Lista 06

5. Em cada exercício a seguir, mostre que existe $\lim_{(x,y)\to(0,0)} f(x,y)$.

(a) $f(x,y) = \dfrac{x^2y+xy^2}{x^2+y^2}$ (b) $f(x,y) = \dfrac{2xy}{\sqrt{x^2+y^2}}$ (c) $f(x,y) = \dfrac{x^2+2xy}{\sqrt{x^2+y^2}}$

(c) Dado $\varepsilon > 0$. Vamos obter $\delta > 0$ tal que,

$\forall (x,y) \in \mathbb{R}^2 : 0 < d((x,y),(0,0)) < \delta \Rightarrow |f(x,y) - 0| < \varepsilon$.

$\sqrt{(x-0)^2+(y-0)^2} < \delta$

$0 < \sqrt{x^2+y^2} < \delta$

Analisando $|f(x,y) - 0|$:

$$0 \leq |f(x,y) - 0| = \left|\frac{x^2+2xy}{\sqrt{x^2+y^2}}\right| \leq \frac{|x|^2 + 2\cdot|x|\cdot|y|}{\sqrt{x^2+y^2}} \Leftrightarrow$$

$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$

$|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$

$$\Leftrightarrow \frac{|x|\cdot(|x| + 2\cdot|y|)}{\sqrt{x^2+y^2}} \leq \frac{\cancel{\sqrt{x^2+y^2}}\cdot\left(\sqrt{x^2+y^2} + 2\sqrt{x^2+y^2}\right)}{\cancel{\sqrt{x^2+y^2}}}$$

$$\Rightarrow \quad |f(x,y) - 0| \leq 3\cdot\underbrace{\sqrt{x^2+y^2}}_{<\delta} = 3\delta := \varepsilon$$

$$\boxed{\delta = \frac{\varepsilon}{3}}$$

L6

1. Use a definição de limite para provar que:

(a) $\lim_{(x,y)\to(1,2)} (2x-4y) = -6$ (b) $\lim_{(x,y)\to(1,1)} (x^2+y^2) = 2$.

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(b) $\lim_{(x,y)\to(1,1)} x^2+y^2 = 2$.

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x^2+y^2$

Dado $\varepsilon > 0$. Precisamos encontrar $\delta > 0$ tal que,

$\forall (x,y) \in \mathbb{R}^2 : 0 < d((x,y),(1,1)) < \delta \Rightarrow |f(x,y) - 2| < \varepsilon$

$\Leftrightarrow \sqrt{(x-1)^2+(y-1)^2} < \delta$

Analisando $|f(x,y) - 2|$:

$$|f(x,y) - 2| = |x^2+y^2-2| = |x^2-1+y^2-1| =$$

$$= |(x+1)(x-1) + (y+1)\cdot(y-1)| \leq$$

$$\leq |x+1|\cdot|x-1| + |y+1|\cdot|y-1| \Leftrightarrow$$

NOTE QUE:

$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2+(y-1)^2} < \delta \quad (*)$

$|y-1| = \sqrt{(y-1)^2} \leq \sqrt{(y-1)^2+(x-1)^2} < \delta \quad (**)$

Disso, temos:

- $|x+1| = |x-1+2| \leq \underbrace{|x-1|}_{<\delta \text{ (por } (*))} + 2 < 2+\delta \quad (\mathrm{I})$

- $|y+1| = |y-1+2| \leq \underbrace{|y-1|}_{<\delta \text{, por } (**)} + 2 < 2+\delta \quad (\mathrm{II})$

$$\Leftrightarrow (2+\delta)\cdot\delta + (2+\delta)\cdot\delta = 2\cdot\delta\cdot(2+\delta) := \varepsilon$$

por $(*)$, $(**)$, (I) e (II)

$4\delta + 2\delta^2 = \varepsilon$

$\delta^2 + 2\delta = \frac{\varepsilon}{2}$

$\delta^2 + 2\delta + 1 = \frac{\varepsilon}{2} + 1$

$(\delta+1)^2 = \frac{\varepsilon+2}{2}$

$\delta = \sqrt{\frac{2+\varepsilon}{2}} - 1 > 0$

LISTA 07.

2. Em cada item a seguir, encontrar o vetor tangente unitário para a curva dada no instante t apresentado.

(a) $\vec{f}(t) = (t+1)\vec{i} - t^2\vec{j} + (1-2t)\vec{k}$, em $t = 1$.

$\vec{f}'(t)$ representa o vetor tangente ao gráfico de $\vec{f}$ em t.

