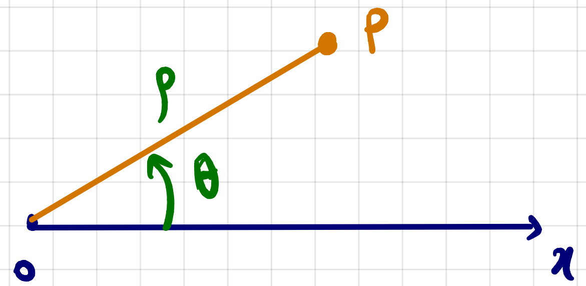
AULA 08 — 07/07/23

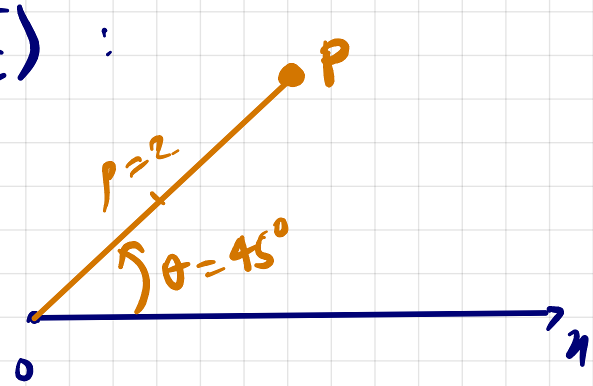
INTEGRAIS EM COORDENADAS POLARES:

Dada $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, quando Ω envolve arcos de circunferência, é conveniente usar o sistema de coordenadas polares (ρ, θ) ao invés do retangular (x, y).

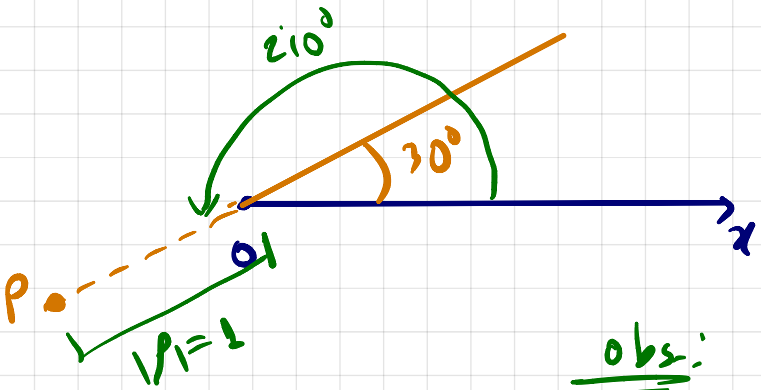
O sistema polar $(\rho\,\theta)$ consiste em um eixo real OX orientado, e um ponto P no plano, que será chamado de plano polar, fica determinado a partir de um raio ρ e um ângulo θ; onde θ é marcado no sentido anti-horário, como na Trigonometria:



Ex.: $P\left(2, \frac{\pi}{4}\right)$:



Obs.: se $\rho < 0$, então indicamos P no prolongamento de $|\rho|$. Ex.: $P(-1, 30°)$

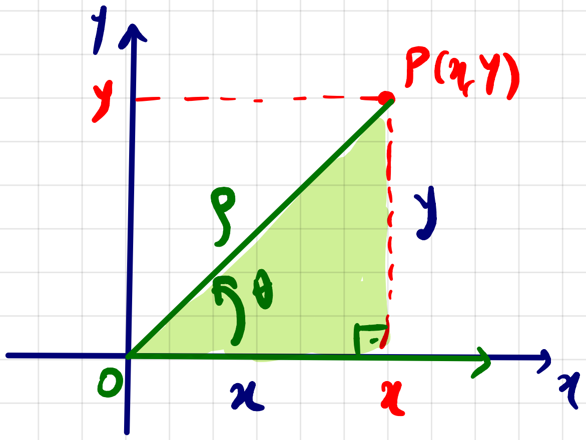


obs: PODERÍAMOS ESCREVER:

$$P(-1, 30°) = P(1, 210°)$$
$$= P(1, -150°)$$

RELAÇÕES DE CONVERSÃO DO SISTEMA POLAR ⟷ RETANGULAR:

Seja $P(x, y)$ no sistema retangular.
Considere o sist. polar (ρ, θ) com origem em comum ao sist. retangular e eixos real e Ox coincidindo.



