Lista 09

3. Ache a matriz de mudança de base da base $\beta = \{(1,1,0);(0,1,0);(0,0,3)\}$ para a base $\gamma = \{(1,1,1);(1,0,1);(1,0,-1)\}$ do $\mathbb{R}^3$. Ache também a matriz de mudança da base γ para a base β.

Vamos determinar $[I]_{\gamma}^{\beta}$

Para isto, sendo $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\gamma = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

$$\vec{v}_1 = a_{11}\cdot\vec{w}_1 + a_{21}\vec{w}_2 + a_{31}\vec{w}_3 \quad (\mathrm{I})$$

$$\vec{v}_2 = a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + a_{32}\vec{w}_3 \quad (\mathrm{II})$$

$$\vec{v}_3 = a_{13}\vec{w}_1 + a_{23}\vec{w}_2 + a_{33}\vec{w}_3 \quad (\mathrm{III})$$

De (I), obtemos:

$$\vec{v}_1 = a_{11}\cdot\vec{w}_1 + a_{21}\vec{w}_2 + a_{31}\vec{w}_3$$

$$(1,1,0) = a_{11}\cdot(1,1,1) + a_{21}\cdot(1,0,1) + a_{31}\cdot(1,0,-1)$$

$$\begin{cases} a_{11} + a_{21} + a_{31} = 1 \\ a_{11} = 1 \\ a_{11} + a_{21} - a_{31} = 0 \end{cases} \sim \begin{cases} \cancel{1} + a_{21} + a_{31} = \cancel{1} \\ 1 + a_{21} - a_{31} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_{21} + \cancel{a_{31}} = 0 \\ a_{21} - \cancel{a_{31}} = -1 \end{cases} +$$

$$2a_{21} = -1 \Rightarrow \boxed{a_{21} = -\frac{1}{2}}$$

$$a_{31} = -a_{21}$$

$$a_{31} = +\frac{1}{2}$$

De (II), temos:

$$\vec{v}_2 = a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + a_{32}\vec{w}_3$$

$$(0,1,0) = a_{12}\cdot(1,1,1) + a_{22}\cdot(1,0,1) + a_{32}\cdot(1,0,-1)$$

$$\begin{cases} a_{12} + a_{22} + a_{32} = 0 \\ a_{12} = 1 \\ a_{12} + a_{22} - a_{32} = 0 \end{cases} \rightarrow \begin{cases} 1 + a_{22} + a_{32} = 0 \\ 1 + a_{22} - a_{32} = 0 \end{cases}$$

$$\sim \begin{cases} a_{22} + \cancel{a_{32}} = -1 \\ a_{22} - \cancel{a_{32}} = -1 \end{cases} +$$

$$2\cdot a_{22} = -2$$

$$a_{22} = -1$$

$$\cancel{-1} + a_{32} = \cancel{-1}$$

$$a_{32} = 0$$

De (III), temos:

$$\vec{v}_3 = a_{13}\vec{w}_1 + a_{23}\vec{w}_2 + a_{33}\vec{w}_3$$

$$(0,0,3) = a_{13}\cdot(1,1,1) + a_{23}\cdot(1,0,1) + a_{33}\cdot(1,0,-1)$$

$$\begin{cases} a_{13} + a_{23} + a_{33} = 0 \\ a_{13} = 0 \\ a_{13} + a_{23} - a_{33} = 3 \end{cases} \sim + \begin{cases} a_{23} + \cancel{a_{33}} = 0 \\ a_{23} - \cancel{a_{33}} = 3 \end{cases}$$

$$2\cdot a_{23} = 3$$

$$\boxed{a_{23} = \frac{3}{2}}$$

$$\frac{3}{2} + a_{33} = 0$$

$$a_{33} = -\frac{3}{2}$$

Assim, a matriz $[I]^{\beta}_{\gamma}$ será:

$$[I]^{\beta}_{\gamma} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix}.$$

Como $[I]^{\gamma}_{\beta} = \left([I]^{\beta}_{\gamma}\right)^{-1}$, então, basta obter a inversa da matriz $[I]^{\beta}_{\gamma}$, que será $[I]^{\gamma}_{\beta}$.

$$\left([I]^{\beta}_{\gamma} \,\vdots\, I\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & -1 & \frac{3}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{3}{2} & 0 & 0 & 1 \end{array}\right) \sim \sim \ldots \sim \left(I \,\vdots\, [I]^{\gamma}_{\beta}\right)$$

EXERCÍCIO

seq. de op. elementares sobre linhas

$$\begin{pmatrix} 1 & 1 & 0 \\ \frac{-1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & \frac{-3}{2} \end{pmatrix}^{(-1)} = ?$$

