LISTA 09

01) (a) $f(x,y) = x^2y - 2xy$

$$\frac{\partial f}{\partial x} = 2xy - 2y \quad \text{e} \quad \frac{\partial f}{\partial y} = x^2 - 2x$$

Como estas derivadas parciais são contínuas, segue que f é diferenciável.

$d_a f = ?$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}$

MATRIZ JACOBIANA DA f.

$\Rightarrow d_a f$ será uma matriz 1×2.

$$d_a f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy - 2y & x^2 - 2x \end{bmatrix}$$

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02) (b) $f(x,y,z) = (\overbrace{x^2+y^2+z^2}^{f_1}, \overbrace{xyz}^{f_2}, \overbrace{\cos xy}^{f_3}, \overbrace{x^2 - yz}^{f_4})$

$f: \mathbb{R}^3 \to \mathbb{R}^4$. Logo, $d_a f$ é matriz 4×3

$$d_a f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} \end{bmatrix}, \quad \text{onde;}$$

sendo: • $f_1(x,y,z) = x^2+y^2+z^2$; temos:

$$\frac{\partial f_1}{\partial x} = 2x \; ; \quad \frac{\partial f_1}{\partial y} = 2y \; ; \quad \frac{\partial f_1}{\partial z} = 2z \; ;$$

• $f_2(x,y,z) = xyz$:

$$\frac{\partial f_2}{\partial x} = yz \quad ; \quad \frac{\partial f_2}{\partial y} = xz \quad ; \quad \frac{\partial f_2}{\partial z} = xy$$

• $f_3(x,y,z) = \cos xy$:

$$\frac{\partial f_3}{\partial x} = -y \operatorname{sen} xy \quad ; \quad \frac{\partial f_3}{\partial y} = -x.\operatorname{sen} xy \quad ; \quad \frac{\partial f_3}{\partial z} = 0$$

• $f_4(x,y,z) = x^2 - yz$:

$$\frac{\partial f_4}{\partial x} = 2x \quad ; \quad \frac{\partial f_4}{\partial y} = -z \quad ; \quad \frac{\partial f_4}{\partial z} = -y .$$

Assim:

$$J_a f = \begin{bmatrix} 2x & 2y & 2z \\ yz & xz & xy \\ -y \operatorname{sen} xy & -x.\operatorname{sen} xy & 0 \\ 2x & -z & -y \end{bmatrix}$$

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03)

• $$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{3\Delta x^2 . 0}{\Delta x^4 + 0} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 = 0 .$$

$$\bullet\ \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{3 \cdot 0 \cdot \Delta y^2}{0 + \Delta y^4} - 0}{\Delta y}$$

$$= \lim_{\Delta y \to 0} 0 = 0$$

Logo, $\frac{\partial f}{\partial x}(0,0) = 0$ e $\frac{\partial f}{\partial y}(0,0) = 0$.

No entanto, f não é diferenciável em $(0,0)$ pois não será contínua na origem. De fato, sejam os caminhos:

$$\bullet\ \underbrace{\lim_{\substack{(x,y) \to (0,0) \\ y = x}}}\ f(x,y) = \lim_{\substack{(x,y) \to (0,0) \\ y = x}} \frac{3x^2 \cdot x^2}{x^4 + x^4} = \lim_{\substack{(x,y) \to (0,0) \\ y = x}} \frac{3\cancel{x^4}}{2\cancel{x^4}} = \frac{3}{2}$$

$$\bullet\ \underbrace{\lim_{\substack{(x,y) \to (0,0) \\ x = 0}}}\ f(x,y) = \lim_{\substack{(x,y) \to (0,0) \\ x = 0}} \frac{3 \cdot 0 \cdot y^2}{0^4 + y^4} = \lim_{\substack{(x,y) \to (0,0) \\ x = 0}} 0 = 0$$

$\neq$

Logo, f não é cont. em $(0,0)$. Portanto, não será diferenciável ali.

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06) (a) $u = \ln xy + y^2$; $x = e^t$; $y = e^{-t}$

$$\frac{\partial u}{\partial t} = ?$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

Obs.: como $x = x(t)$ e $y = y(t)$, i.e., funções de uma única variável, poderíamos escrever:

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$$

No nosso caso, temos: $\frac{du}{dx} = \frac{\cancel{y}}{x\cancel{y}} = \frac{1}{x}$

$$\frac{du}{dy} = \frac{\cancel{x}}{\cancel{x}y} + 2y = \frac{1}{y} + 2y = \frac{1+2y^2}{y}$$

$\frac{dx}{dt} = e^t$; $\frac{dy}{dt} = -e^{-t}$. Logo:

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$$

$$= \frac{1}{x} \cdot e^t + \frac{1+2y^2}{y} \cdot (-e^{-t}) =$$

$x = e^t$
$y = e^{-t}$

$$= \frac{1}{\cancel{e^t}} \cdot \cancel{e^t} + \frac{1+2 \cdot e^{-2t}}{\cancel{e^{-t}}} \cdot (-\cancel{e^{-t}}) =$$

$$= 1 + (1 + 2e^{-2t}) \cdot (-1) = \cancel{1} - \cancel{1} - 2e^{-2t}$$

$$= -2e^{-2t}.$$

12) (a) $f(x,y) = \ln\sqrt{x^2+y^2}$; $P(2,1)$; $\theta = 60^\circ$

Com $\theta = 60^\circ$, o vetor unitário básico será $\vec{u} = (\cos\theta, \operatorname{sen}\theta)$, ou seja:

$$\vec{u} = (\cos 60^\circ, \operatorname{sen} 60^\circ) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Queremos, então, obter: $\dfrac{\partial f}{\partial \vec{u}}(2,1)$

Sendo $f(x,y) = \dfrac{1}{2}\ln(x^2+y^2)$

Disso:
$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{1}{2}\dfrac{2x}{x^2+y^2} = \dfrac{x}{x^2+y^2} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{1}{2}\dfrac{2y}{x^2+y^2} = \dfrac{y}{x^2+y^2} \end{cases}$$

Logo: $\nabla f = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$;

Então:
$$\frac{\partial f}{\partial \vec{u}} = \nabla f \cdot \vec{u} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)\cdot\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

VEJA Obs. ABAIXO

$$= \frac{x}{2(x^2+y^2)} + \frac{\sqrt{3}\,y}{2(x^2+y^2)} = \frac{x+\sqrt{3}\,y}{2(x^2+y^2)}$$

Portanto:
$$\frac{\partial f}{\partial \vec{u}}(2,1) = \frac{2+\sqrt{3}\cdot 1}{2[(2)^2+(1)^2]} = \frac{2+\sqrt{3}}{10}$$

Obs.: $f: \mathbb{R}^m \to \mathbb{R}$, $\vec{u}$ vetor unitário.

$\forall a \in \mathbb{R}^m$:

$$\frac{\partial f}{\partial \vec{u}}(a) = f'(a) \cdot \vec{u} = \nabla f(a) \cdot \vec{u}$$

PROVADO EM AULA

NESTE CASO, O PRODUTO É O PROD. ESCALAR, E $\vec{u} = (u_1, u_2, \ldots, u_m)$

$$f'(a) = \left[\frac{\partial f}{\partial x_1}(a) \quad \frac{\partial f}{\partial x_2}(a) \quad \cdots \quad \frac{\partial f}{\partial x_m}(a) \right]_{1 \times m}$$

EQUIVALE AO VETOR GRADIENTE

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \ldots, \frac{\partial f}{\partial x_m}(a) \right)$$

ESTE PRODUTO É ENTRE MATRIZES, E NESTE CASO,

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$