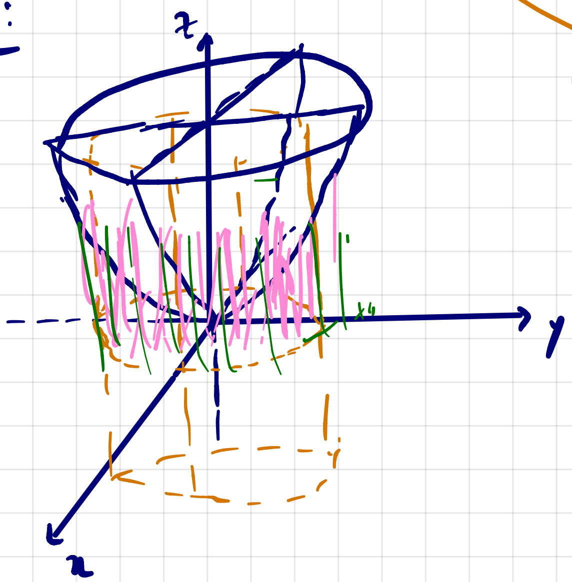
30/06/23 – AULA 06

OUTROS EXEMPLOS:

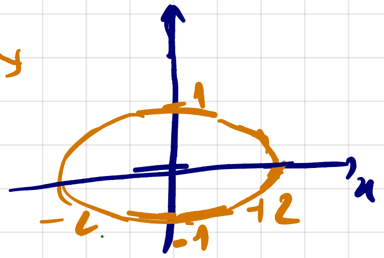
03) Obter o volume do sólido acima do plano xy, limitado pelo parabolóide elíptico $z = x^2 + 4y^2$ e pelo cilindro $x^2 + 4y^2 = 4$.

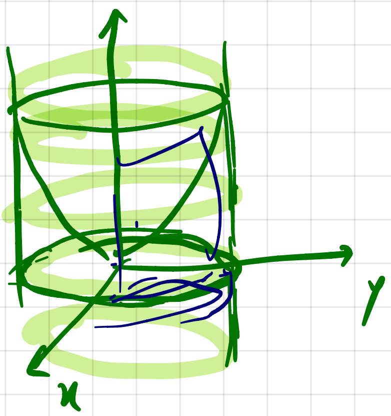
$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

SOLUÇÃO:



TRAÇO NO PLANO xy:

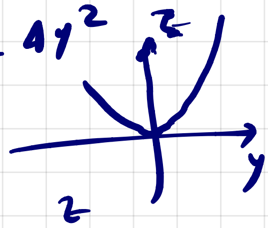




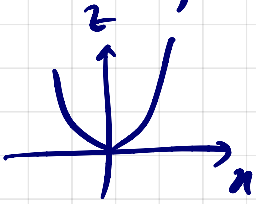
$z = f(x,y) = x^2 + 4y^2$

traços:

- $x = 0$: $z = 4y^2$



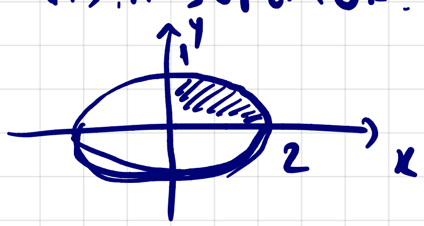
- $y = 0$: $z = x^2$



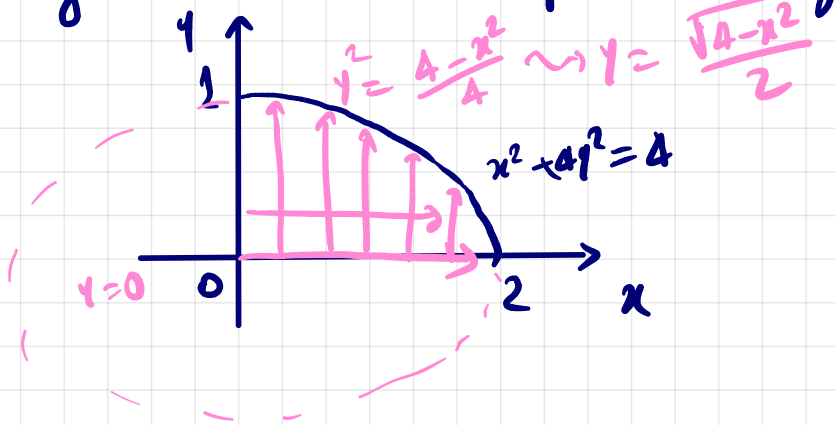
- $z = 0$: $x^2 + 4y^2 = 0$ (ORIGEM)

Em virtude da simetria do problema, vamos considerar o problema no 1º q. (Depois multiplicamos por 4, por força da simetria)