O cálculo da Matriz Inversa utilizando a eliminação de Gauss-Jordan:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ \frac{-1}{2} & -1 & \frac{3}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-3}{2} & 0 & 0 & 1 \end{pmatrix} \times \left(\frac{1}{2}\right) \underset{L_2 - (\frac{-1}{2}) \cdot L_1 \to L_2}{\sim} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-3}{2} & 0 & 0 & 1 \end{pmatrix} \times \left(\frac{-1}{2}\right) \underset{L_3 - (\frac{1}{2}) \cdot L_1 \to L_3}{\sim}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{-1}{2} & \frac{-3}{2} & \frac{-1}{2} & 0 & 1 \end{pmatrix} \times (-2) \underset{L_2 / (\frac{-1}{2}) \to L_2}{\sim} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & -2 & 0 \\ 0 & \frac{-1}{2} & \frac{-3}{2} & \frac{-1}{2} & 0 & 1 \end{pmatrix} \times \left(\frac{1}{2}\right) \underset{L_3 - (\frac{-1}{2}) \cdot L_2 \to L_3}{\sim}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & -2 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 \end{pmatrix} \times \left(\frac{-1}{3}\right) \underset{L_3 / (-3) \to L_3}{\sim} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \end{pmatrix} \times (3) \underset{L_2 - (-3) \cdot L_3 \to L_2}{\sim}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \end{pmatrix} \times (-1) \underset{L_1 - 1 \cdot L_2 \to L_1}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ \frac{-1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & \frac{-3}{2} \end{pmatrix}^{(-1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \end{pmatrix}$$

$$[I]^{\gamma}_{\beta} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

EXTRA:

sendo $[\vec{u}]_{\beta} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$; obter $[\vec{u}]_{\gamma}$.

$$[\vec{u}]_{\gamma} = [I]^{\beta}_{\gamma} \cdot [\vec{u}]_{\beta}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 0 & 0 \\ +1 & +0 + \frac{3}{2} \\ -1 & +0 - \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix}$$

LISTA 10

5. Ache a transformação linear $T : \mathbb{R}^3 \to \mathbb{R}^2$ tal que $T(1,0,0) = (2,0)$; $T(0,1,0) = (1,1)$ e $T(0,0,1) = (0,-1)$. Em seguida, obtenha $\vec{v} \in \mathbb{R}^3$ tal que $T(\vec{v}) = (3,2)$.

$$(*) \begin{cases} T(1,0,0) = (2,0) \\ T(0,1,0) = (1,1) \\ T(0,0,1) = (0,-1) \end{cases}$$

$\{(1,0,0);(0,1,0);(0,0,1)\}$ forma uma base do $\mathbb{R}^3$ (base canônica)

Então, existe uma única $T : \mathbb{R}^3 \to \mathbb{R}^2$ que cumpre $(*)$.

Dado $(x,y,z) \in \mathbb{R}^3$, então

$$(x,y,z) = a\cdot(1,0,0) + b(0,1,0) + c\cdot(0,0,1)$$

$$\begin{cases} a = x \\ b = y \\ c = z \end{cases}$$

Então:

$$(x,y,z) = x\cdot(1,0,0) + y\cdot(0,1,0) + z\cdot(0,0,1)$$

Como T deve ser linear, então:

$$T(x,y,z) = T\big(x\cdot(1,0,0) + y\cdot(0,1,0) + z\cdot(0,0,1)\big)$$

$$= T(x\cdot(1,0,0)) + T(y\cdot(0,1,0)) + T(z(0,0,1))$$

$$= x\cdot\underbrace{T(1,0,0)}_{(2,0)} + y\cdot\underbrace{T(0,1,0)}_{(1,1)} + z\cdot\underbrace{T(0,0,1)}_{(0,-1)}$$

POR $(*)$

$$= x\cdot(2,0) + y\cdot(1,1) + z\cdot(0,-1)$$

$$= (2x+y,\ y-z)$$

Resp.: $T : \mathbb{R}^3 \to \mathbb{R}^2$

$T(x,y,z) = (2x+y,\ y-z)$

LISTA 10

8. Determine a transformação linear $T : \mathbb{R}^2 \to \mathbb{R}^3$ tal que $T(-1,1) = (3,2,1)$ e $T(0,1) = (1,1,0)$.

Note que $\{(-1,1) ; (0,1)\}$ é L.I ; e como $\dim \mathbb{R}^2 = 2$, segue que tal conj. é uma base para o $\mathbb{R}^2$.

