Seguindo o que explicamos na aula anterior, veremos alguns exemplos de aplicação de integrais triplas.

Antes, vale observar que o volume V de um sólido S, via integrais triplas, é definido por

$$V = \iiint_S dV = \iiint_S dx\,dy\,dz$$

Por exemplo, poderíamos tomar um paralelepípedo A (bloco do $\mathbb{R}^3$) tal que $S \subset A$.

Assim,

$$\underbrace{V}_{} = \text{Vol}\, S = \iiint_A \xi_S(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_S 1 \cdot dx\,dy\,dz + \underbrace{\iiint_{A \setminus S} 0 \cdot dx\,dy\,dz}_{=0}$$

$$= \underbrace{\iiint_S dx\,dy\,dz}_{}$$

Obs.: Do mesmo modo que na integral dupla, uma integral tripla é calculada via integrais iteradas.

EXEMPLOS:

01) [LISTA 03, EXERCÍCIO 4-d]

$$\int_0^1 \int_0^z \int_0^y z \cdot e^{-y^2} \cdot dx\,dy\,dz =$$

$$\uparrow z \quad \uparrow y \quad \uparrow x$$

$$\int_{z=0}^{z=1} z \cdot \int_{y=0}^{y=z} e^{-y^2} \cdot \left(\int_{x=0}^{x=y} 1 \cdot dx \right) dy\,dz =$$

$$= \int_{z=0}^{z=1} z \int_{y=0}^{y=z} e^{-y^2} \cdot x \Big|_{x=0}^{x=y} dy\,dz =$$

$$= -\frac{1}{2} \int_{z=0}^{z=1} z \left(\int_{y=0}^{y=z} e^{-y^2} (-2y) \, dy \right) dz =$$

$$\int e^{v} dv \qquad v = -y^2$$
$$dv = -2y\,dy$$

$$= -\frac{1}{2}\int_{z=0}^{z=1} z \cdot e^{-y^2}\Big|_{y=0}^{y=z} dz =$$

$$= -\frac{1}{2}\int_{z=0}^{z=1} z\cdot\left(e^{-z^2}-1\right) dz = -\frac{1}{2}\left(-\frac{1}{2}\right)\int_0^1 e^{-z^2}(-2z)\,dz + \frac{1}{2}\int_0^1 z\,dz$$

$$= +\frac{1}{4}\; e^{-z^2}\Big|_0^1 + \frac{1}{2}\;\frac{z^2}{2}\Big|_0^1 =$$

$$\frac{1}{4}\left(e^{-1}-1\right) + \frac{1}{4}\cdot(1-0) = \frac{1}{4}\left(\frac{1}{e}\cancel{-1}\cancel{+1}\right) = \frac{1}{4e}$$

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02) $$\int_{-1}^{0}\int_{e}^{2e}\int_{0}^{\frac{\pi}{3}} y\cdot \ln z\cdot \tan x\cdot dx\,dz\,dy =$$

y z x

$$= \int_{y=-1}^{y=0} y \int_{z=e}^{z=2e} \ln z\left(\int_{x=0}^{x=\frac{\pi}{3}} \tan x\; dx\right) dz\,dy \;=$$

$$\int \tan x\, dx = ?$$

Obs.: $\int \tan x \, dx = \ln|\sec x + \tan x| + C$

De fato:

$$[\ln(\sec x + \tan x)]' = \frac{\sec x \cdot \tan x \cdot x' + \sec^2 x \cdot x'}{\sec x + \tan x}$$

$$= \frac{\sec x \cdot \cancel{(\tan x + \sec x)} \cdot x'}{\cancel{\sec x + \tan x}} = \tan x \cdot x'$$

$$\textcircled{=} \int_{y=-1}^{y=0} y \int_{z=e}^{z=2e} \ln z \left(\ln|\sec x + \tan x| \right) \Big|_0^{\frac{\pi}{3}} dz\, dy$$

$$= \int_{y=-1}^{y=0} y \int_{z=e}^{z=2e} \ln z \cdot \left[\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| \right] dz\, dy$$

$$= \int_{y=-1}^{y=0} y \int_{z=e}^{z=2e} \ln z \left[\ln|2 + \sqrt{3}| - \underbrace{\ln|1 + 0|}_{= \ln 1 = 0} \right] dz\, dy$$

$$= \ln(2 + \sqrt{3}) \cdot \int_{y=-1}^{y=0} y \, dz \underbrace{\int_{z=e}^{z=2e} \ln z \, dz}_{} \textcircled{=}$$

$$\int \ln z \, dz = \int u \cdot dv = u \cdot v - \int v \, du$$

(INTEGRAÇÃO POR PARTES)