VISTA SUPERIOR:



Região do domínio para integrar:



Disso, o volume V será dado por

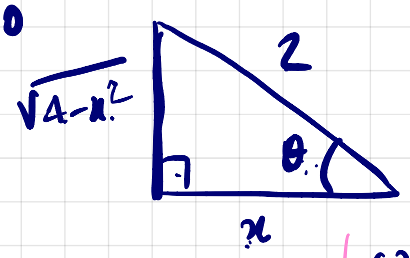
$$V = 4 \cdot \int_{x=0}^{x=2} \left(\int_{y=0}^{y=\frac{\sqrt{4-x^2}}{2}} (x^2+4y^2)\, dy \right) dx$$

DEVIDO À SIMETRIA

$$= 4 \cdot \int_{x=0}^{x=2} \left(x^2 y + 4\frac{y^3}{3} \right) \Bigg|_{y=0}^{y=\frac{\sqrt{4-x^2}}{2}} dx =$$

$$= 4 \cdot \int_{x=0}^{x=2} \left(x^2 \frac{\sqrt{4-x^2}}{2} + \frac{4}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 \right) dx$$

$$= 2\int_0^2 \sqrt{4-x^2} \cdot x^2 dx + \frac{2}{3}\int_0^2 (4-x^2)\sqrt{4-x^2}\, dx$$



SUBSTITUIÇÃO TRIGONOMÉTRICA.

$$\cos\theta = \frac{x}{2}$$

$$\boxed{x = 2\cos\theta.}$$

$$\operatorname{sen}\theta = \frac{\sqrt{4-x^2}}{2}$$

$$\Rightarrow \boxed{\sqrt{4-x^2} = 2\operatorname{sen}\theta}$$

$$\boxed{dx = -2\operatorname{sen}\theta\, d\theta}$$

Assim:

$$\bullet \int \sqrt{4-x^2}\cdot x^2\,dx = \int 2\,\text{sen}\,\theta\cdot(2\cos\theta)^2\cdot(-2\,\text{sen}\,\theta\,d\theta)$$

$$= -16\int \text{sen}^2\theta\cdot\cos^2\theta\,d\theta = -16\int \text{sen}^2\theta\cdot(1-\text{sen}^2\theta)\,d\theta$$

$$= -16\int \text{sen}^2\theta\,d\theta + 16\int \text{sen}^3\theta\,d\theta$$

$$= -16\cdot\int \frac{1-\cos 2\theta}{2}\,d\theta + 16\int \text{sen}\,\theta\;\text{sen}^2\theta\,d\theta$$

$$= -8\int d\theta + 8\int \cos 2\theta\,d\theta + 16\int (1-\cos^2\theta)\,\text{sen}\,\theta\,d\theta$$

$$= -8\theta + 4\cdot 2\int \cos 2\theta\,d\theta + 16\int \text{sen}\,\theta\,d\theta - 16\int (\cos\theta)^2\,\text{sen}\,\theta\,d\theta$$

$\int \cos r\,dr$

" $\int r^k\,dr$ "

$\frac{r^{k+1}}{k+1} + c$

$= 2\cdot\text{sen}\,\theta\cdot\cos\theta$ (ARCO DUPLO)

$$= -8\theta + 4\cdot\text{sen}\,2\theta - 16\cos\theta + 16\,\frac{\cos^3\theta}{3} + C$$

sendo:

$$\cos\theta = \frac{x}{2} \Rightarrow \theta = \arccos\left(\frac{x}{2}\right)\cdot\ \text{e}$$

$$\text{sen}\,\theta = \frac{\sqrt{4-x^2}}{2};\ \text{teremos:}$$

(... etc.)

- EXERCÍCIO: terminar os cálculos.

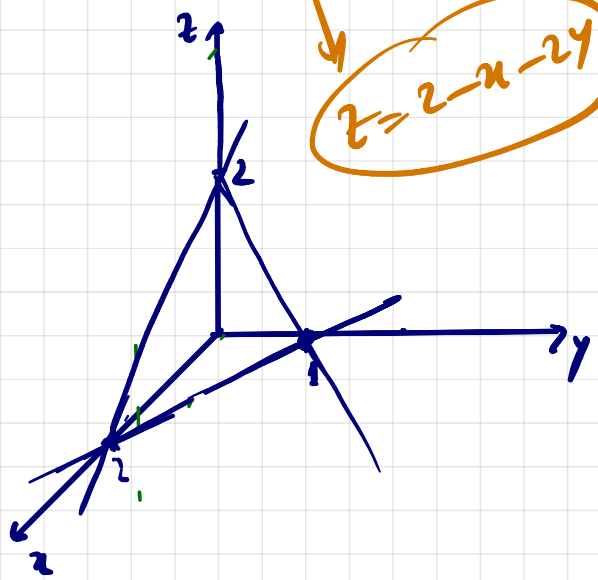
(CÁLCULO II)

