26/07/23 – AULA 12

AULA DE EXERCÍCIOS:

LISTA 01:

09) Sejam (a_n) e (b_n) seq. tais que $\lim_{n\to\infty} a_n = 0$ e $\exists\, K>0$ tal que $|b_n| \leq K$, $\forall n \in \mathbb{N}$. $(*)$

mostrar: $\lim_{n\to\infty} a_n \cdot b_n = 0$.

Dado $\varepsilon>0$, precisamos encontrar $n_0 \in \mathbb{N}$ tal que, $\forall n \geq n_0 \Rightarrow |a_n \cdot b_n - 0| < \varepsilon$.

Como $a_n \to 0$, então, para o $\varepsilon>0$ dado acima, segue que $\exists\, n_1 \in \mathbb{N}$, tal que, $\forall n \geq n_1$, implique em

$$|a_n - 0| < \frac{\varepsilon}{K} \qquad (**)$$

Assim, $\forall n \geq n_1$, temos:

$$|a_n \cdot b_n - 0| = |a_n \cdot b_n| = \underbrace{|a_n|}_{< \frac{\varepsilon}{K}} \cdot \underbrace{|b_n|}_{\leq K} < \frac{\varepsilon}{\cancel{K}} \cdot \cancel{K} = \varepsilon$$

de $(*)$ e $(**)$

$$\Rightarrow |a_n \cdot b_n| < \varepsilon, \quad \forall n \geq n_1, \text{ i.e.;}$$

$$a_n \cdot b_n \to 0.$$

Se (b_n) não limitada, o resultado acima é falso. De fato: tome $a_n = \frac{1}{n}$ e $b_n = n$.

Então $a_n \to 0$ e b_n não é limitada; e não tais que

$$a_n \cdot b_n = \cancel{\frac{1}{n}} \cdot \cancel{n} = 1, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow a_n \cdot b_n \not\to 0.$$

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14) (x_n) seq. tal que

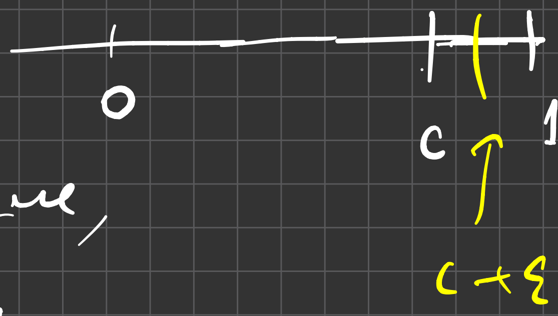
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = c < 1$$

mostrar: $\exists \lambda \in (0,1)$ e $\exists n_0 \in \mathbb{N}$ tais que

$$|x_{n+1}| \leq \lambda \cdot |x_n|, \quad \forall n \geq n_0.$$

Como $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = c < 1$, dado $\varepsilon > 0$,

tal que $c + \varepsilon < 1$.



segue que $\exists n_0 \in \mathbb{N}$ tal que,

$\forall n \geq n_0$, implique em:

$$\left| \frac{x_{n+1}}{x_n} - c \right| < \varepsilon$$

$$-\varepsilon < \frac{x_{n+1}}{x_n} - c < \varepsilon, \quad \forall n \geq n_0.$$

$$\frac{x_{n+1}}{x_n} < c+\varepsilon \quad , \forall n \geq n_0$$

$$x_{n+1} < (c+\varepsilon) \cdot x_n \quad , \forall n \geq n_0$$

$$\Longrightarrow |x_{n+1}| < \underbrace{(c+\varepsilon)}_{=:\lambda} \cdot |x_n| , \forall n \geq n_0$$

pois $x_n > 0, \forall n$.

Ou seja, tomando $\lambda = c+\varepsilon \in (0,1)$, segue que $\forall n \geq n_0$,

$$|x_{n+1}| < \lambda \cdot |x_n| .$$

Isto conclui a 1ª parte do exercício.