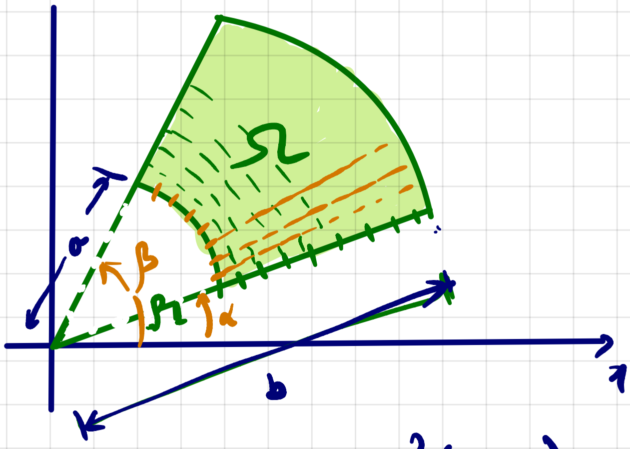
Temos as seguintes relações:

$$\boxed{\rho^2 = x^2 + y^2}$$

$$\cos\theta = \frac{x}{\rho} \Rightarrow \boxed{x = \rho \cdot \cos\theta}$$

$$\operatorname{sen}\theta = \frac{y}{\rho} \Rightarrow \boxed{y = \rho \cdot \operatorname{sen}\theta}$$

Seja $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ função integrável; e Ω uma <u>região polar</u> (formada por semicírculos)



Queremos determinar

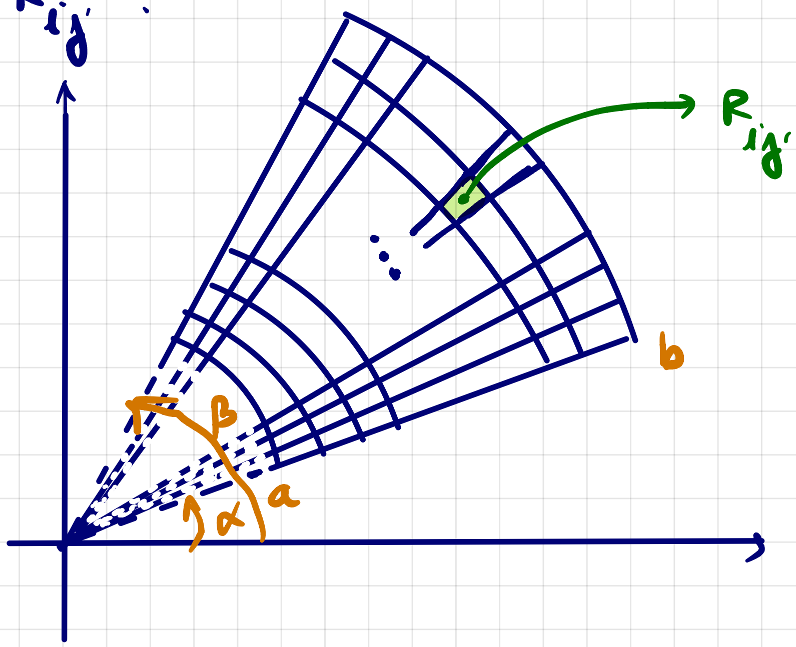
$$\iint_{\Omega} f(x,y) \underbrace{dx\,dy}_{dA}$$

$$\Omega = \{(\rho,\theta): a \leq \rho \leq b \,;\, \alpha \leq \theta \leq \beta\}$$

