AULA DE EXERCÍCIOS

26/07/23 AULA 12

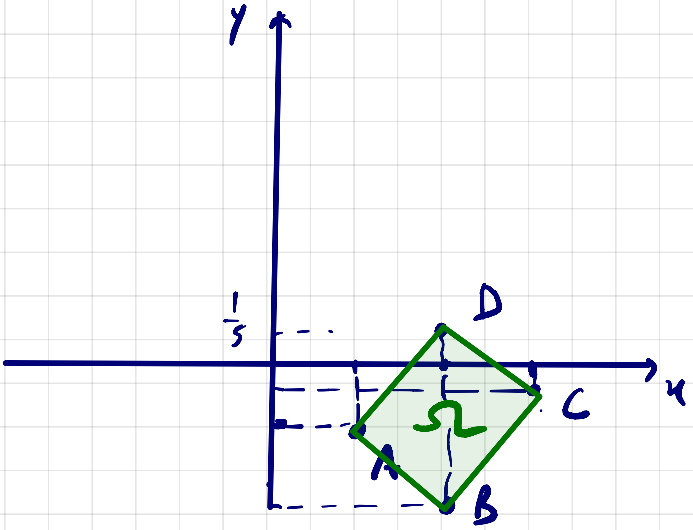
QUESTÃO EXTRA: (DE UMA PROVA ANTIGA)

Calcule $\displaystyle\iint_{\Omega} \frac{1}{x-y} \cdot \cos(2x+3y)\, dx\, dy$, onde

Ω é a região do plano xy definida pelo quadrilátero

$A\left(\frac{3}{5}, -\frac{2}{5}\right)$; $B\left(\frac{6}{5}, -\frac{4}{5}\right)$, $C\left(\frac{9}{5}, -\frac{1}{5}\right)$ e $D\left(\frac{6}{5}, \frac{1}{5}\right)$

SOLUÇÃO:



Escreva $\begin{cases} u = x - y \\ v = 2x + 3y \end{cases} \rightsquigarrow x = u + y$

$v = 2.(u+y) + 3y$

$y = \dfrac{-2u + v}{5}$

$$\boxed{y = -\frac{2}{5}u + \frac{1}{5}v}$$

$\Rightarrow x = u + y$

$x = u - \frac{2}{5}u + \frac{1}{5}v$

$$\boxed{x = \frac{3}{5}u + \frac{1}{5}v}$$

$$A\left(\frac{3}{5}, -\frac{2}{5}\right) \rightsquigarrow A'\left(\overbrace{\frac{3}{5}+\frac{2}{5}}^{u=x-y},\ \overbrace{2\cdot\left(\frac{3}{5}\right)+3\cdot\left(\frac{-2}{5}\right)}^{v=2x+3y}\right)$$

$x \quad y \qquad\qquad u,\ v$

$$A'(1, 0)$$

$$B\left(\frac{6}{5}, -\frac{4}{5}\right) \rightsquigarrow B'\left(\frac{6}{5}+\frac{4}{5},\ 2\cdot\frac{6}{5}+3\cdot\left(-\frac{4}{5}\right)\right)$$

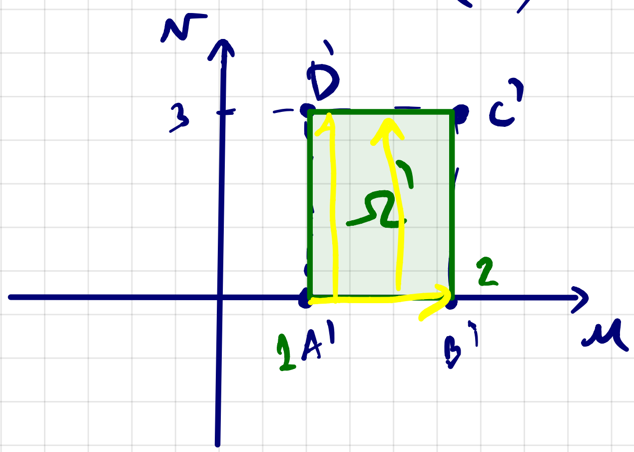
$$B'(2, 0)$$

$$C\left(\frac{9}{5}, -\frac{1}{5}\right) \rightsquigarrow C'\left(\frac{9}{5}+\frac{1}{5},\ 2\cdot\frac{9}{5}+3\cdot\left(-\frac{1}{5}\right)\right)$$