$$\begin{cases} u = \ln z \Rightarrow du = \dfrac{dz}{z} \\ dv = dz \Rightarrow v = z \end{cases}$$

$$\int \ln z\, dz = z\cdot \ln z - \int \cancel{z}\cdot\frac{dz}{\cancel{z}} = z\ln z - \int dz$$

$$= z\ln z - z + C$$

$$\Leftrightarrow \ln(2+\sqrt{3})\cdot \frac{y^2}{2}\Big|_{-1}^{0}\cdot (z\cdot\ln z - z)\Big|_{e}^{2e} =$$

$$= \ln(2+\sqrt{3})\cdot\left(0-\frac{1}{2}\right)\cdot\left(2e\cdot\ln 2e - 2e - [e\underbrace{\ln e}_{=1} - e]\right)$$

$$= \ln(2+\sqrt{3})\cdot\left(-\frac{1}{2}\right)\cdot\left[2e\cdot(\ln 2 + \underbrace{\ln e}_{=1}) - 2e - \cancel{e}\cdot 1 + \cancel{e}\right)$$

$$= -\frac{1}{\cancel{2}}\ln(2+\sqrt{3})\cdot\left[\cancel{2}e\cdot\ln 2 + \cancel{2e} - \cancel{2e}\right]$$

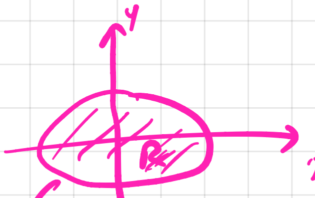
$$= -e\cdot\ln 2\cdot\ln(2+\sqrt{3})$$

03) Use integração tripla para calcular o volume do sólido S delimitado pelos paraboloides

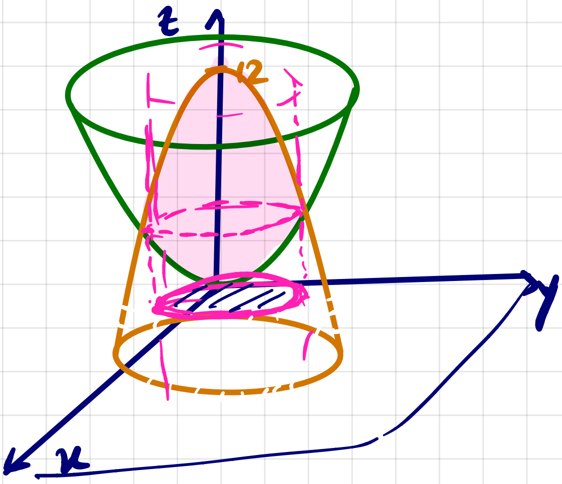
$$z = x^2+y^2 \quad e \quad \boxed{z = 12 - x^2 - 3y^2}.$$

Solução:

$$V = \iiint_S dV$$



↳ R é determinada pela interseção das duas superfícies.



R é dada por: $\begin{cases} z = x^2+y^2 \\ z = 12-x^2-3y^2 \end{cases}$

$$\Downarrow$$

$$x^2+y^2 = 12-x^2-3y^2$$

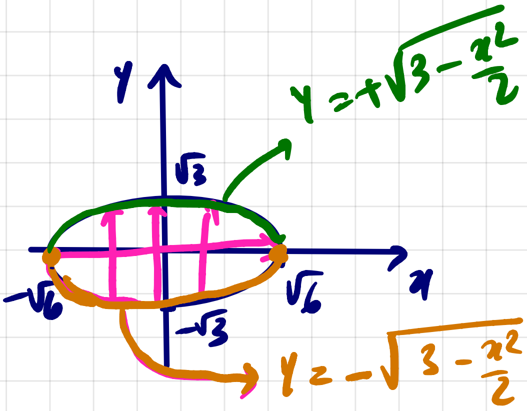
$$2x^2+4y^2=12 \quad \div 12$$

$$\frac{x^2}{6}+\frac{y^2}{3}=1 \quad \times 6 \quad \text{(elipse no plano } xy\text{)}$$