Seja $(x,y) \in \mathbb{R}^2$. Então

$$(x,y) = a.(-1,1) + b.(0,1)$$

$$\Rightarrow \begin{cases} -a = x \rightsquigarrow a = -x \\ a + b = y \end{cases}$$

$$a + b = y$$
$$-x + b = y \rightsquigarrow \boxed{b = x + y}$$

Logo: $(x,y) = -x.(-1,1) + (x+y)(0,1)$

Como T a ser determinada DEVE ser linear:

$$T(x,y) = -x.\underbrace{T(-1,1)}_{=(3,2,1)} + (x+y).\underbrace{T(0,1)}_{=(1,1,0)}$$

$$\Rightarrow T(x,y) = -x.(3,2,1) + (x+y).(1,1,0)$$

$$T(x,y) = (-3x + x + y, -2x + x + y, -x + 0)$$

$$\Rightarrow T(x,y) = (-2x + y, -x + y, -x)$$

EXTRA: $T : \mathbb{R}^2 \to \mathbb{R}^3$, T é injetiva?

$$\ker(T) = \{(x,y) \in \mathbb{R}^2 : T(x,y) = (0,0,0)\}$$
$$= \{(x,y) \in \mathbb{R}^2 : (-2x+y, -x+y, -x) = (0,0,0)\} ;$$

ou seja, $\ker(T)$ é a solução do sistema homogêneo:

$$\begin{cases} -2x + y = 0 \\ -x + y = 0 \\ -x = 0 \rightsquigarrow x = 0 \end{cases} \qquad y = 0$$

Logo, $\ker(T) = \{(0,0)\}$, donde segue por proposição que T é injetiva.

Lista 10

12. Seja $T : \mathbb{R}^3 \to \mathbb{R}^3$ a aplicação linear dada por

$$T(x, y, z) = (x + 2y - z,\ y + z,\ x + y - 2z).$$

Determine uma base e a dimensão para o núcleo e para a imagem de T.

$\operatorname{Ker} T = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$

$= \{(x,y,z) \in \mathbb{R}^3 : (x+2y-z,\ y+z,\ x+y-2z) = (0,0,0)\}$;

ou seja, $\operatorname{Ker}(T)$ será a solução do sistema homogêneo:

$$\begin{cases} x + 2y - z = 0 \\ y + z = 0 \rightsquigarrow y = -z \\ x + y - 2z = 0 \end{cases} \rightsquigarrow \begin{cases} x + 2\cdot(-z) - z = 0 \\ x - z - 2z = 0 \end{cases}$$

$$\longrightarrow \begin{cases} x - 3z = 0 \\ x - 3z = 0 \end{cases} \Rightarrow x = 3z$$

$\Rightarrow \operatorname{Ker}(T) = \{(x,y,z) \in \mathbb{R}^3 : x = 3z \text{ e } y = -z\}$

$= \{(3z, -z, z) : z \in \mathbb{R}\} = \{z \cdot (3,-1,1) : z \in \mathbb{R}\}$

$= [(3,-1,1)]$.

Como $\operatorname{Ker}(T)$ é formado por um único vetor não nulo, segue que $\{(3,-1,1)\}$ é uma base para $\operatorname{Ker}(T)$. (pois será L.I.).

Disso, segue que $\dim \operatorname{Ker}(T) = 1$

$\operatorname{Im}(T) = ?$

$\operatorname{Im}(T) = \{T(x,y,z) : x,y,z \in \mathbb{R}\}$

$= \{(x+2y-z,\ y+z,\ x+y-2z) : x,y,z \in \mathbb{R}\}$

$= \{(x, 0, x) + (2y, y, y) + (-z, z, -2z) : x,y,z \in \mathbb{R}\}$

$= \{x\cdot(1,0,1) + y\cdot(2,1,1) + z\cdot(-1,1,-2) : x,y,z \in \mathbb{R}\}$

$= [(1,0,1);(2,1,1);(-1,1,2)]$

Verifiquemos se o conj.: $\{(1,0,1);(2,1,1);(-1,1,2)\}$ é L.I.:

$$a\cdot(1,0,1) + b\cdot(2,1,1) + c\cdot(-1,1,2) = (0,0,0)$$

$$\Longleftrightarrow \begin{cases} a + 2b - c = 0 \\ b + c = 0 \rightsquigarrow b = -c \\ a + b + 2c = 0 \end{cases}$$

$$\rightsquigarrow \begin{cases} a + 2\cdot(-c) - c = 0 \\ a - c + 2c = 0 \end{cases} \rightsquigarrow \begin{cases} a - 3c = 0 \\ a + c = 0 \rightarrow a = -c \end{cases}$$

$a - 3c = 0$

$-c - 3c = 0$

$-4c = 0 \Rightarrow c = 0$

$\Rightarrow a = 0$ e $b = 0$

Logo, o conj. é L.I. e disso, o conj. $\{(1,0,1);(2,1,1);(-1,1,2)\}$ é uma base para $\operatorname{Im}(T)$. Logo, $\dim \operatorname{Im}(T) = 3$.