05/05/23

TEOREMA (Regra da Cadeia)

Seja $u = f(x,y)$ uma função $\mathbb{R}^2 \to \mathbb{R}$ diferenciável, e $x = x(r,s)$, $y = y(r,s)$ funções de r e s, tais que $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial r}$, $\frac{\partial y}{\partial s}$ existam. Então:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r},$$

e

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

DEMONSTR.: Como $u = f(x,y)$ é diferenciável, então,

$$\boxed{\Delta u = \Delta f(x,y) = \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \Delta y,} \quad (*)$$

onde $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_1 = 0$ e $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_2 = 0$

Tome $\Delta r \neq 0$ um incremento para a variável r, e mantenha s fixo.

Note que $\begin{cases} \Delta x = x(r+\Delta r, s) - x(r,s) \\ \Delta y = y(r+\Delta r, s) - y(r,s) \end{cases}$

Dividindo $(*)$ por $\Delta r \neq 0$, vamos obter:

$$\frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial x} \cdot \frac{\Delta x}{\Delta r} + \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta r} + \varepsilon_1 \cdot \frac{\Delta x}{\Delta r} + \varepsilon_2 \cdot \frac{\Delta y}{\Delta r}$$

Fazendo a passagem ao limite com $\Delta r \to 0$, vamos obter:

- $$\lim_{\Delta r \to 0} \frac{\Delta u}{\Delta r} = \lim_{\Delta r \to 0} \frac{u(r+\Delta r, s) - u(r,s)}{\Delta r}$$

$$= \frac{\partial u}{\partial r}$$

- $$\lim_{\Delta r \to 0} \frac{\Delta x}{\Delta r} = \lim_{\Delta r \to 0} \frac{x(r+\Delta r, s) - x(r,s)}{\Delta r} = \frac{\partial x}{\partial r}$$

- $$\lim_{\Delta r \to 0} \frac{\Delta y}{\Delta r} = \lim_{\Delta r \to 0} \frac{y(r+\Delta r, s) - y(r,s)}{\Delta r} = \frac{\partial y}{\partial r}$$

Então, teremos:

$$\lim_{\Delta r \to 0} \frac{\Delta u}{\Delta r} = \lim_{\Delta r \to 0} \left(\frac{\partial u}{\partial x} \cdot \frac{\Delta x}{\Delta r} + \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta r} + \underbrace{\varepsilon_1 \cdot \frac{\Delta x}{\Delta r}}_{\to 0} + \underbrace{\varepsilon_2 \cdot \frac{\Delta y}{\Delta r}}_{\to 0} \right)$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}}.$$

A prova da outra igualdade é análoga.

$\square$

Ex: $f(x,y) = 3x^2y - y$; $x = e^{rs}$, $y = r^2s^2$

Obter $\frac{\partial f}{\partial r}$; $\frac{\partial f}{\partial s}$.

Solução:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$; onde:

$$\begin{cases} \frac{\partial f}{\partial x} = 6xy \ ; \ \frac{\partial f}{\partial y} = 3x^2 - 1 \\ \frac{\partial x}{\partial r} = s\cdot e^{rs} \\ \frac{\partial y}{\partial r} = 2rs^2 \end{cases}$$

Assim, teremos:

- $\frac{\partial f}{\partial r} = 6xy\cdot s\cdot e^{rs} + (3x^2 - 1)\cdot 2rs^2$;

e como $x = e^{rs}$ e $y = r^2s^2$, teremos:

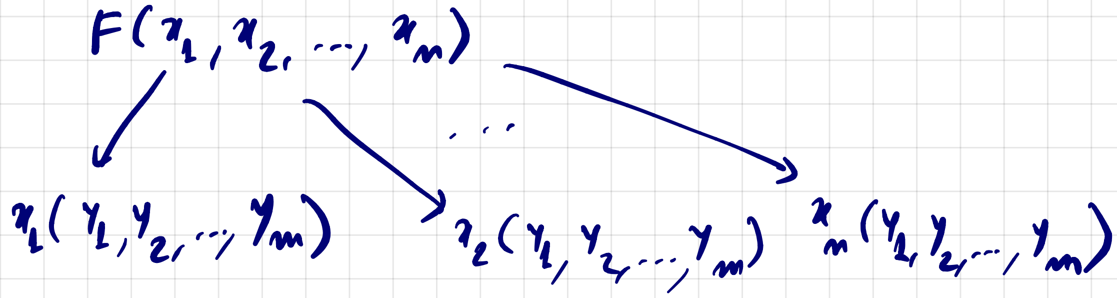
$$= 6\cdot e^{rs}\cdot r^2\cdot s^2 s\cdot e^{rs} + (3\cdot(e^{rs})^2 - 1)\cdot 2rs^2$$

