03/05/23

Vimos na aula passada que, dada

$$f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n, \quad \Omega \text{ aberto de } \mathbb{R}^m,$$

$a \in int(\Omega)$; f é diferenciável se

$$\lim_{x \to a} \frac{f(x) - f(a) - L(x-a)}{\|x-a\|} = 0,$$

onde $L : \mathbb{R}^m \to \mathbb{R}^n$ é transf. linear; e denotamos $L = d_a f$.

Vimos que: $f = (f_1, f_2, \ldots, f_n)$

$$d_a f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ - & - & - & - \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

— MATRIZ JACOBIANA

$$\Rightarrow (d_a f)(x) = L(x).$$

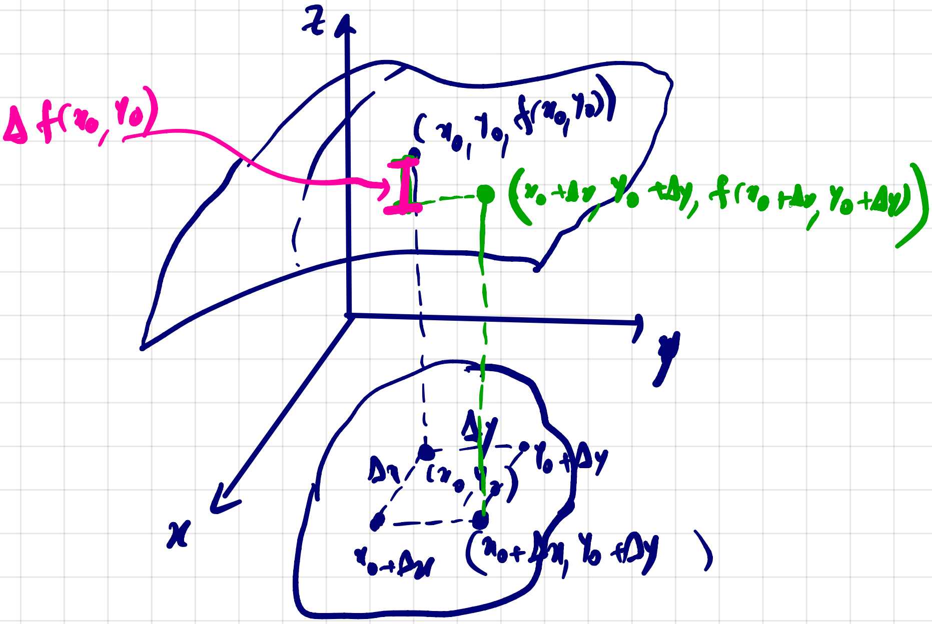
Def.: Seja $f : \mathbb{R}^m \to \mathbb{R}^n$, definimos o <u>incremento</u> de f em a por

$$\Delta f(a) = f(a+h) - f(a).$$

Ex: $f:\mathbb{R}^2\to\mathbb{R}$; $a=(x_0,y_0)$; $h=(\Delta x,\Delta y)$

$$\Delta f(a) = f(a+h) - f(a)$$

$$\Delta f(x_0,y_0) = f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0,y_0)$$



Def: Seja $f:\Omega\subset\mathbb{R}^m\to\mathbb{R}$; Ω um aberto de $\mathbb{R}^m$ e $a\in\Omega$. Dizemos que f é diferenciável em a se o incremento $\Delta f(a)$ for escrito por: pondo $h\in\mathbb{R}^m$, $h=(h_1,h_2,\dots,h_m)$,

então:

$$\Delta f(a) = \frac{\partial f}{\partial x_1}\cdot h_1 + \frac{\partial f}{\partial x_2} h_2 + \dots + \frac{\partial f}{\partial x_m} h_m +$$

$$\varepsilon_1 \cdot h_1 + \varepsilon_2 h_2 + \dots + \varepsilon_m h_m \;;$$

onde $\varepsilon_i \xrightarrow[h\to 0]{} 0$

Ex.: $f: \mathbb{R}^2 \to \mathbb{R}$,

$(x,y) \mapsto f(x,y)$ $\quad a = (x_0, y_0)$

$h = (\Delta x, \Delta y)$

$$\Delta f(a) = \Delta f(x_0, y_0)$$

$$\Rightarrow \boxed{\begin{gathered}\Delta f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\cdot\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \varepsilon_1\cdot\Delta x + \varepsilon_2 \Delta y, \\ \text{com } (\varepsilon_1, \varepsilon_2) \to (0,0) \\ (\Delta x, \Delta y) \to (0,0)\end{gathered}}$$

Note que este conceito de diferenciabilidade usando incremento para funções $\mathbb{R}^m \to \mathbb{R}$ (no caso acima ilustramos de $\mathbb{R}^2$ em $\mathbb{R}$) será equivalente ao conceito usando a matriz jacobiana das derivadas parciais relembrada acima.

