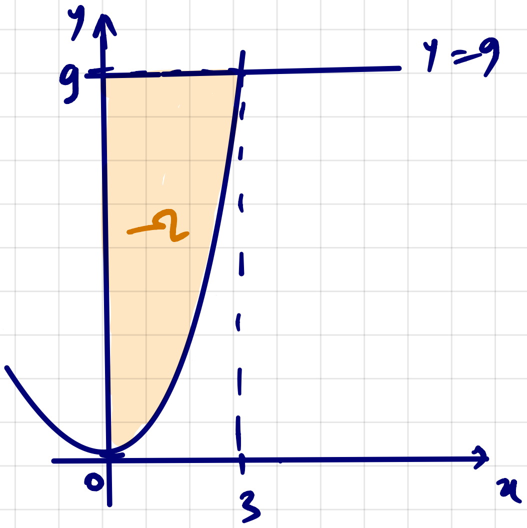
CÁLCULO 3 — GABARITO DA P3.

01) (a) $$\int_{x=0}^{x=3}\int_{y=x^2}^{y=9} x^3 \cdot e^{y^3}\, dy\, dx = \int_{y=0}^{y=9}\int_{x=0}^{x=\sqrt{y}} x^3 \cdot e^{y^3} \cdot dx\, dy$$



$$= \int_{y=0}^{y=9} e^{y^3} \cdot \left(\int_{x=0}^{x=\sqrt{y}} x^3\, dx \right) dy =$$

$$= \int_{y=0}^{y=9} e^{y^3} \cdot \left(\frac{x^4}{4} \right) \Big|_0^{\sqrt{y}} dy =$$

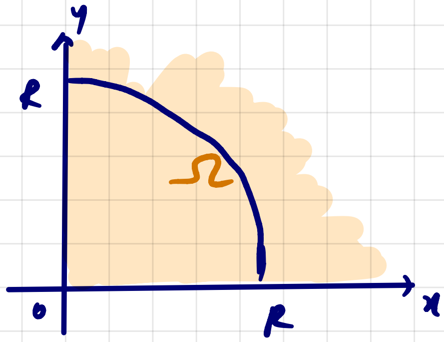
$$= \frac{1}{4} \cdot \int_0^9 e^{y^3} \cdot \left((\sqrt{y})^4 - 0^4 \right) dy = \frac{1}{4} \cdot \frac{1}{3} \int_0^9 e^{y^3} \cdot (3y^2\, dy)$$

$\left[\int e^u du \quad u = y^3 \quad du = 3y^2 dy \right.$

$$= \frac{1}{12} \left[e^{y^3} \right] \Big|_0^9 =$$

$$= \frac{1}{12} \left(e^{9^3} - e^0 \right) = \frac{1}{12} \left(e^{729} - 1 \right)$$

(b) $\int_0^\infty \int_0^\infty e^{-x-y}\,dx\,dy$.



Consider $\Omega_R = B_R(0) \cap \Omega$.

$$\iint_{\Omega_R} e^{-x-y}\,dx\,dy = \lim_{R\to+\infty} \int_0^R \int_0^R e^{-x-y}\,dx\,dy$$

$$= \lim_{R\to+\infty} \int_0^R e^{-x}\,dx \int_0^R e^{-y}\,dy =$$

$$= \lim_{R\to+\infty} -e^{-x}\Big|_0^R \cdot \left(-e^{-y}\right)\Big|_0^R =$$

$$= \lim_{R\to+\infty} \left(-\frac{1}{e^R}+1\right)^2 = 1.$$

(c) $$\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx = \int_{x=0}^{x=1} 2x \int_{y=x}^{y=2x} y \left(\int_{z=0}^{z=y} z\,dz\right) dy\,dx$$

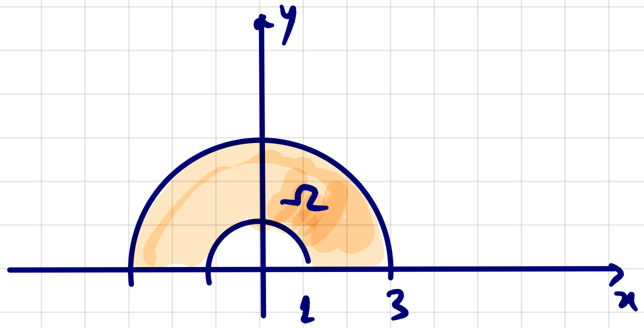
$$= \int_{x=0}^{x=1} \cancel{2}x \int_{y=x}^{y=2x} y \cdot \frac{z^2}{\cancel{2}}\Big|_0^y\,dy\,dx =$$