$$C'(2, 3)$$

$$D\left(\frac{6}{5}, \frac{1}{5}\right) \rightsquigarrow D'\left(\frac{6}{5}-\frac{1}{5},\ 2\cdot\frac{6}{5}+3\cdot\left(\frac{1}{5}\right)\right)$$

$$D'(1, 3)$$



$$\iint_{\Omega} \frac{1}{x-y}\cos(2x+3y)\,dx\,dy = \iint_{\Omega'} \frac{1}{u}\cos v \cdot \left|\det J(T)(u,v)\right| \cdot du\,dv \ ;$$

onde; sendo $x = \frac{3}{5}u + \frac{1}{5}v$ e $y = -\frac{2}{5}u + \frac{1}{5}v$:

$$\det(J)(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix}$$

$-$ $+$

$$= \frac{3}{5}\cdot\frac{1}{5} - \left(\frac{1}{5}\right)\left(-\frac{2}{5}\right) = \frac{3}{25} + \frac{2}{25} = \frac{1}{5}$$

$$\Rightarrow \int_{v=0}^{v=3}\int_{u=1}^{u=2} \frac{1}{u}\cdot\cos v\cdot\frac{1}{5}\cdot du\,dv =$$

$$= \frac{1}{5}\cdot\int_{v=0}^{v=3}\cos v\,dv\cdot\int_{u=1}^{u=2}\frac{du}{u}$$

$$= \frac{1}{5}\cdot \operatorname{sen} v\Big|_0^3 \cdot \ln|u|\Big|_1^2 = \frac{1}{5}\cdot(\operatorname{sen}3 - \underbrace{\operatorname{sen}0}_{0})\cdot(\ln 2 - \underbrace{\ln 1}_{0})$$

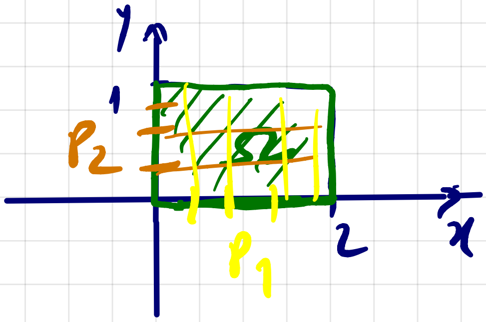
$$= \frac{1}{5}\operatorname{sen}3\cdot\ln 2$$

LISTA 01.

05) – b $f(x,y) = 3x^2 + 2y$;

no bloco $[0,2]\times[0,1]$

(pela def.)



