27/07/23 – AULA 13

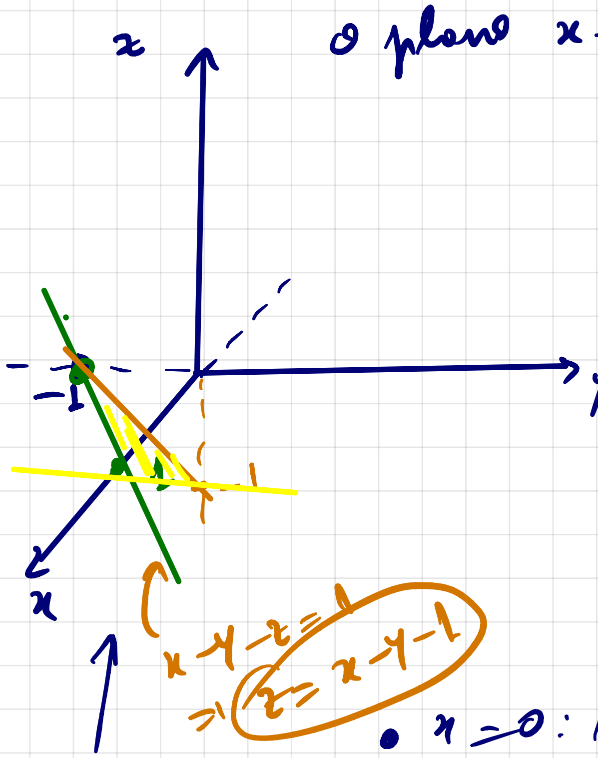
(AULA EXTRA DE EXERCÍCIOS)

LISTA 03.

06 – c)

$w = f(x,y,z) = z$

$\int_D f = ?$ D – tetraedro limitado pelos planos coord. e o plano $x - y - z = 1$.

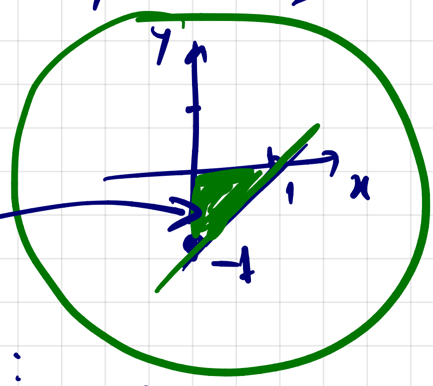


traços: • plano xy: ($z = 0$)

$x - y = -1$

$y = x - 1$

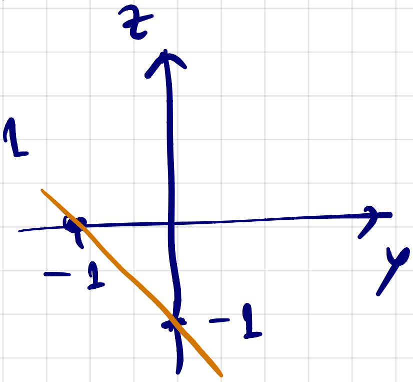
R



• $x = 0$: plano yz:

$-y - z = 1$

$y = -z - 1$

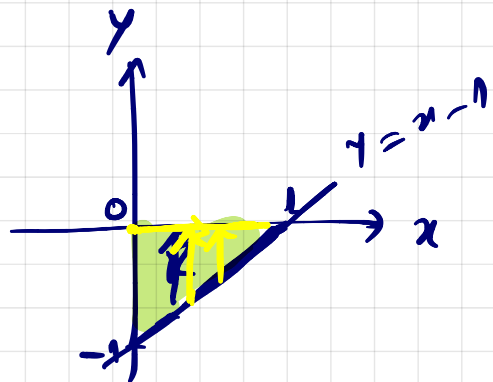


NOTAMOS QUE O TETRAEDRO FICA NO 5º OCTANTE.

Então, teremos:

$$\iiint_{\Omega} \underbrace{f(x,y,z)}_{z}\, dx\,dy\,dz =$$

$$= \iint_R dy\,dx \int_{z=x-y-1}^{z=0} f(x,y,z)\,dz =$$



$$= \int_{x=0}^{x=1} \int_{y=x-1}^{y=0} dy\, dx \int_{z=x-y-1}^{z=0} z \cdot dz =$$

$$= \int_{x=0}^{x=1} \int_{y=x-1}^{y=0} dy\, dx . \left. \frac{z^2}{2} \right|_{z=x-y-1}^{z=0} =$$

$$\int_{x=0}^{x=1} \int_{y=x-1}^{y=0} \left(0 - \frac{(x-y-1)^2}{2} \right) dy\, dx$$

$$= \cancel{-}+\frac{1}{2} \int_{x=0}^{x=1} \left(\int_{y=x-1}^{y=0} - (\underbrace{x-y-1}_{r})^2 dy \right) dx = +\frac{1}{2} \int_{x=0}^{x=1} \left. \frac{(x-y-1)^3}{3} \right|_{y=x-1}^{y=0} dx$$

r^2

$dr = -dy$

$$= \frac{1}{2} \int_{x=0}^{x=1} \left[\frac{(x-1)^3}{3} - \frac{(\cancel{x} - \cancel{x} + \cancel{1} - \cancel{1})^3}{3} \right] dx$$

$$= \frac{1}{6} \cdot \int_0^1 (x-1)^3 dx = \frac{1}{6} \left. \frac{(x-1)^4}{4} \right|_0^1 =$$