$$= 6\cdot r^2s^3\cdot e^{2rs} + 6rs^2\cdot e^{2rs} - 2rs^2$$

$$= 6rs^2e^{2rs}(rs + 1) - 2rs^2$$

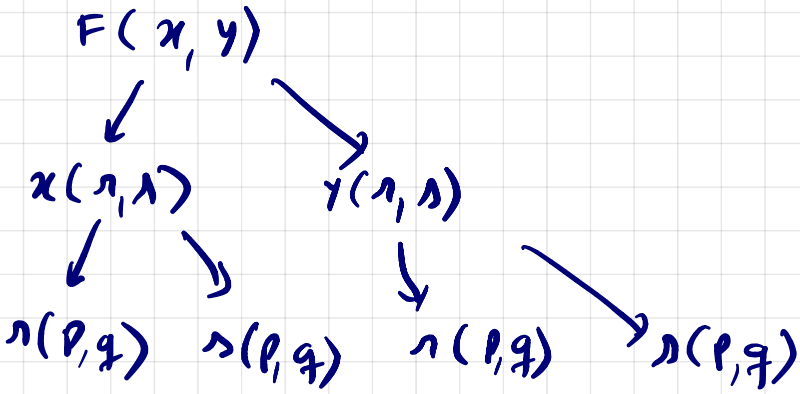
- $\frac{\partial f}{\partial s} = \ldots$ exercício.

Obs.: Podemos ampliar a regra da cadeia para mais variáveis, c.f. o esquema em "árvore":



$$\frac{\partial F}{\partial y_i} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial y_i} + \frac{\partial F}{\partial x_3} \cdot \frac{\partial x_3}{\partial y_i} + \dots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial y_i}$$

Além disso:



$$\frac{\partial F}{\partial p} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial p} =$$

$$= \frac{\partial F}{\partial x} \cdot \left(\frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial p} + \frac{\partial x}{\partial s} \cdot \frac{\partial s}{\partial p} \right) + \frac{\partial F}{\partial y} \cdot \left(\frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial p} + \frac{\partial y}{\partial s} \cdot \frac{\partial s}{\partial p} \right)$$

Obs.: Vimos na aula passada que, em geral;

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a) \, ;$$

i.e.;

$$d_a(g \circ f) = d_{f(a)} g \cdot d_a f$$

(MATRICIALMENTE)

Tome $u = g(x,y)$

$$\left.\begin{array}{l} x = x(r,s) \\ y = y(r,s) \end{array}\right\} f$$

$$u = g(x(r,s), y(r,s)) = (g \circ f)\underbrace{(r,s)}_{=a}$$

$$d_a u = d_a\, g \circ f = d_{f(a)} g \cdot d_a f$$

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{bmatrix}$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \, ;$$

e

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \, ;$$

as fórmulas do teorema anterior.

DERIVADA DIRECIONAL:

Def.: Seja $f: \mathbb{R}^m \to \mathbb{R}$, diferenciável em $a \in \mathbb{R}^m$.