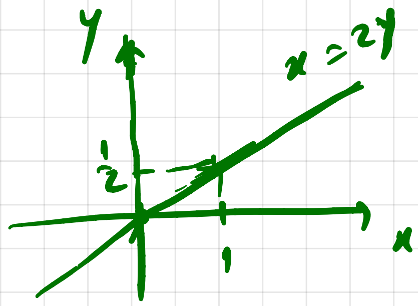
04) Obtenha o volume do tetraedro limitado pelos planos $x+2y+z=2$, $x=2y$, $x=0$ e $z=0$.

PLANO YZ (x = 0) · PLANO XY (z = 0)

$z = 2 - x - 2y$

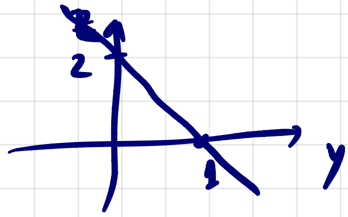
SOLUÇÃO:



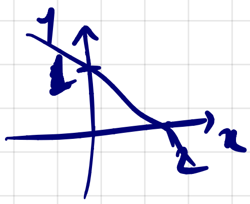


$x+2y+z=2$

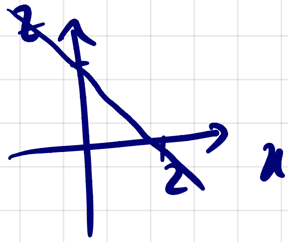
• $x=0$: $2y+z=2$



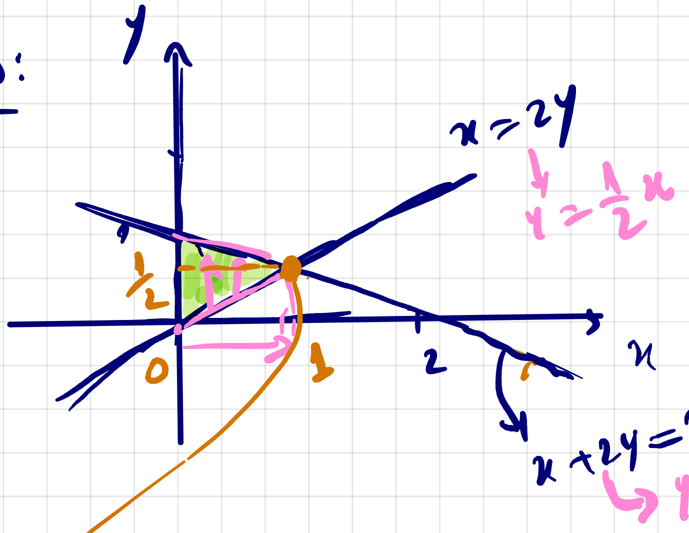
• $z=0$:
$x+2y=2$



• $y=0$: $x+z=2$

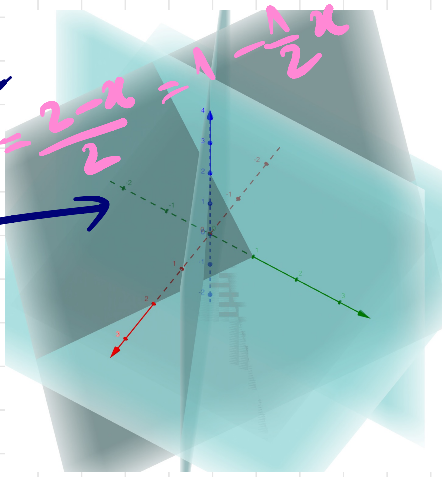


REGIÃO:



$z = f(x,y) = 2 - x - 2y$

NO GEOGEBRA.



INTERSEÇÃO:

$$\begin{cases} x = 2y \\ x + 2y = 2 \end{cases}$$ (INTERCEPTO DO PLANO COM O PLANO XY

$$2y + 2y = 2$$
$$4y = 2 \Rightarrow y = \frac{1}{2} \quad \begin{cases} x = 2 \cdot \frac{1}{2} \\ x = 1 \end{cases}$$

