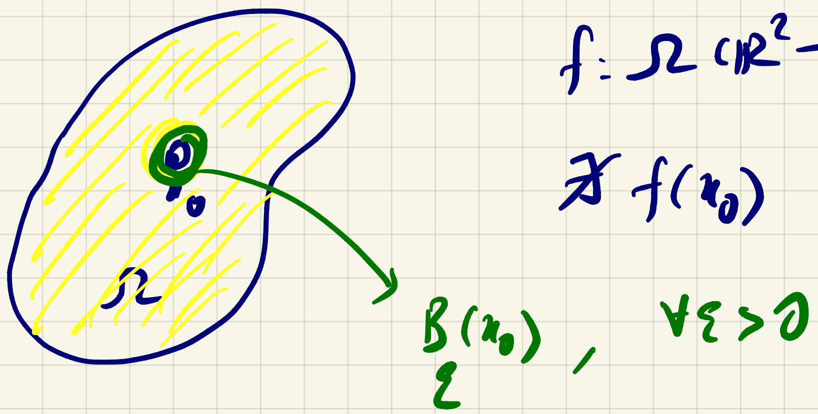
19/07/23 – AULA 10

INTEGRAIS IMPRÓPRIAS:

Do mesmo modo que em cálculo II, onde tínhamos integrais em regiões ilimitadas ou em regiões onde havia alguma singularidade [i.e., ponto onde f não está definida], vamos estudar para integrais duplas, chamadas de integrais impróprias.

Serão dois casos de integrais impróprias.

CASO 1: f possui algum tipo de singularidade em Ω.



$f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

$\nexists f(x_0)$

Seja $B_\varepsilon(x_0)$ a bola centrada em x_0 e raio $\varepsilon > 0$. Então, f fica definida em

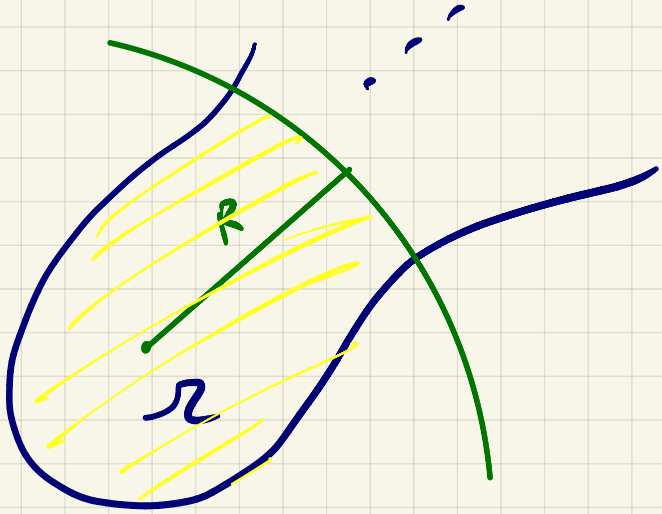
$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon(x_0).$$

Então,

$$\iint_\Omega f = \lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon} f$$

Existindo este limite, a integral será convergente, e converge para o valor do limite. Caso contrário, será divergente.

CASO 2: Quando Ω é ilimitado.



Tome $R>0$ e considere

Ω_R uma região

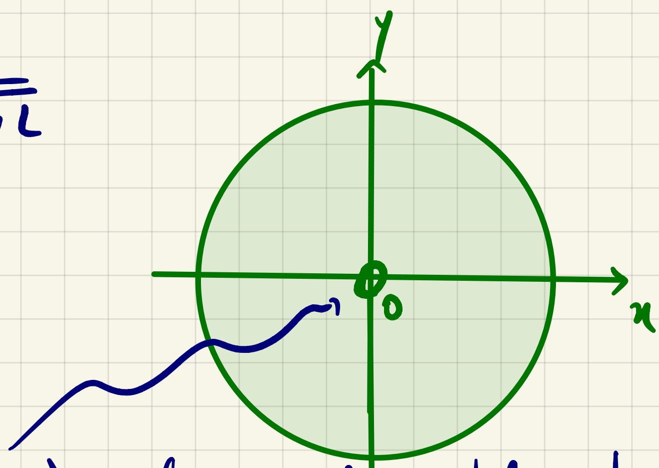
tal que $\lim_{R\to\infty} \Omega_R = \Omega$

Então $\iint_{\Omega} f = \lim_{R\to\infty} \iint_{\Omega_R} f$

Como antes, a integral converge se o limite existir. Do contrário, diverge.