↳ cilindro no $\mathbb{R}^3$ elíptico

$$x^2+2y^2=6$$

$$y^2=\frac{6-x^2}{2}$$



Assim:

$$V=\iiint_S dV=\iint_R dx\,dy\int_{z=x^2+y^2}^{z=12-x^2-3y^2} dz =$$

$$=\int_{x=-\sqrt{6}}^{x=\sqrt{6}}\int_{y=-\sqrt{3-\frac{x^2}{2}}}^{y=\sqrt{3-\frac{x^2}{2}}} dx\,dy \cdot z\Big|_{x^2+y^2}^{12-x^2-3y^2} =$$

$$=\int_{x=-\sqrt{6}}^{x=\sqrt{6}}\int_{y=-\sqrt{3-\frac{x^2}{2}}}^{y=\sqrt{3-\frac{x^2}{2}}} dy\,dx \cdot \left(12-x^2-3y^2-x^2-y^2\right)$$

$$=\int_{x=-\sqrt{6}}^{x=\sqrt{6}}\int_{y=-\sqrt{3-\frac{x^2}{2}}}^{y=\sqrt{3-\frac{x^2}{2}}} \left(12-2x^2-4y^2\right) dy \cdot dx =$$

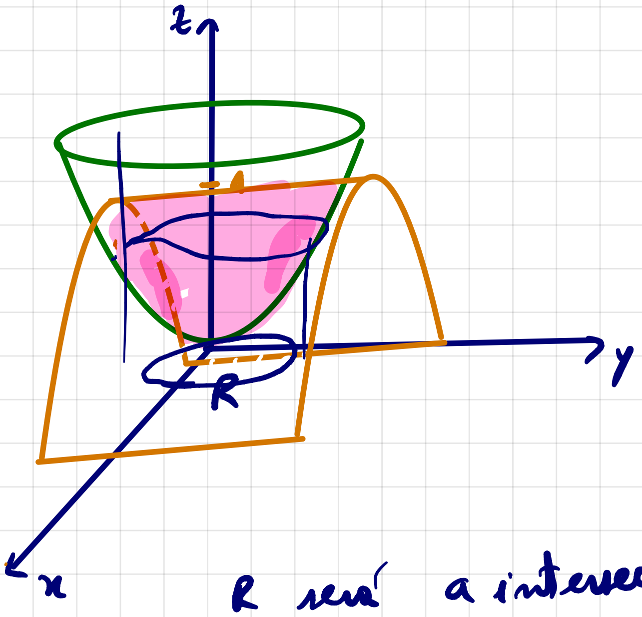
$$\int_{x=-\sqrt{6}}^{x=\sqrt{6}} \left(12y-2x^2y-\frac{4y^3}{3}\right)\Bigg|_{y=-\sqrt{3-\frac{x^2}{2}}}^{y=\sqrt{3-\frac{x^2}{2}}} \cdot dx = \dots$$

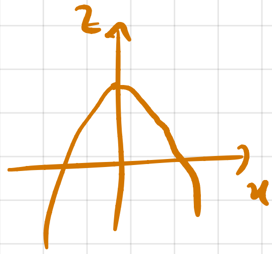
et cetera

04) Ache o volume do sólido acima do paraboloide elíptico $3x^2+y^2=z$ e abaixo do cilindro $x^2+z=4$.

↳ $z=4-x^2$

Solução:





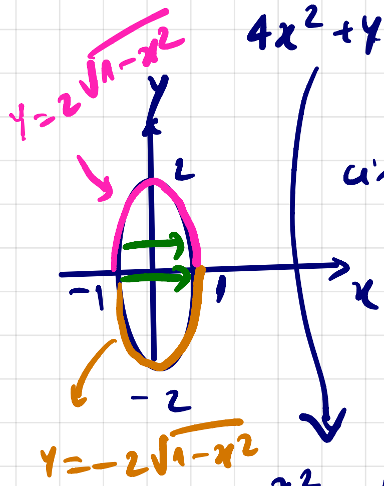
R será a interseção das superfícies:

$$\begin{cases} z=3x^2+y^2 \\ z=4-x^2 \end{cases} \quad 3x^2+y^2=4-x^2$$

$4x^2+y^2=4$ (elipse)

cilindro elíptico no $\mathbb{R}^3$

$$\frac{x^2}{1}+\frac{y^2}{4}=1$$



$$\begin{aligned} y^2 &= 4-4x^2 \\ y &= \pm\sqrt{4(1-x^2)} \\ y &= \pm 2\sqrt{1-x^2} \end{aligned}$$

$$V=\iiint_S dV = \iint_R dx\,dy \int_{z=3x^2+y^2}^{z=4-x^2} dz =$$

$$= \int_{x=-1}^{x=1}\int_{y=-2\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}} dy\,dx \, . \, z\Big|_{3x^2+y^2}^{4-x^2} =$$

$$= \int_{x=-1}^{x=1}\int_{y=-2\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}} (4-x^2-3x^2-y^2)\,dy\,dx =$$

$$= \int_{x=-1}^{x=1}\int_{y=-2\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}} (4-4x^2-y^2)\,dy\,dx =$$ et cetera ...