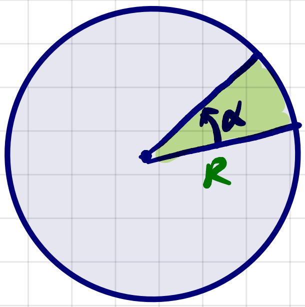
Seja $P = P_1 \times P_2$ uma partição regular de Ω, onde P_1 divide $[a,b]$ em m subintervalos de mesmo comprimento $\Delta\rho = \dfrac{b-a}{m}$;

e P_2 divide $[\alpha,\beta]$ em n subintervalos de mesmo comprimento $\Delta\theta = \dfrac{\beta-\alpha}{n}$.

Esta partição $P = P_1 \times P_2$ divide Ω em subregiões (subblocos) R_{ij}:



A área de cada subbloco R_{ij} é a área de um setor circular.



$A_{setor} = ?$

$\pi R^2 \;\text{—}\; 2\pi \text{ rad}$

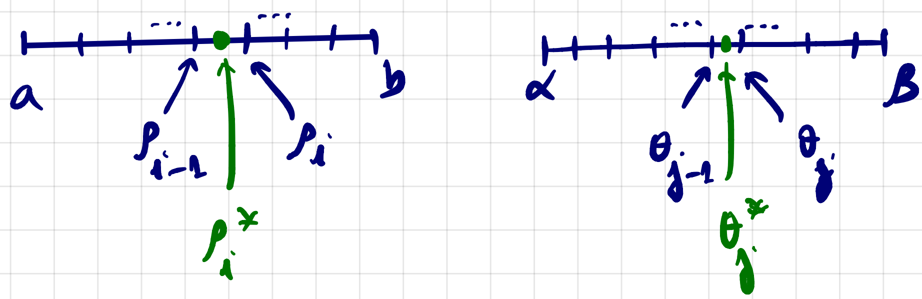
$A \;\text{—}\; \alpha$

$2\cancel{\pi} \cdot A = \alpha \cdot \cancel{\pi} R^2$

$$\boxed{A = \frac{1}{2} \cdot \alpha \cdot R^2}$$

Seja (ρ_i^*, θ_j^*) o ponto médio em R_{ij} ; i.e.,

$$\rho_i^* = \frac{\rho_i + \rho_{i-1}}{2} \quad ; \quad \theta_j^* = \frac{\theta_j + \theta_{j-1}}{2}$$

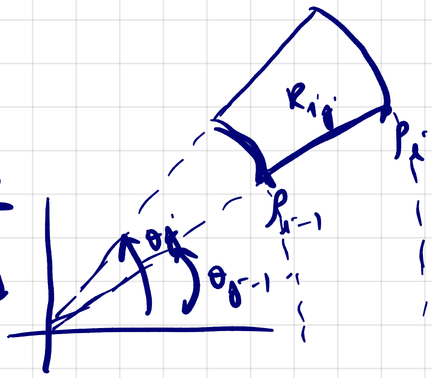


A área de R_{ij} será:

$$A_i = \frac{1}{2} \cdot (\theta_j - \theta_{j-1}) \cdot \rho_i^2 - \frac{1}{2} (\theta_j - \theta_{j-1}) \cdot \rho_{i-1}^2$$

ÁREA SETOR CIRCULAR

(DIFERENÇA)



$$= \frac{1}{2} \cdot (\theta_j - \theta_{j-1}) \cdot \left[\rho_i^2 - \rho_{i-1}^2 \right]$$

$$= \underbrace{\frac{1}{2}\overbrace{(\theta_j - \theta_{j-1})}^{\Delta\theta} \cdot \overbrace{(\rho_i - \rho_{i-1})}^{\Delta\rho} \cdot (\rho_i + \rho_{i-1})}_{\rho_i^*}$$

$$\Rightarrow \boxed{A_i = \rho_i^* \Delta\rho . \Delta\theta}$$

Seja $D^* = (P, f(x_i^*, y_j^*)) = (P, f(\underbrace{\rho_i^* \cos\theta_j^*}_{x_i^*}, \underbrace{\rho_i^* \operatorname{sen}\theta_j^*}_{y_j^*}))$

a decomposição de Ω.