$\vec{f}'(t) = 1 \cdot \vec{i} - 2t\,\vec{j} - 2\vec{k}$; $t = 1$:

$\vec{f}'(1) = 1\vec{i} - 2\cdot(1)\vec{j} - 2\vec{k}$

$\vec{f}'(1) = (1, -2, -2)$

O vetor unitário $\vec{u}$ será:

$$\vec{u} = \frac{\vec{f}'(1)}{\|\vec{f}'(1)\|} \text{ , onde :}$$

$$\|\vec{f}'(1)\| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$\Rightarrow \vec{u} = \frac{1}{3}\cdot(1, -2, -2) = \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$$

L7

9. Determinar o comprimento de arco das seguintes curvas:

(a) $\vec{f}(t) = (e^t \cos t,\ e^t \sin t,\ e^t)$, $0 \le t \le 1$.

$$l = \int_0^1 \|\vec{f}'(t)\| \cdot dt$$

$$\vec{f}'(t) = (-\operatorname{sen} t \cdot e^t + e^t \cdot \cos t,\ e^t \cdot \cos t + e^t \cdot \operatorname{sen} t,\ e^t)$$

$$\|\vec{f}'(t)\| = \sqrt{(-\operatorname{sen} t \cdot e^t + e^t \cdot \cos t)^2 + (e^t \cos t + e^t \operatorname{sen} t)^2 + e^{2t}}$$

$$= \sqrt{e^{2t} \cdot [\cos t - \operatorname{sen} t]^2 + e^{2t} \cdot [\cos t + \operatorname{sen} t]^2 + e^{2t}}$$

$$= \sqrt{e^{2t} \cdot [\underbrace{\cos^2 t - \cancel{2 \operatorname{sen} t \cos t} + \operatorname{sen}^2 t}_{1} + \underbrace{\cos^2 t + \cancel{2 \cdot \operatorname{sen} t \cdot \cos t} + \operatorname{sen}^2 t}_{1} + 1]}$$

$$= e^t \cdot \sqrt{3}$$

Assim, teremos:

$$l = \int_0^1 \|\vec{f}'(t)\| \cdot dt = \int_0^1 \sqrt{3} \cdot e^t \cdot dt = \sqrt{3} \cdot \int_0^1 e^t dt = \sqrt{3} \cdot e^t \Big|_0^1$$

$$= \sqrt{3} \cdot (e - 1)$$

LISTA 08.

04) (c) $f(x,y) = \arctan\sqrt{\dfrac{x^2-y^2}{x^2+y^2}}$

$\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$

$(\arctan u)' = \dfrac{u'}{1+u^2}$

$$\frac{\partial f}{\partial x} = \frac{\left(\sqrt{\frac{x^2-y^2}{x^2+y^2}}\right)_x}{1+\left(\sqrt{\frac{x^2-y^2}{x^2+y^2}}\right)^2} =$$

$$= \frac{\frac{1}{2}\cdot\left(\frac{x^2-y^2}{x^2+y^2}\right)^{-\frac{1}{2}}\cdot\frac{(x^2+y^2).2x-(x^2-y^2).2x}{x^2+y^2}}{1+\frac{x^2-y^2}{x^2+y^2}}$$

$$= \frac{\frac{1}{2}\cdot\left(\frac{x^2-y^2}{x^2+y^2}\right)^{-\frac{1}{2}}\cdot\frac{\cancel{2x^3}+2xy^2-\cancel{2x^3}+2xy^2}{x^2+y^2}}{\frac{x^2+\cancel{y^2}+x^2\cancel{-y^2}}{x^2+y^2}}$$

$$= \frac{\frac{1}{\cancel{2}}\cdot\frac{\cancel{4}^{2}\cancel{x}y^2}{x^2+\cancel{y^2}}\cdot\frac{1}{\sqrt{\frac{x^2-y^2}{x^2+y^2}}}}{\frac{2x^{\cancel{2}}}{x^2+\cancel{y^2}}} = \frac{y^2}{\sqrt{\frac{x^2-y^2}{x^2+y^2}}}\cdot\frac{1}{2x}$$

$$= \frac{y^2}{2x\sqrt{\frac{x^2-y^2}{x^2+y^2}}}$$

13. Se $z = \sqrt{y+ax} + \arctan(y-ax)$, verifique que

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}.$$

- $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$
- $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$

$$z = (y+ax)^{\frac{1}{2}} + \arctan(y-ax)$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(y+ax)^{-\frac{1}{2}}\cdot a + \frac{-a}{1+(y-ax)^2}$$

$$= \frac{a}{2\sqrt{y+ax}} - \frac{a}{1+(y-ax)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{a}{2\sqrt{y+ax}} - \frac{a}{1+(y-ax)^2}\right) = \ldots$$

(...)