Vejamos alguns exemplos:

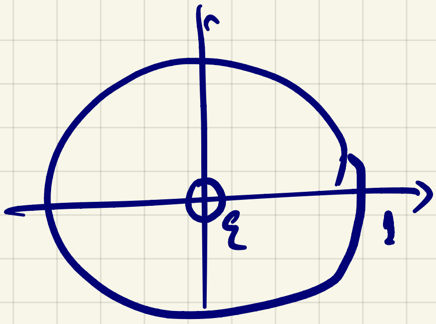
01) $$\iint_{x^2+y^2\le 1} \frac{dx\,dy}{\sqrt{x^2+y^2}}$$



$(x,y)=(0,0)$, f não está definida (singularidade)

Tome $0 < \varepsilon < 1$ e considere

$$\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^2 : \varepsilon \leq x^2+y^2 \leq 1\}$$



Passando para coordenadas polares, teremos:

$$\iint_{x^2+y^2\leq 1} f = \lim_{\varepsilon\to 0} \iint_{\Omega_\varepsilon} \frac{dx\,dy}{\sqrt{x^2+y^2}} =$$

$$= \lim_{\varepsilon\to 0} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \frac{\cancel{\rho}\, d\rho\, d\theta}{\sqrt{\cancel{\rho^2}}} =$$

$$= \lim_{\varepsilon\to 0} \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\rho=\varepsilon}^{\rho=1} d\rho = 2\pi \cdot \lim_{\varepsilon\to 0} \rho \Big|_{\varepsilon}^{1}$$

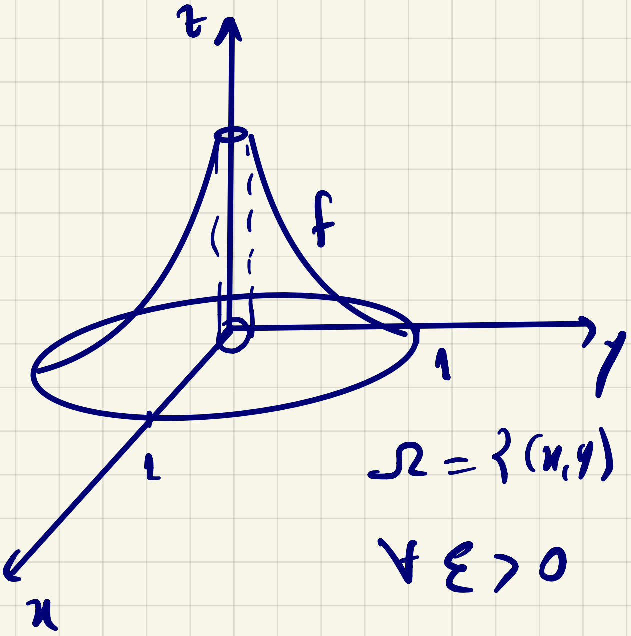
$$= 2\pi \underbrace{\lim_{\varepsilon\to 0} (1-\varepsilon)}_{1} = 2\pi.$$

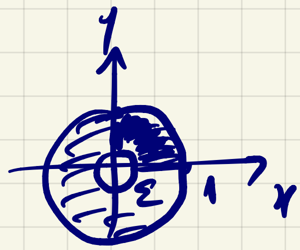
$$\Longrightarrow \iint_{x^2+y^2\leq 1} \frac{dx\,dy}{\sqrt{x^2+y^2}} = 2\pi.$$

02) $$\iint_{x^2+y^2\leq 1} -\ln(x^2+y^2)\,dx\,dy = \iint_{x^2+y^2\leq 1} \frac{dx\,dy}{\ln(x^2+y^2)}$$