Dado $\vec{u} \in \mathbb{R}^m$ um vetor unitário, i.e.; $\|\vec{u}\| = 1$; definimos a DERIVADA DIRECIONAL de f na direção do vetor $\vec{u}$ no ponto a por:

$$\frac{\partial f}{\partial \vec{u}}(a) = \lim_{t \to 0} \frac{f(a + t.\vec{u}) - f(a)}{t}$$

Em particular em $\mathbb{R}^2$; $\vec{i} = (1,0)$ é unitário

Então, dado $f: \mathbb{R}^2 \to \mathbb{R}$; $a \in \mathbb{R}^2$:

$$\frac{\partial f}{\partial \vec{i}}(a) = \lim_{t \to 0} \frac{f(a + t.\vec{i}) - f(a)}{t};$$

sendo $a = (x_0, y_0)$, então

$$a + t_i = (x_0, y_0) + (t, 0) = (x_0 + t, y_0);$$

e disso teremos:

$$\underbrace{\frac{\partial f}{\partial \vec{i}}(a)} = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} = \underbrace{\frac{\partial f}{\partial x}(a)}$$

Do mesmo modo segue que

$$\frac{\partial f}{\partial \vec{j}}(a) = \frac{\partial f}{\partial y}.$$

Ex.: Dada $f(x,y) = 3xy^2$,

calcule $\dfrac{\partial f}{\partial \vec{u}}$, onde $\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Solução:

$$\frac{\partial f}{\partial \vec{u}} = \lim_{t\to 0} \frac{f\Big(\overbrace{(x,y) + t\left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)}^{a + t\vec{u}}\Big) - f(\overbrace{x,y}^{a})}{t}$$

$$= \lim_{t\to 0} \frac{f\left(x + \frac{t}{\sqrt{2}}, y - \frac{t}{\sqrt{2}}\right) - f(x,y)}{t}$$

$$= \lim_{t\to 0} \frac{3\left(x + \frac{t}{\sqrt{2}}\right)\cdot\left(y - \frac{t}{\sqrt{2}}\right)^2 - 3xy^2}{t}$$

$$= \lim_{t\to 0} \frac{3\left(x + \frac{t}{\sqrt{2}}\right)\cdot\left(y^2 - \frac{2ty}{\sqrt{2}} + \frac{t^2}{2}\right) - 3xy^2}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{3xy^2} - \frac{6xyt}{\sqrt{2}} + 3\frac{xt^2}{2} + \frac{3ty^2}{\sqrt{2}} - \frac{3t^2y}{2} + \frac{3t^3}{2\sqrt{2}} \cancel{-3xy^2}}{t}$$

$$= \lim_{t\to 0} \left(\frac{6xy}{\sqrt{2}} + \underbrace{\frac{3xt}{2}}_{\to 0} + \frac{3y^2}{\sqrt{2}} - \underbrace{\frac{3ty}{2}}_{\to 0} + \underbrace{\frac{3t^2}{2\sqrt{2}}}_{\to 0} \right)$$

$$= \frac{6xy}{\sqrt{2}} + \frac{3y^2}{\sqrt{2}} = \frac{3y}{\sqrt{2}}(2x + y)$$

$$\Rightarrow \frac{\partial f}{\partial \vec{u}} = \frac{3y}{\sqrt{2}} \cdot (2x+y)$$

PROPOSIÇÃO: $$\frac{\partial f}{\partial \vec{u}}(a) = f'(a) \cdot \vec{u}$$

DEMONSTR.: De facto:

$$\frac{\partial f}{\partial \vec{u}}(a) = \lim_{t \to 0} \frac{f(a+t\vec{u}) - f(a)}{t} =$$

$$f(a+\overbrace{tu}^{h}) - f(a) = f'(a).h + \|h\|.r(h)$$

$$= \lim_{t \to 0} \frac{f'(a).(\cancel{t}\vec{u})}{\cancel{t}} + \frac{\|t\vec{u}\|}{t} \cdot r(t\vec{u})$$

$$= \lim_{t \to 0} \Bigg(f'(a).\vec{u} + \underbrace{\frac{|t|}{t}}_{\text{LIMITADA}} \cdot \underbrace{\|\vec{u}\|}_{1} \underbrace{r(tu)}_{\to 0} \Bigg)$$

$$= f'(a).\vec{u}$$

Def.: O gradiente de f é definido por: $f = f(x_1, x_2, \dots, x_n)$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

Ex.: $f(x, y) = x^2 y$

$$\Rightarrow \frac{\partial f}{\partial x} = 2xy \quad \text{e} \quad \frac{\partial f}{\partial y} = x^2$$

Logo: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left(2xy, x^2 \right)$