Temos, com isso, a soma de Riemann:

$$\Sigma(D^*, f(\rho_i^* \cos\theta_j^*, \rho_i^* \operatorname{sen}\theta_j^*)).$$

Então,

$$\iint_\Omega f(x,y)\,dx\,dy = \lim_{\|D\|\to 0} \Sigma(D^*, f(\rho_i^* \cos\theta_j^*, \rho_i^* \operatorname{sen}\theta_j^*))$$

$$= \lim_{\|D\|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\rho_i^* \cos\theta_j^*, \rho_i^* \operatorname{sen}\theta_j^*) . \underbrace{\operatorname{Vol}(R_{ij})}_{A_i}$$

$$= \lim_{\|D\|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\rho_i^* \cos\theta_j^*, \rho_i^* \operatorname{sen}\theta_j^*) . A_i$$

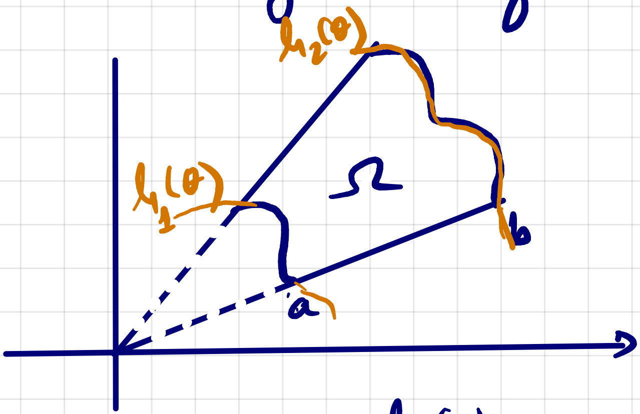
$$= \lim_{\|D\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\rho_i^* \cos\theta_j^*, \rho_i^* \operatorname{sen}\theta_j^*) \cdot \rho_i^* \cdot \Delta\rho \, \Delta\theta$$

$$\int_{\rho=a}^{\rho=b} \int_{\theta=\alpha}^{\theta=\beta} f(\rho\cos\theta, \rho \operatorname{sen}\theta) \cdot \rho \cdot d\rho \, d\theta$$

Portanto, mostramos que:

$$\iint_{\Omega} f(x,y) \, dx \, dy = \int_{\rho=a}^{\rho=b} \int_{\theta=\alpha}^{\theta=\beta} f(\rho\cos\theta, \rho \operatorname{sen}\theta) \cdot \rho \, d\rho \, d\theta$$

<u>Obs.:</u> podemos generalizar a região como segue:

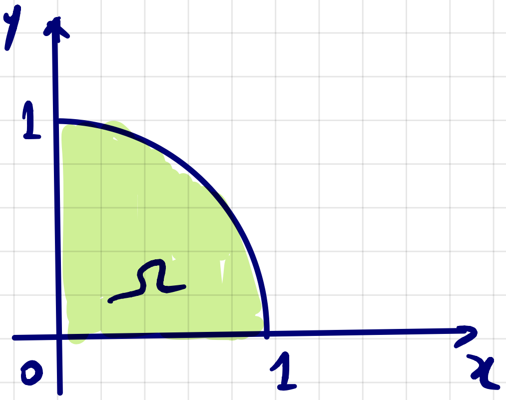


$$\iint_{\Omega} f = \int_{\rho=a}^{\rho=b} \int_{\theta=h_1(\theta)}^{\theta=h_2(\theta)} f(\rho\cos\theta, \rho \operatorname{sen}\theta) \cdot \rho \, d\rho \, d\theta \, .$$

Ex.:

01) Calcule $\iint_{\Omega} \frac{x}{\sqrt{x^2+y^2}}\, dA$, onde Ω é a região do 1º quadrante, limitada pela circunferência $x^2+y^2=1$ e pelos eixos coordenados.