POSSUI SINGULARIDADE NA ORIGEM.

$$f(x,y) = \frac{1}{\ln(x^2+y^2)}$$





$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$\forall\, \varepsilon > 0 \quad (\varepsilon < 1)$, seja

$$\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^2 : \varepsilon \leq x^2+y^2 \leq 1\}$$

Assim, teremos:

$$\iint_{\Omega} -\ln(x^2+y^2)\,dx\,dy = \lim_{\varepsilon\to 0} \iint_{\Omega_\varepsilon} -\ln(x^2+y^2)\,dx\,dy$$

$$= \lim_{\varepsilon\to 0} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} -\ln(\rho^2)\cdot\rho\,d\rho\,d\theta =$$

COORD. POLARES

$$= \lim_{\varepsilon\to 0} \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\rho=\varepsilon}^{\rho=1} -\ln\rho^2 \cdot \rho\,d\rho$$

$$= 2\pi \cdot \lim_{\varepsilon \to 0} \underbrace{\int_{\rho=\varepsilon}^{\rho=1} -2\ln\rho \cdot \rho \, d\rho}_{(*)}$$

$$\cdot \int \ln\rho \cdot \rho \, d\rho = \int u \, dv = u \cdot v - \int v \, du$$

(INT. POR PARTES)

$$\begin{cases} u = \ln\rho \Rightarrow du = \dfrac{d\rho}{\rho} \\ dv = \rho \, d\rho \Rightarrow v = \dfrac{\rho^2}{2} \end{cases}$$

$$\Rightarrow \int \ln\rho \cdot \rho \, d\rho = \frac{\rho^2}{2} \cdot \ln\rho - \int \frac{\rho^{\cancel{2}}}{2} \cdot \frac{d\rho}{\cancel{\rho}}$$

$$= \frac{\rho^2}{2} \ln\rho - \frac{1}{2} \int \rho \, d\rho$$

$$= \frac{\rho^2}{2} \ln\rho - \frac{\rho^2}{4} + C$$

$$= \frac{\rho^2}{2} \cdot \left(\ln\rho - \frac{1}{2} \right) + C$$

Com isso, temos:

$$\iint_\Omega -\ln(x^2+y^2) \, dx \, dy = 2\pi \cdot \lim_{\varepsilon \to 0} \int_{\rho=\varepsilon}^{\rho=1} -2\ln\rho \cdot \rho \, d\rho$$

$$= -4\pi \lim_{\varepsilon \to 0} \int_{\rho=\varepsilon}^{\rho=1} \ln\rho \cdot \rho \, d\rho =$$

$$= -4\pi \cdot \lim_{\varepsilon \to 0} \frac{\rho^2}{2}\left(\ln \rho - \frac{1}{2}\right)\Bigg|_{\varepsilon}^{1}$$

$$= -4\pi \lim_{\varepsilon \to 0} \left[\frac{1}{2}\left(0 - \frac{1}{2}\right) - \frac{\varepsilon^2}{2}\left(\ln \varepsilon - \frac{1}{2}\right)\right]$$

$$= -4\pi \lim_{\varepsilon \to 0} \left[-\frac{1}{4} - \frac{\varepsilon^2}{2} \cdot \ln \varepsilon + \underbrace{\frac{\varepsilon^2}{4}}_{\to 0}\right]$$

$$= -4\pi \left(-\frac{1}{4} - \frac{1}{2} \underbrace{\lim_{\varepsilon \to 0} \frac{\ln \varepsilon}{\frac{2}{\varepsilon^2}}}_{\text{L'HOPITAL}} + \lim \frac{\varepsilon^2}{4}\right)$$

$$= \pi - \frac{1}{2} \cdot \lim_{\varepsilon \to 0} \frac{\frac{1}{\varepsilon}}{-\frac{4}{\varepsilon^3}} + 0$$

$$= \pi - \frac{1}{2} \cdot \underbrace{\lim_{\varepsilon \to 0} -\left(\frac{1}{\varepsilon} \cdot \frac{\varepsilon^3}{4}\right)}_{= 0} = \pi$$