De fato, para simplificar considere $f: \mathbb{R}^2 \to \mathbb{R}$.
$a = (x_0, y_0)$; $h = (\Delta x, \Delta y)$
Vimos na aula passada que
$$f(a+h) = f(a) + d_a f(h) + \|h\| \cdot r(h) \quad ;$$

Ou seja:

$$\Delta f(a) = f(a+h) - f(a) = d_a f(h) + \|h\| \cdot r(h)$$

Como $f: \mathbb{R}^2 \to \mathbb{R}^1$, então temos.

$$d_a f = \left[\frac{\partial f}{\partial x}(a) \quad \frac{\partial f}{\partial y}(a) \right]_{1\times 2} \quad . \text{ Disso:}$$

- $$d_a f(h) = \left[\frac{\partial f}{\partial x}(a) \quad \frac{\partial f}{\partial y}(a) \right] \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$= \frac{\partial f}{\partial x}(a) \cdot \Delta x + \frac{\partial f}{\partial y}(a) \cdot \Delta y$$

E:

- $$\|h\| \cdot r(h) = \Delta x \cdot \underbrace{r(h)}_{\varepsilon_1} + \Delta y \cdot \underbrace{r(h)}_{\varepsilon_2} = \varepsilon_1 \cdot \Delta x + \varepsilon_2 \Delta y$$

TOME:

$$\|h\| = |\Delta x| + |\Delta y|$$

$$\Delta x, \Delta y > 0$$

com: $\frac{r(h)}{h} \to 0$, $h \to 0$

$$\varepsilon_1 = \varepsilon_1(\Delta x, \Delta y)$$

$$\varepsilon_2 = \varepsilon_2(\Delta x, \Delta y)$$

Logo:

$$\Delta f(a) = d_a f(h) + \|h\| \cdot r(h)$$

$$= \frac{\partial f}{\partial x}(a) \cdot \Delta x + \frac{\partial f}{\partial y}(a) \cdot \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y$$

TEOREMA: Seja $f: \mathbb{R}^m \to \mathbb{R}^n$, $a \in \mathbb{R}^m$

tal que $\frac{\partial f_i}{\partial x_j}$ existem em a e são contínuas.

Então f será diferenciável em a.

DEMONSTR.: exercício.

Def.: Definimos o <u>diferencial total</u> de

$f: \mathbb{R}^m \to \mathbb{R}$ em $a \in \mathbb{R}^m$ por

$$df(a) = \frac{\partial f}{\partial x_1}(a)\Delta x_1 + \frac{\partial f}{\partial x_2}(a)\Delta x_2 + \cdots + \frac{\partial f}{\partial x_m}(a)\Delta x_m$$

Como nas variáveis dependentes, $\Delta x_i = dx_i$

então,

$$df = \frac{\partial f}{\partial x_1}(a) \cdot dx_1 + \frac{\partial f}{\partial x_2}(a) \cdot dx_2 + \cdots + \frac{\partial f}{\partial x_m}(a) \cdot dx_m$$

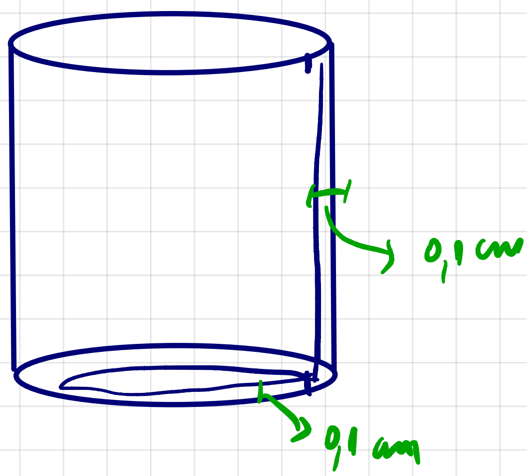
$$\boxed{df \cong \Delta f}$$

Ou seja, o diferencial total de f fica

$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \cdot dx_i$$

EXEMPLO: Um recipiente de metal, fechado, na forma de um cilindro circular reto, tem uma altura interna de 6 cm, um raio interno de 2 cm, e uma espessura de 0,1 cm. Se o custo do metal a ser usado é de R$ 10,00 por cm^3, ache por diferenciais o custo aproximado do metal que será empregado na produção do recipiente.

SOLUÇÃO:



$V = Ab \cdot h$

$V = \pi R^2 h$

$h = 6\,cm \; ; \; \Delta h = 0,2\,cm$

$R = 2\,cm \; . \; \Delta R = 0,1\,cm$

$dV = ?$

$$V = V(R, h) = \pi R^2 h.$$

$$dV = \frac{\partial V}{\partial R} \cdot \Delta R + \frac{\partial V}{\partial h} \cdot \Delta h$$

$$dV = 2\pi R h \cdot \Delta R + \pi R^2 \cdot \Delta h$$

$$dV = 2\pi \cdot 2 \cdot 6 \, (0,1) + \pi \cdot (2)^2 \cdot (0,2)$$

$$dV = 2,4\pi + 0,8\pi$$

$$dV = 3,2\pi \text{ cm}^3$$

$$\Delta V \cong dV = 3,2\pi \text{ cm}^3$$

$$\text{CUSTO} = 10 \cdot \Delta V = 10 \times 3,2\pi$$

$$\text{CUSTO} = 32\pi \text{ reais.}$$

TEOREMA: (REGRA DA CADEIA)