Disso, temos:

$$V = \int_{x=0}^{x=1} \int_{y=\frac{1}{2}x}^{y=1-\frac{1}{2}x} f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left(\int_{y=\frac{1}{2}x}^{y=1-\frac{1}{2}x} (2-x-2y)\, dy \right) dx =$$

$$= \int_{x=0}^{x=1} \left(2y - xy - y^2\right)\Big|_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} dx =$$

$$= \int_0^1 2\cdot\left(1-\frac{x}{2}\right) - x\cdot\left(1-\frac{x}{2}\right) - \left(1-\frac{x}{2}\right)^2 - \left[\cancel{2}\cdot\frac{x}{\cancel{2}} - x\cdot\frac{x}{2} - \left(\frac{x}{2}\right)^2\right] dx$$

$$= \int_0^1 \left(2 - x - x + \frac{x^2}{2} - 1 + \cancel{x} - \cancel{\frac{x^2}{4}} - \cancel{x} + \frac{x^2}{2} + \cancel{\frac{x^2}{4}}\right) dx$$

$$= \int_0^1 \left(1 - 2x + x^2\right) dx = \left(x - x^2 + \frac{x^3}{3}\right)\Big|_0^1 =$$

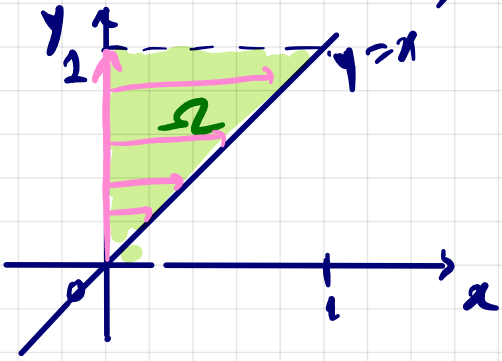
$$1 - 1 + \frac{1}{3} - 0 = \frac{1}{3} \Rightarrow \boxed{V = \frac{1}{3}}$$

05) Calcule $\displaystyle\int_0^1 \int_x^1 \operatorname{sen} y^2 \, dy \, dx.$

<u>SOLUÇÃO:</u> Note que na ordem em que estão as integrações não é possível calcular $\int \operatorname{sen} y^2 dy$.

Neste caso, faz-se necessário trocar a ordem de integração. Vejamos a região:

$$\Omega := \{ (x,y) \in \mathbb{R}^2 ;\ 0 \leq x \leq 1 \ ;\ x \leq y \leq 1 \}$$



FAREMOS VARIAR
y DE 0 ATÉ 1 E
x DE 0 ATÉ y.
(PARA TROCAR A ORDEM DE INTEGRAÇÃO)

Assim:

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \operatorname{sen}^2 y \, dy \, dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} \operatorname{sen}^2 y \, dx \, dy =$$

$$= \int_{y=0}^{y=1} \operatorname{sen} y^2 \left(\int_{x=0}^{x=y} dx \right) dy = \int_{y=0}^{y=1} \operatorname{sen} y^2 \cdot x \Big|_0^y dy$$

$$= \frac{1}{2} \int_0^1 \operatorname{sen} y^2 \, 2y \, . \, dy = \frac{1}{2} \left(-\cos y^2 \right) \Big|_0^1 = -\frac{1}{2} \cos 1 +$$

$\int \operatorname{sen} u \, du = -\cos u + c$

$u = y^2 \Rightarrow du = 2y\,dy$

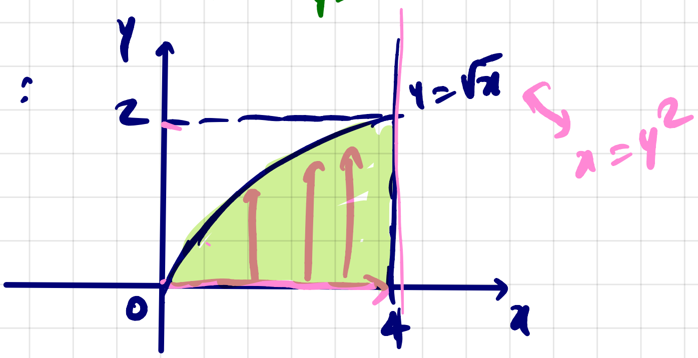
$$= -\frac{1}{2}\cos 1 - \left(-\frac{1}{2}\cos 0\right) = -\frac{1}{2}\cos 1 + \frac{1}{2}$$

$$= \frac{1 - \cos 1}{2}$$

06) Esboce a região de integração e faça a mudança da ordem de integração para

$$\int_{x=0}^{4} \int_{y=0}^{\sqrt{x}} f(x,y)\,.\,dy\,dx$$

SOLUÇÃO:



$$\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{x}} f(x,y)\,dy\,dx = \int_{y=0}^{y=2} \int_{x=y^2}^{x=4} f(x,y)\,.\,dx\,dy$$

07) Idem para $\displaystyle\int_{1}^{2} \int_{0}^{\ln x} f(x,y)\,.\,dy\,dx$.

(EXERCÍCIO)