Além disso, sendo $x_n > 0$, $\forall n \in \mathbb{N}$; e como $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = c < 1$; então:

$$|x_{n+1}| < \lambda \cdot |x_n| \quad , \forall n \geq n_0 ;$$

$$|x_{n+1}| < \lambda \cdot \underbrace{|x_n|}_{< \lambda \cdot |x_{n-1}|} < \lambda \cdot \lambda \cdot |x_{n-1}| =$$

$$= \lambda^2 \cdot \underline{|x_{n-1}|} < \lambda^2 \cdot \lambda \cdot |x_{n-2}| =$$

$$< \lambda \cdot |x_{n-2}|$$

$$= \lambda^3 \cdot |x_{n-2}| < \cdots < \lambda^{n-n_0+1} \cdot |x_{n_0}|$$

$$\Rightarrow \quad |x_{n+1}| < \underbrace{\lambda^{1-n_0} \cdot |x_{n_0}|}_{\text{CONSTANTE}} \cdot \lambda^n \underset{n \to \infty}{\longrightarrow} 0$$

pois $0 < \lambda < 1$.

$$\Rightarrow \quad x_n \to 0.$$

Vamos mostrar, por fim, que

$$\frac{a^n}{n!} \underset{n \to \infty}{\longrightarrow} 0 \quad ; \quad a > 0.$$

Pelo feito acima, observando que

$$x_n := \frac{a^n}{n!} > 0, \ \forall n,$$

basta mostrar que $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} < 1.$

[ISTO MOSTRARÁ QUE $x_n \to 0$]

De fato:

$$\frac{x_{n+1}}{x_n} = \frac{\dfrac{a^{n+1}}{(n+1)!}}{\dfrac{a^n}{n!}} = \frac{a^{n+1}}{(n+1)!} \times \frac{n!}{a^n}$$

$$= \frac{\cancel{a^n} \cdot a \cdot \cancel{n!}}{(n+1) \cdot (\cancel{n!}) \cdot \cancel{a^n}} = \frac{a}{n+1}$$

$$\Rightarrow \quad \frac{x_{n+1}}{x_n} = \frac{a}{n+1} \xrightarrow[n \to \infty]{} 0 < 1$$

Então, $x_n \to 0$.

<u>QUESTÃO 13</u>: segue exatamente mesma

ideia:

(x_n) seq. tal que $\exists \lambda \in (0,1)$

e $\exists n_0 \in \mathbb{N}$: $|x_{n+1}| \leq \lambda \cdot |x_n|$, $\forall n \geq n_0$

mostrar: $x_n \to 0$.

De fato, note que:

$|x_{n+1}| < \lambda \cdot |x_n|$, $\forall n \geq n_0$; então:

$$|x_{n+1}| < \lambda \cdot \underbrace{|x_n|}_{< \lambda \cdot |x_{n-1}|} < \lambda \cdot \lambda \cdot |x_{n-1}| =$$

$$= \lambda^2 \cdot \underbrace{|x_{n-1}|}_{< \lambda \cdot |x_{n-2}|} < \lambda^2 \cdot \lambda \cdot |x_{n-2}| =$$

$$= \lambda^3 \cdot |x_{n-2}| < \dots < \lambda^{n-n_0+1} \cdot |x_{n_0}|$$

$$\Longrightarrow \quad |x_{n+1}| < \underbrace{\lambda^{1-n_0} \cdot |x_{n_0}|}_{\text{CONSTANTE}} \cdot \lambda^n \xrightarrow[n \to \infty]{} 0$$

pois $0 < \lambda < 1$.

$$\Longrightarrow \quad x_n \to 0.$$

QUESTÃO 01

c) $\lim\limits_{n \to \infty} \left(\sqrt{n+1} - \sqrt{n} \right) = 0.$

Dado $\varepsilon > 0$, precisamos achar $n_0 \in \mathbb{N}$, tal que, $\forall n \geq n_0 \Longrightarrow$

$$\left| \sqrt{n+1} - \sqrt{n} - 0 \right| < \varepsilon .$$

Analisando $|\sqrt{n+1}-\sqrt{n}|$:

$$|\sqrt{n+1}-\sqrt{n}| = \left|(\sqrt{n+1}-\sqrt{n})\cdot\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}\right|$$