Vamos calcular

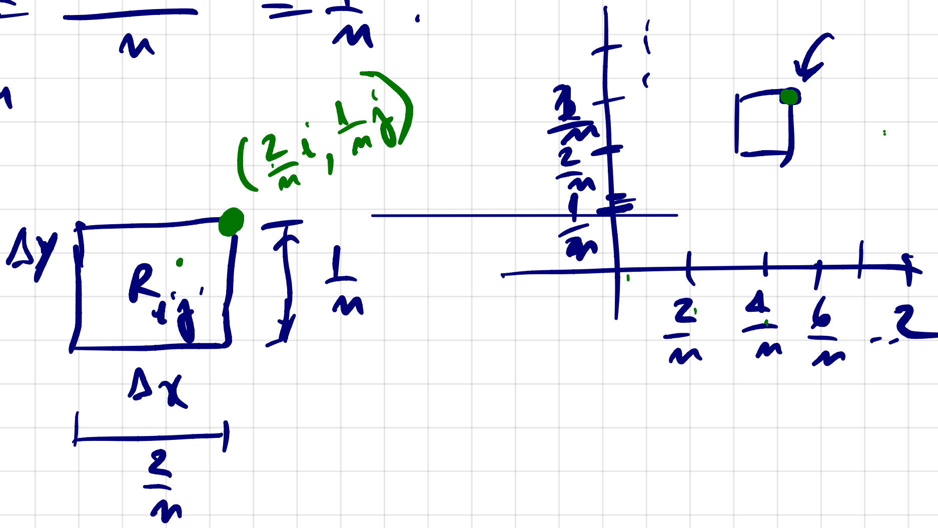
$$S(f\,;P)$$

seja $P = P_1 \times P_2$ partição do bloco $[0,2]\times[0,1]$, onde P_1 se divide em n subintervalos de comprimento $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$;

e P_2 se divide em n subintervalos de comprim

$$\Delta y = \frac{1-0}{n} = \frac{1}{n}.$$

Teremos subblocos



Em cada subretângulo R_{ij}, temos que, sendo $f(x,y) = 3x^2 + 2y$, o supremo é atingido no canto superior direito, i.e.,

$$M_B = \sup_{x \in B} f(x) = (3x^2+2y)\Big| = 3\cdot\left(\frac{2i}{n}\right)^2 + 2\cdot\left(\frac{j}{n}\right)$$

$$x = \frac{2}{n}i;\ y = \frac{1}{n}j$$

$$= \underbrace{\frac{12}{n^2}i^2 + \frac{2}{n}j}$$

Sendo

$$S(f,P) = \sum_{B\in P} M_B \cdot \mathrm{Vol}(B);$$ onde

$$\mathrm{Vol}(B) = \Delta x \cdot \Delta y = \frac{2}{n}\cdot\frac{1}{n} = \frac{2}{n^2};$$

(é uma área)

então:

$$S(f,P) = \sum_{B\in P}\left(\frac{12}{n^2}i^2 + \frac{2}{n}j\right)\cdot\frac{2}{n^2} =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{24}{n^4}i^2 + \frac{4}{n^3}j\right) =$$

$$\sum_{i=1}^{n}\left(\frac{24}{n^4}i^2\sum_{j=1}^{n}1 + \sum_{j=1}^{n}\frac{4}{n^3}j\right) =$$

$$= \sum_{i=1}^{n}\left(\frac{24 i^2}{\cancel{n^4}}\cdot\cancel{n} + \frac{4}{n^3}\sum_{j=1}^{n}j\right) =$$

$$= \frac{24}{m^3} \sum_{i=1}^{m} i^2 + \sum_{i=1}^{m} \frac{4}{m^3} \cdot \sum_{j=1}^{m} j$$

$$= \frac{24}{m^3} \sum_{i=1}^{m} i^2 + \frac{4}{m^3} \cdot \sum_{j=1}^{m} j \underbrace{\sum_{i=1}^{m} 1}_{m}$$

$$= \frac{24}{m^3} \underbrace{\sum_{i=1}^{m} i^2}_{\frac{m(m+1)(2m+1)}{6}} + \frac{4}{m^{\cancel{3}}} \cdot \cancel{m} \cdot \underbrace{\sum_{j=1}^{m} j}_{\frac{(1+m)m}{2}} =$$

$$= \frac{\overset{4}{\cancel{24}}}{m^{\cancel{3}}{}_{2}} \cdot \frac{\cancel{m} \cdot (m+1)(2m+1)}{\cancel{6}} + \frac{\cancel{4}}{m^{\cancel{2}}} \frac{\cancel{m}(m+1)}{\cancel{2}}$$

$$= 4 \cdot \left(\frac{m+1}{m}\right) \cdot \left(\frac{2m+1}{m}\right) + 2 \cdot \left(\frac{m+1}{m}\right)$$

$$s(f, P) = 4 \cdot \left(1 + \frac{1}{m}\right) \cdot \left(2 + \frac{1}{m}\right) + 2 \cdot \left(1 + \frac{1}{m}\right)$$