Solução:



$$\iint_{\Omega} \frac{x}{\sqrt{x^2+y^2}}\, dx\,dy = \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{\cancel{\rho}\cos\theta}{\cancel{\rho}} \cdot \rho\, d\rho\, d\theta =$$

$$\begin{cases} x = \rho\cos\theta \\ \rho^2 = x^2+y^2 \end{cases}$$

$\hookrightarrow \rho = \sqrt{x^2+y^2}$

$$= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta \cdot \rho\, d\rho\, d\theta = \int_{\rho=0}^{\rho=1} \rho\, d\rho \cdot \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta\, d\theta$$

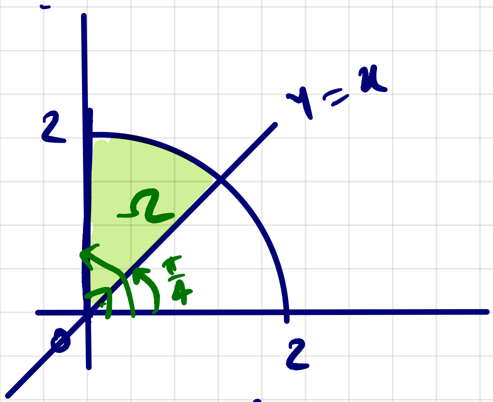
$$= \left.\frac{\rho^2}{2}\right|_0^1 \cdot \left.\operatorname{sen}\theta\right|_0^{\frac{\pi}{2}} = \left(\frac{1}{2} - 0\right) \cdot \left(\underbrace{\operatorname{sen}\frac{\pi}{2}}_{1} - \underbrace{\operatorname{sen} 0}_{0}\right)$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2}$$

02) Calcule $\iint_\Omega (2x-y)\,dx\,dy$, onde Ω é

a região do 1º quadrante limitada pelo círculo $x^2+y^2=4$ e as retas $x=0$ e $y=x$.

Solução:



$$\iint_\Omega (2x-y)\,dx\,dy = \int_{\rho=0}^{\rho=2}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} (2.\rho\cos\theta - \rho\,\text{sen}\theta)\rho\, d\rho\, d\theta$$

$$= \int_{\rho=0}^{\rho=2}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} 2\rho^2\cos\theta\, d\rho\, d\theta - \int_{\rho=0}^{\rho=2}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \rho^2\,\text{sen}\,\theta\, d\rho\, d\theta$$

$$\int_{\rho=0}^{\rho=2} 2\rho^2 d\rho . \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cos\theta\, d\theta - \int_{\rho=0}^{\rho=2} \rho^2 d\rho . \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \text{sen}\,\theta\, d\theta =$$

$$= \frac{2\rho^3}{3}\Big|_0^2 \cdot \operatorname{sen}\theta\Big|_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} - \frac{\rho^3}{3}\Big|_0^2 \cdot (-\cos\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{16}{3} - 0\right)\cdot\left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\frac{\pi}{4}\right) - \frac{8}{3}\cdot\left(-\cos\frac{\pi}{2} + \cos\frac{\pi}{4}\right)$$

$$= \frac{16}{3}\cdot\left(1 - \frac{\sqrt{2}}{2}\right) - \frac{8}{3}\cdot\left(0 + \frac{\sqrt{2}}{2}\right)$$

$$= \frac{16}{3} - \frac{16\sqrt{2}}{6} - \frac{8\sqrt{2}}{6} = \frac{16}{3} - \frac{24\sqrt{2}}{6} = \frac{16}{3} - 4\sqrt{2}$$

03) $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \operatorname{sen}(x^2+y^2)\,dy\,dx.$ (EXERCÍCIO).