$$= \int_{x=0}^{x=1} x \cdot \left(\int_{y=x}^{y=2x} y \cdot (y^2 - 0^2)\,dy\right) dx =$$

$$= \int_{x=0}^{x=1} x \cdot \left.\frac{y^4}{4}\right|_{x}^{2x} dx = \int_0^1 x \cdot \frac{16x^4 - x^4}{4} dx =$$

$$= \frac{15}{4} \int_0^1 x^5 dx = \frac{15}{4} \cdot \left.\frac{x^6}{6}\right|_0^1 = \frac{15}{4} \cdot \frac{1}{6} = \frac{\cancel{3} \cdot 5}{2 \cdot 2 \cdot \cancel{3} \cdot 2} = \frac{5}{8}$$

02) $\iint_\Omega \arctan \frac{y}{x} \, dy\, dx = ?$



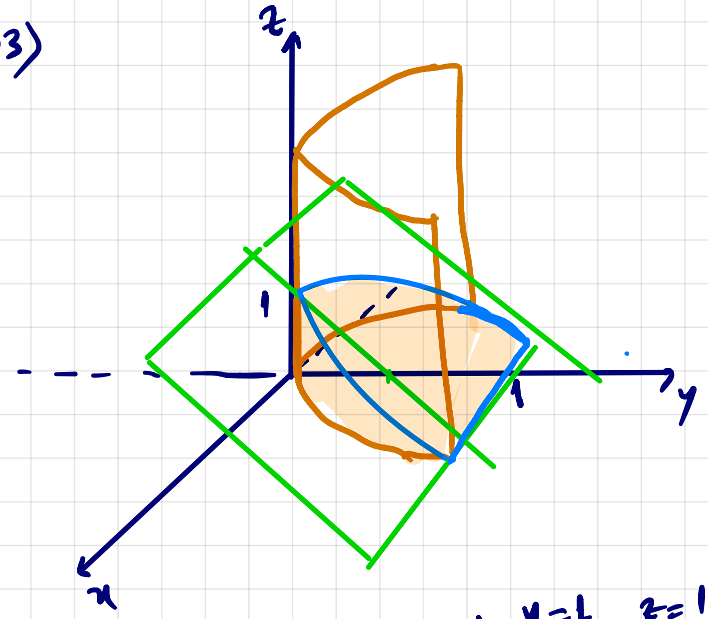
$$\iint_\Omega \arctan \frac{y}{x} \, dy\, dx = \int_{\theta=0}^{\theta=\pi} \int_{\rho=1}^{\rho=3} \arctan\left(\frac{\cancel{\rho} \operatorname{sen}\theta}{\cancel{\rho} \cos\theta}\right) \cdot \rho \, d\rho \, d\theta =$$

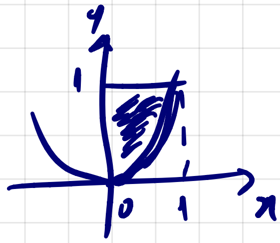
$$= \int_{\theta=0}^{\theta=\pi} \int_{\rho=1}^{\rho=3} \arctan(\tan\theta) \cdot \rho \, d\rho \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{\rho=1}^{\rho=3} \theta \rho \cdot d\rho \, d\theta =$$

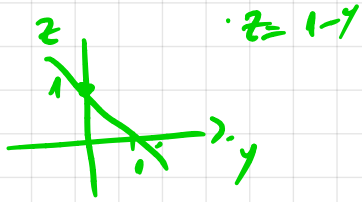
$$= \int_{\theta=0}^{\theta=\pi} \theta \, d\theta \cdot \int_{\rho=1}^{\rho=3} \rho \, d\rho = \left.\frac{\theta^2}{2}\right|_0^{\pi} \cdot \left.\frac{\rho^2}{2}\right|_1^3 = \frac{\pi^2}{2} \cdot \frac{1}{2}(9-1)$$

$$= 2\pi^2$$

03)







$$V = \iiint_{\Omega} dV = \underbrace{2}_{} \cdot \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} \int_{z=0}^{z=1-y} dz\, dy\, dx =$$

DEVIDO À SIMETRIA

$$= 2 \cdot \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} z \Big|_{z=0}^{z=1-y} dy\, dx = 2 \cdot \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} (1-y)\, dy\, dx =$$

$$= 2 \cdot \int_{x=0}^{x=1} \left(y - \frac{y^2}{2}\right) \Big|_{y=x^2}^{y=1} dx = 2 \cdot \int_{x=0}^{x=1} \left[1 - \frac{1}{2} - \left(x^2 - \frac{x^4}{2}\right)\right] dx =$$

$$= 2 \int_0^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx = \int_0^1 (1 - 2x^2 + x^4)\, dx =$$

$$= \left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right) \Big|_0^1 = 1 - \frac{2}{3} + \frac{1}{5} - 0 = \frac{15 - 10 + 3}{15} = \frac{8}{15}$$

