \int_{u=-\pi}^{u=\pi} u^2\, du =$$

$$= \frac{1}{2} (-\cos v)\Big|_{\pi}^{3\pi} \cdot \frac{u^3}{3}\Big|_{-\pi}^{\pi} = \frac{1}{2}(-\cos 3\pi + \cos \pi) \cdot \left(\frac{\pi^3}{3} - \left(-\frac{\pi^3}{3}\right)\right)$$

$$= \frac{1}{2} \underbrace{\left(-(-1) + (-1)\right)}_{=0} \cdot \frac{2\pi^3}{3} = 0$$