$$\underbrace{\overline{\int_A} f} = \lim_{m \to \infty} s(f, P) =$$

$$= \lim_{m \to \infty} 4\left(1 + \underbrace{\frac{1}{m}}_{0}\right)\left(2 + \underbrace{\frac{1}{m}}_{0}\right) + 2 \cdot \left(1 + \underbrace{\frac{1}{m}}_{0}\right)$$

$$\underbrace{= 10}$$

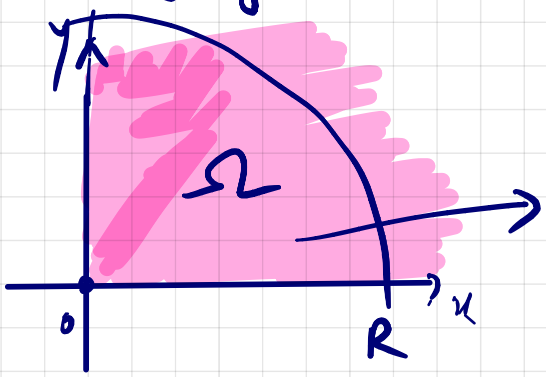
Analogamente, se mostra que $\underline{\int}_A f = 10$

Portanto, f é integrável e

$$\int_A f = 10.$$

LISTA 03

01) -b- $\displaystyle\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2} dx\, dy$



$\rho^2 = x^2 + y^2$

$dx\,dy \longrightarrow \rho\, d\rho\, d\theta$

$x = \rho \cos\theta$

$$\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2} dx\, dy = \lim_{R \to 0} \int_0^R \int_0^R x^2 \cdot e^{-x^2-y^2} dx\, dy$$

$$= \lim_{R\to\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} (\rho\cos\theta)^2 \cdot e^{-\rho^2} \cdot \rho\, d\rho\, d\theta$$

$$= \lim_{R\to\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos^2\theta\, d\theta \cdot \int_{\rho=0}^{\rho=R} \rho^3 \cdot e^{-\rho^2} d\rho =$$

- $\int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} \, d\theta$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\cdot\frac{1}{2}\int \cos 2\theta \, 2d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\cdot \operatorname{sen} 2\theta + C$$

- $\int \rho^3 e^{-\rho^2} d\rho = \int \rho^2 \cdot e^{-\rho^2} \cdot \rho \, d\rho = \int u \, dv =$

$$= u\cdot v - \int v \cdot du =$$

$$u = \rho^2 \Rightarrow du = 2\rho \, d\rho$$

$$dv = e^{-\rho^2} \cdot \rho \, d\rho$$

$$v = -\frac{1}{2}\int e^{-\rho^2} \cdot (-2\rho \, d\rho) = -\frac{1}{2} e^{-\rho^2}$$

$$= -\frac{1}{2} e^{-\rho^2} \cdot \rho^2 - \int -\frac{1}{2} e^{-\rho^2} \cdot 2\rho \, d\rho$$

$$= -\frac{\rho^2}{2} \cdot e^{-\rho^2} - \frac{1}{2} \cdot e^{-\rho^2} + C$$

$$= -\frac{e^{-\rho^2}}{2}\left[1+\rho^2\right] + C$$

Assim, voltando ao problema original:

$$\Leftrightarrow \lim_{R\to\infty} \left(\frac{\theta}{2} + \frac{1}{4}\operatorname{sen} 2\theta\right)\Bigg|_0^{\frac{\pi}{2}} \cdot \left(-\frac{e^{-\rho^2}}{2}(1+\rho^2)\right)\Bigg|_0^R$$

$$\left(\frac{\pi}{4} + \underbrace{\frac{1}{4}\operatorname{sen}\pi}_{=0} - 0\right) \cdot \lim_{R\to\infty}\left(-\frac{e^{-R^2}}{2}\cdot(1+R^2) + 1\right)$$

$$\frac{\pi}{4} \cdot \lim_{R\to\infty}\left(1 - \underbrace{\frac{1+R^2}{2\cdot e^{R^2}}}_{\to 0}\right) = \frac{\pi}{4}\cdot 1 = \frac{\pi}{4}$$