04) $$\iint_{\Omega} \sqrt{2x+3y}\cdot \cos(x-y)\,dx\,dy .$$

$$\begin{cases} u = 2x+3y \\ v = x-y \end{cases} \longrightarrow \quad x = v+y$$

$$u = 2(v+y)+3y$$

$$u = 2v+2y+3y \Rightarrow \boxed{y = \frac{1}{5}u - \frac{2}{5}v}$$

$$\Rightarrow \quad x = v+y$$

$$x = v + \frac{1}{5}u - \frac{2}{5}v \Rightarrow \boxed{x = \frac{1}{5}u + \frac{3}{5}v}$$

$$\det J(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{vmatrix}$$

$$= -\frac{2}{25} - \frac{3}{25} = -\frac{5}{25} = -\frac{1}{5}$$

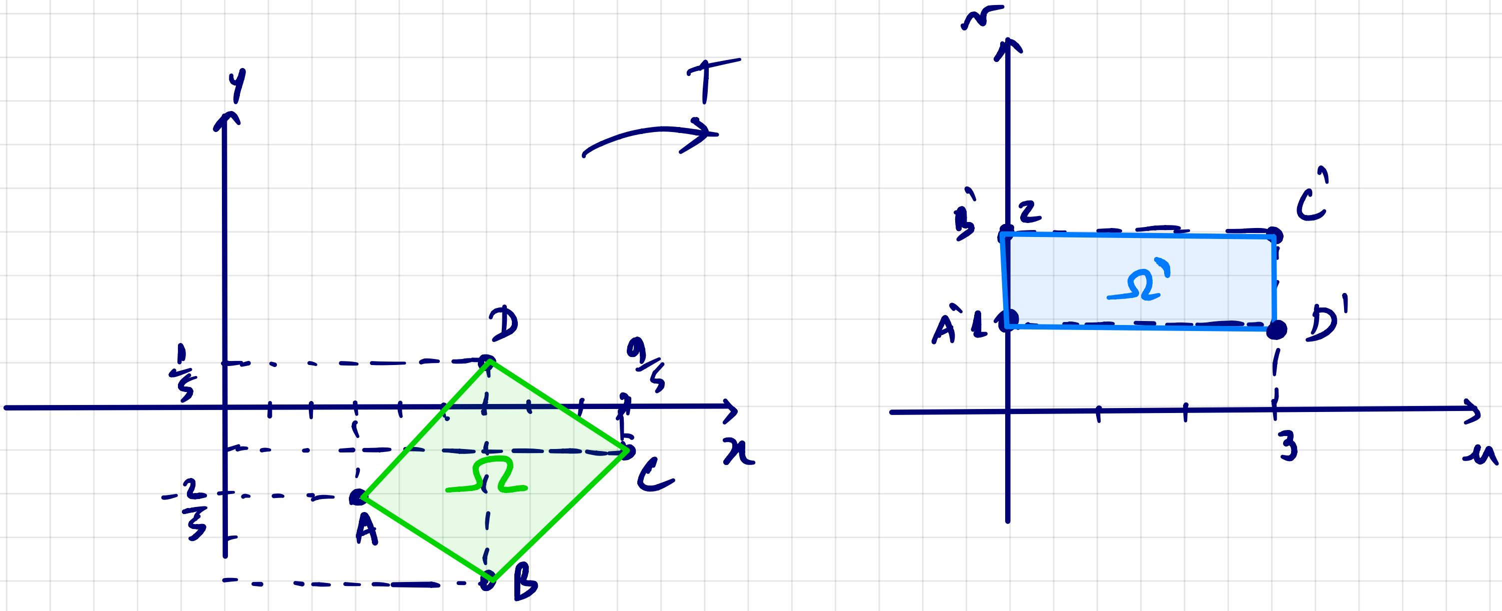
$$(x,y) \longmapsto (u,v) = (2x+3y,\ x-y)$$

$$A\left(\frac{3}{5}, -\frac{2}{5}\right) \longmapsto \left(\frac{6}{5} - \frac{6}{5},\ \frac{3}{5} + \frac{2}{5}\right) = (0, 1) = A'$$

$$B\left(\frac{6}{5}, -\frac{4}{5}\right) \longmapsto \left(\frac{12}{5} - \frac{12}{5},\ \frac{6}{5} + \frac{4}{5}\right) = (0, 2) = B'$$

$$C\left(\frac{9}{5}, -\frac{1}{5}\right) \longmapsto \left(\frac{18}{5} - \frac{3}{5},\ \frac{9}{5} + \frac{1}{5}\right) = (3, 2) = C'$$

$$D\left(\frac{6}{5}, \frac{1}{5}\right) \longmapsto \left(\frac{12}{5} + \frac{3}{5},\ \frac{6}{5} - \frac{1}{5}\right) = (3, 1) = D'$$