$\infty - \infty$
(INDETERM)

$$= \left|\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

RASCUNHO:

$\sqrt{n+1}+\sqrt{n} > \sqrt{n}$

$\Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n_0}} < \varepsilon$

$\frac{1}{n_0} < \varepsilon^2$

então, será a escolha. $\rightarrow$ $n_0 > \frac{1}{\varepsilon^2}$

Tome $n_0 \in \mathbb{N}$ tal que $n_0 > \frac{1}{\varepsilon^2}$

Assim, $\forall n \geq n_0$, teremos:

$$|\sqrt{n+1}-\sqrt{n}| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n_0}} <$$

$< \varepsilon$

$n \geq n_0$

$\Rightarrow \frac{1}{n} \leq \frac{1}{n_0}$

$\Rightarrow \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n_0}}$

$n_0 > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{n_0} < \varepsilon^2$

$\frac{1}{\sqrt{n_0}} < \sqrt{\varepsilon^2} = \varepsilon$

Ou seja, $\forall \varepsilon > 0$, tome $n_0 > \frac{1}{\varepsilon^2}$,

e daí

$$|\sqrt{n+1} - \sqrt{n} - 0| < \varepsilon, \quad \forall n \geq n_0,$$

i.e.,

$$\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0.$$

LISTA 03

11) séries para $\ln(1+x)$ e $\ln(1-x)$.

Note que

- $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \quad \text{(série geométrica)}$$

Logo:

$$\ln(1-x) = -\int_0^x \frac{dt}{1-t} = -\int_0^x \sum_{n=0}^{\infty} t^n \, dt =$$

$\int \frac{du}{u}$

$u = 1-x$

$du = -dx$

$$= -\sum_{n=0}^{\infty} \int_0^x t^n \, dt = -\sum_{n=0}^{\infty} \left. \frac{t^{n+1}}{n+1} \right|_0^x =$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n} \; ; \; R = 1.$$

$$\Longrightarrow \boxed{-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}}$$

Do mesmo modo, temos:

$$\bullet \quad \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad ; \quad r = 1$$

$$\Rightarrow \ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \sum_{n=0}^{\infty} (-1)^n \cdot t^n \, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \int_0^x t^n \, dt = \sum_{n=0}^{\infty} (-1)^n \cdot \left. \frac{t^{n+1}}{n+1} \right|_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$$

$$\boxed{R=1}$$

Disso, teremos:

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) =$$

PROPRIEDADE DOS LOGARITMOS

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \sum_{n=1}^{\infty} \left((-1)^{n+1} \frac{x^n}{n} + \frac{x^n}{n} \right) =$$

$$= \sum_{n=1}^{\infty} \left((-1)^{n+1} + 1 \right) \cdot \frac{x^n}{n} =$$

$$= 2\cdot\frac{x}{1} + 0 + 2\cdot\frac{x^3}{3} + 0 + 2\cdot\frac{x^5}{5} + \cdots$$

$$= 2\cdot\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \right) \quad ; \quad R = 1.$$

Valor aprox. para $\ln 5$, usando os 3 primeiros termos:

$$\ln\left(\frac{1+x}{1-x}\right) = \ln 5$$

$$\Leftrightarrow \frac{1+x}{1-x} = 5$$

$$\Leftrightarrow 1 + x = 5 - 5x$$

$$6x = 4 \Rightarrow x = \frac{2}{3}$$

$$\Rightarrow \ln 5 = \ln\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) \cong \frac{2}{3} + \frac{\left(\frac{2}{3}\right)^3}{3} + \frac{\left(\frac{2}{3}\right)^5}{5}$$

QUESTÃO: $f(x) = e^{-x^2}$. série de Taylor?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad ; \quad R = \infty$$

$$e^{-x^2} = \sum_{n=0}^{+\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!} \quad ;$$