Sejam $f: \mathbb{R}^m \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^k$ funções, com f diferenciável em $a \in \mathbb{R}^m$ e g diferenciável em $g(a) = b$. Então,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

EX-1 $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (f_1(x,y), f_2(x,y))$;
$g: \mathbb{R}^2 \to \mathbb{R}^3$; $g(u,v) = (g_1(u,v); g_2(u,v); g_3(u,v))$

$$(g \circ f)'(a) = \underbrace{\left[d_{f(a)} g \right]}_{3\times 2} \cdot \underbrace{\left[d_a f \right]}_{2\times 2} =$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial u} f(a) & \frac{\partial g_1}{\partial v} f(a) \\ \frac{\partial g_2}{\partial u} f(a) & \frac{\partial g_2}{\partial v} f(a) \\ \frac{\partial g_3}{\partial u} f(a) & \frac{\partial g_3}{\partial v} f(a) \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} \frac{\partial f_1}{\partial x}(a) & \frac{\partial f_1}{\partial y}(a) \\ \frac{\partial f_2}{\partial x}(a) & \frac{\partial f_2}{\partial y}(a) \end{bmatrix}_{2\times 2} =$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial u} f(a) \cdot \frac{\partial f_1}{\partial x}(a) + \frac{\partial g_1}{\partial v} f(a) \cdot \frac{\partial f_2}{\partial x}(a) & \frac{\partial g_1}{\partial u} f(a) \cdot \frac{\partial f_1}{\partial y}(a) + \frac{\partial g_1}{\partial v} f(a) \cdot \frac{\partial f_2}{\partial y}(a) \\ \frac{\partial g_2}{\partial u} f(a) \cdot \frac{\partial f_1}{\partial x}(a) + \frac{\partial g_2}{\partial v} f(a) \cdot \frac{\partial f_2}{\partial x}(a) & \frac{\partial g_2}{\partial u} f(a) \cdot \frac{\partial f_1}{\partial y}(a) + \frac{\partial g_2}{\partial v} f(a) \cdot \frac{\partial f_2}{\partial y}(a) \\ \frac{\partial g_3}{\partial u} f(a) \cdot \frac{\partial f_1}{\partial x}(a) + \frac{\partial g_3}{\partial v} f(a) \cdot \frac{\partial f_2}{\partial x}(a) & \frac{\partial g_3}{\partial u} f(a) \cdot \frac{\partial f_1}{\partial y}(a) + \frac{\partial g_3}{\partial v} f(a) \cdot \frac{\partial f_2}{\partial y}(a) \end{bmatrix}_{3\times 2}$$

Ex.: $f: \mathbb{R}^2 \to \mathbb{R}^2$;

$f(x,y) = (x^2+y^2;\ x^2-y^2)$

$g: \mathbb{R}^2 \to \mathbb{R}^2$;

$g(u,v) = (u \cdot v,\ u+v)$

Ache $(g \circ f)'(x,y)$.

$$(g\circ f)'(x,y) = g'(f(x,y))\cdot f'(x,y) \quad ; \quad \text{onde}$$

- $$g'(u,v) = \begin{bmatrix} \frac{\partial}{\partial u}(u\cdot v) & \frac{\partial}{\partial v}(u\cdot v) \\ \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \end{bmatrix}$$

$$= \begin{bmatrix} v & u \\ 1 & 1 \end{bmatrix}$$

- $$f'(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}(x^2+y^2) & \frac{\partial}{\partial y}(x^2+y^2) \\ \frac{\partial}{\partial x}(x^2-y^2) & \frac{\partial}{\partial y}(x^2-y^2) \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$\Rightarrow (g\circ f)'(x,y) = g'(f(x,y))\cdot f'(x,y) =$$

$$= \begin{bmatrix} x^2-y^2 & x^2+y^2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix} =$$

$$g'(u,v) = \begin{bmatrix} v & u \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow g'(f(x,y)) = g'(x^2+y^2,\ x^2-y^2)$$

$$\Leftrightarrow \begin{bmatrix} 2x(x^2-y^2) + 2x(x^2-y^2) & 2y(x^2-y^2) - 2y(x^2+y^2) \\ 2x \cdot 1 + 2x \cdot 1 & 2y \cdot 1 + (-2y)\, 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4x^3 - 4y^2x & -4y^3 \\ 4x & 0 \end{bmatrix}$$

Assim, por exemplo:

$$(g \circ f)'(2,1) = \begin{bmatrix} 4 \cdot (2)^3 - 4 \cdot 1 \cdot 2 & -4 \cdot (1)^3 \\ 4 \cdot 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -4 \\ 8 & 0 \end{bmatrix}$$