Logo, a integral converge, e

$$\iint_{x^2+y^2 \leq 1} -\ln(x^2+y^2)\, dx\, dy = \pi .$$

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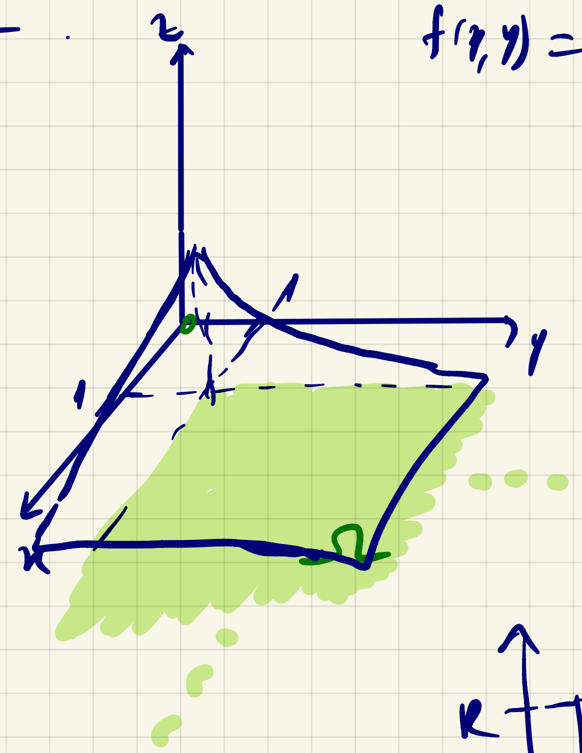
03) $\displaystyle\int_1^{+\infty}\int_1^{+\infty} \frac{dx\,dy}{x^2y^2}$.

$f(x,y) = \dfrac{1}{x^2y^2}$

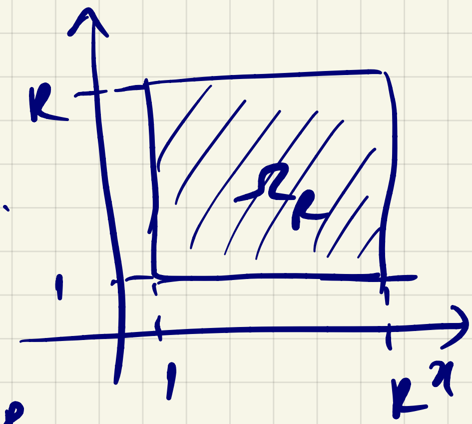
$\Omega = [1,+\infty)\times[1,+\infty)$



$\Omega_R = [1,R]\times[1,R]$

$R \to \infty$

$\Omega = \lim\limits_{R\to\infty} \Omega_R$.



Assim

$$\int_1^{\infty}\int_1^{\infty} \frac{dx\,dy}{x^2y^2} = \lim_{R\to+\infty}\int_1^{R}\int_1^{R}\frac{dx\,dy}{x^2y^2}$$

$$= \lim_{R\to+\infty}\int_1^{R}\frac{dx}{x^2}\int_1^{R}\frac{dy}{y^2} = \lim_{R\to\infty}\left(\frac{x^{-1}}{-1}\right)\Bigg|_1^R\cdot\left(\frac{y^{-1}}{-1}\right)\Bigg|_1^R$$

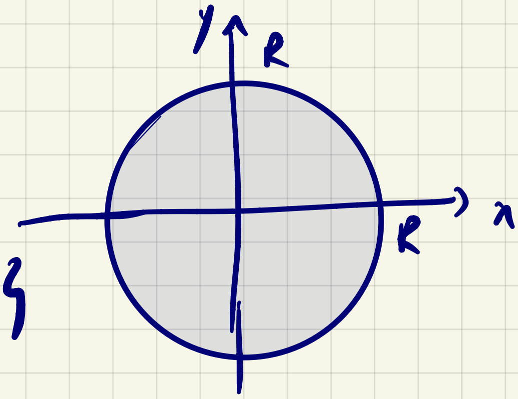
$$= \lim_{R\to+\infty} -\frac{1}{x}\Bigg|_1^R\cdot\left(-\frac{1}{y}\Bigg|_1^R\right)$$

$$= \lim_{R\to+\infty}\left(-\frac{1}{R}+1\right)\cdot\left(-\frac{1}{R}+1\right) = \lim_{R\to+\infty}\Big(1-\underbrace{\frac{1}{R}}_{\to 0}\Big)^2 = 1.$$

04) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy = \iint_{R^2} e^{-x^2-y^2}\,dx\,dy$$

Seja $R>0$ e considere

$$\Omega_R = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq R^2\}$$



Assim $\mathbb{R}^2 = \Omega = \lim_{R\to\infty}\Omega_R$. Com isso, temos:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy = \lim_{R\to+\infty}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=R} e^{-\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$\rho^2 = x^2+y^2$$

$$= \lim_{R\to+\infty}\int_{\theta=0}^{\theta=2\pi} d\theta\cdot\left(-\frac{1}{2}\right)\int_{\rho=0}^{\rho=R} e^{-\rho^2}(-2\rho)\,d\rho$$

$$= 2\pi\cdot\left(-\frac{1}{2}\right)\cdot\lim_{R\to+\infty} e^{-\rho^2}\Big|_0^R = -\pi\cdot\lim_{R\to+\infty}\Big(\underbrace{e^{-R^2}}_{0}-1\Big)$$

$$=\pi.$$

05) INTEGRAL DE POISSON : $\int_{-\infty}^{+\infty} e^{-x^2} dx = ?$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-y^2} dy$$

$$= \int_{-\infty}^{+\infty} e^{-x^2-y^2} dx\, dy = \pi$$

PELO EXEMPLO ACIMA

$$\Longrightarrow \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \pi$$

$$\Longrightarrow \boxed{\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}}$$

INTEGRAIS TRIPLAS:

Seja $w = f(x,y,z)$; $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável

Ω - J - mensurável.

Então, sendo D uma decomposição de Ω (em $\mathbb{R}^3$)

$D = X_1 \cup X_2 \cup \dots \cup X_m$; X_k conjuntos J-mensuráveis,

para $k = 1, 2, \dots, m$. Temos então, $\forall\, c_i \in X_i$,

a soma de Riemann:

$$\Sigma(f; D^*) = \sum_{i=1}^{m} f(c_i) \cdot Vol(X_i)$$

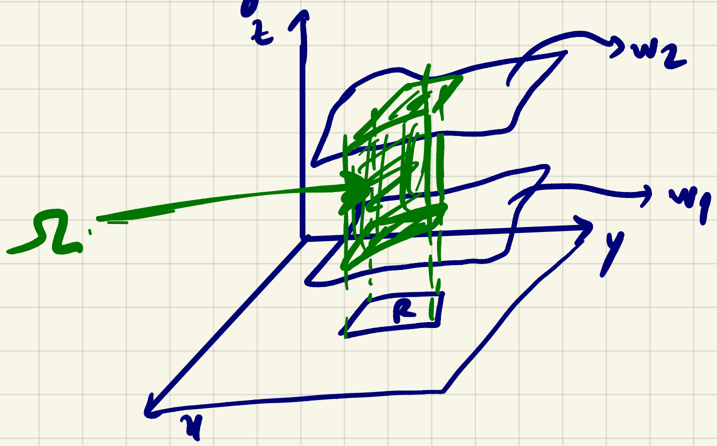
Então,

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\cdot \mathrm{Vol}(X_i)$$

Ou seja, tudo que fizemos no $\mathbb{R}^2$ pode ser aplicado para $\mathbb{R}^3$, com as devidas adaptações.

Na prática, uma integral tripla costuma ser transformada numa integral dupla e uma integral simples.

Em geral: Ω fica limitada por 2 superfícies w_1 e w_2 do $\mathbb{R}^3$ e uma região R do $\mathbb{R}^2$. Assim:



$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_{R} dx\,dy \underbrace{\int_{w_1}^{w_2} f(x,y,z)\,dz}_{F(x,y)}$$

Na próxima aula veremos exemplos de aplicações.