$\int r^3 dr$

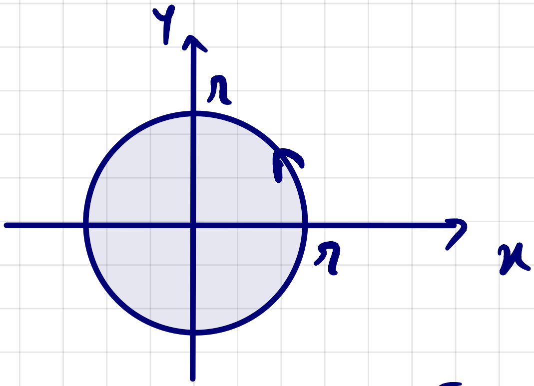
$dr = dx$

$$= \frac{1}{6}\cdot\frac{1}{4}\cdot\left[0^4-(-1)^4\right] = -\frac{1}{24}$$

LISTA 02 |

08) a.

$$\iint\limits_{x^2+y^2\le r^2} e^{-x^2-y^2}\,dx\,dy$$



$x^2+y^2=\rho^2$

ρ: 0 até r

θ: 0 até 2π

$$\iint\limits_{x^2+y^2\le r^2} e^{-x^2-y^2}\,dx\,dy = -\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=r} e^{-\rho^2}(-2)\underbrace{\rho\,d\rho\,d\theta}_{dx\,dy} =$$

$$\int e^{r}\,dr$$

$$r=-\rho^2 \rightarrow dr = -2\rho\,d\rho$$

$$= -\frac{1}{2}\int_{\theta=0}^{\theta=2\pi} e^{-\rho^2}\Big|_0^{r}\,d\theta = -\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\left(e^{-r^2}-\underbrace{e^{0}}_{=1}\right)d\theta$$

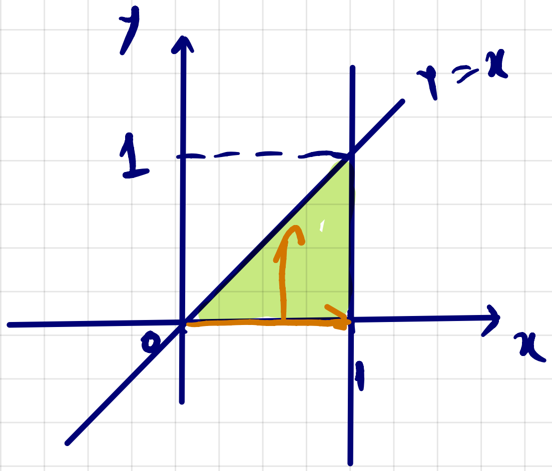
$$-\frac{e^{-r^2}-1}{2}\int_{\theta=0}^{\theta=2\pi} d\theta = \frac{1-e^{-r^2}}{2}\cdot \theta\Big|_0^{2\pi}$$

$$= \frac{1-e^{-r^2}}{\cancel{2}}\cdot \cancel{2}\pi = \underline{(1-e^{-r^2})\cdot\pi}.$$

EXTRA: Calcule a integral:

$$\int_0^1\int_y^1 \sqrt{1+x^2}\,dx\,dy$$

solução: $\displaystyle\int_{y=0}^{y=1}\int_{x=y}^{x=1}\sqrt{1+x^2}\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=x}\sqrt{1+x^2}\,dy\,dx$



$$= \int_{x=0}^{x=1}\sqrt{1+x^2}\int_{y=0}^{y=x} dy\; dx =$$

$$= \int_{x=0}^{x=1}\sqrt{1+x^2}\cdot y\Big|_{y=0}^{y=x} dx = \int_{x=0}^{x=1}\sqrt{1+x^2}\,(x-0)\,dx$$

$$= \frac{1}{2}\int_0^1 (1+x^2)^{\frac{1}{2}} \cdot 2x\, dx = \frac{1}{\cancel{2}}\, \frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}\Bigg|_0^1 =$$

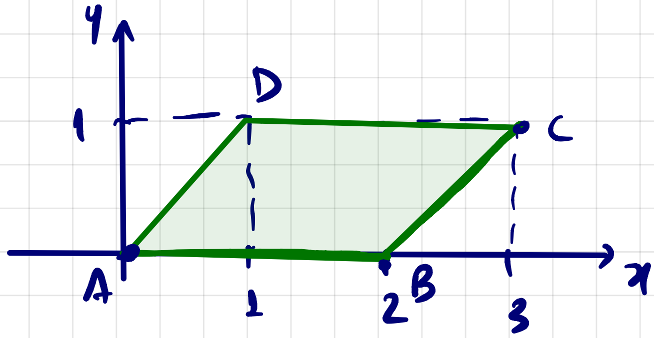
$$\int u^{\frac{1}{2}}\, du$$

$$u = 1+x^2 \Rightarrow du = 2x\,dx$$

$$= \frac{1}{3}\,(1+x^2)^{\frac{3}{2}}\Bigg|_0^1 = \frac{1}{3}\left[(2)^{\frac{3}{2}} - 1\right]$$

$$= \frac{\sqrt{2^3}-1}{3} = \frac{2\sqrt{2}-1}{3}$$

EXTRA: Calcule $\displaystyle\int_{\Omega} (x^2-y^2)\cos(x-y)\,dx\,dy$, onde Ω é o domínio dado pelo paralelogramo da figura:



Solução:

A(0,0) ; B(2,0) ; C(3,1) ; D(1,1)

Note que $x^2-y^2 = (x+y)\cdot(x-y)$. Assim:

$$\int_{\Omega} (x^2-y^2)\cos(x+y)\,dx\,dy =$$

$$\int_{\Omega} \underbrace{(x+y)}_{u}\cdot\underbrace{(x-y)}_{v}\cdot \cos\underbrace{(x+y)}_{u}\cdot dx\,dy$$

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \quad +$$

$$u+v = 2x \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$$

$$y = u - x = u - \frac{1}{2}u - \frac{1}{2}v \Rightarrow \boxed{y = \frac{1}{2}u - \frac{1}{2}v}$$

$$T(u,v) = (x,y) = \left(\overbrace{\frac{1}{2}u + \frac{1}{2}v}^{x}, \overbrace{\frac{1}{2}u - \frac{1}{2}v}^{y}\right)$$

$$\det J(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

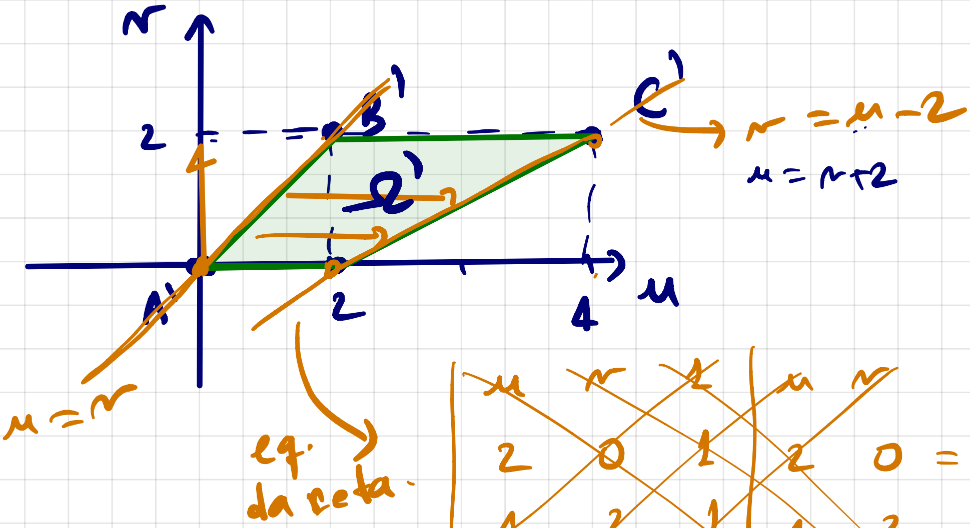
x y $\qquad (u,v) = (x+y, x-y)$

$A(0,0) \rightsquigarrow A'(0+0, 0-0) \rightsquigarrow A'(0,0)$

$B(2,0) \rightsquigarrow B'(2+0, 2-0) \rightsquigarrow B'(2,2)$

$C(3,1) \rightsquigarrow C'(3+1, 3-1) \rightsquigarrow C'(4,2)$

$D(1,1) \rightsquigarrow D'(1+1, 1-1) \rightsquigarrow D'(2,0)$



$$\begin{array}{ccc|cc} u & v & 1 & u & v \\ 2 & 0 & 1 & 2 & 0 \\ 4 & 2 & 1 & 4 & 2 \end{array} = 0$$

$$0 + 4v + 4 - 0 - 2u - 2v = 0$$

$$2v - 2u + 4 = 0 \quad \div 2$$

$$v - u + 2 = 0$$

$$\boxed{v = u - 2}$$

Assim, temos:

$$\int_{\Omega} (x^2 - y^2)\cos(x-y)\, dx\, dy = \int_{\Omega'} u \cdot v \cdot \cos u \cdot \underbrace{|\det J(T)(u,v)|}_{= \frac{1}{2}} \cdot du\, dv$$

$$= \int_{v=0}^{v=2} \int_{u=v}^{u=v+2} u \cdot v \cdot \cos u \cdot \frac{1}{2} \cdot du\, dv =$$

$$\frac{1}{2} \int_{v=0}^{v=2} v \left(\underbrace{\int_{u=v}^{u=v+2} u \cdot \cos u \, du}_{} \right) dv$$

(et cetera)

$$\int u \cdot \cos u \, du = \int U . dV = U . V - \int V dU .$$