Assim, teremos:

$$\iint_{\Omega} \sqrt{2x+3y}\cdot \cos(x-y)\,dx\,dy = \iint_{\Omega'} \sqrt{u}\cdot \cos v \cdot \Big|\underbrace{\det J(T)(u,v)}_{-\frac{1}{5}}\Big|\cdot du\,dv$$

$$= \int_{v=1}^{v=2}\int_{u=0}^{u=3} u^{\frac{1}{2}}\cdot \cos v \cdot \frac{1}{5}\cdot du\,dv =$$

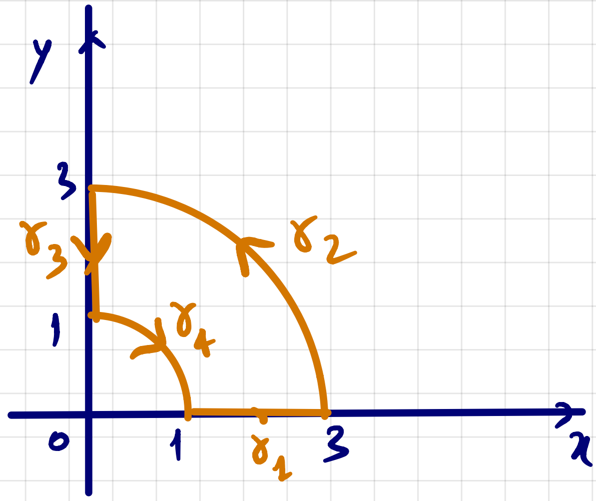
$$= \frac{1}{5}\cdot \int_{v=1}^{v=2} \cos v\cdot dv \cdot \int_{u=0}^{u=3} u^{\frac{1}{2}}\,du = \frac{1}{5}(\operatorname{sen} v)\Big|_1^2 \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^3 =$$

$$= \frac{1}{5}\cdot(\operatorname{sen} 2 - \operatorname{sen} 1)\cdot \frac{2}{3}\cdot\left[3^{\frac{3}{2}} - 0^{\frac{3}{2}}\right]$$

$$= \frac{1}{5}(\operatorname{sen} 2 - \operatorname{sen} 1)\cdot\frac{2}{3}\cdot\left(\sqrt{3^{\cancel{2}}\cdot 3}\right) = \frac{1}{5}\cdot(\operatorname{sen} 2 - \operatorname{sen} 1)\cdot\frac{2}{\cancel{3}}\cdot\cancel{3}\sqrt{3}$$

$$= \frac{2\sqrt{3}}{5}\cdot(\operatorname{sen} 2 - \operatorname{sen} 1)$$

105)



$\vec{F}(x,y) = (xy, x+y)$

$\oint_\gamma \vec{F}\, d\vec{r}$.

(a) $\gamma_1 : [1,3] \to \mathbb{R}^2$

$\gamma_1(t) = (t, 0)$. Disso: $x = t \Rightarrow dx = dt$

$y = 0 \Rightarrow dy = 0$

$\hookrightarrow \gamma_1'(t) = (1, 0)$

Logo:

$$\underbrace{\int_{\gamma_1} \vec{F}\, d\vec{r}} = \int_1^3 \vec{F}(\gamma_1(t)) \cdot \gamma_1'(t)\, dt$$

$$= \int_1^3 P\, dx + Q\, dy = \int_1^3 xy\, dx + (x+y)\, dy$$

$$= \int_1^3 0 + (x+0) \cdot 0 = \underline{0}$$

$\gamma_2 : [0, \frac{\pi}{2}] \to \mathbb{R}^2$;

$\gamma_2(t) = (3\cos t, 3\,\text{sen}\, t)$

$\Rightarrow x = 3\cos t \Rightarrow dx = -3\,\text{sen}\, t\, dt$

$y = 3\,\text{sen}\, t \Rightarrow dy = 3\cos t\, dt$

Logo:

$$\int_{\gamma_2} P\,dx + Q\,dy = \int_0^{\frac{\pi}{2}} xy\,dx + (x+y)\,dy =$$

$$= \int_0^{\frac{\pi}{2}} 3\cos t \cdot 3\,\text{sen}\,t \cdot (-3\,\text{sen}\,t\,dt) + (3\cos t + 3\,\text{sen}\,t) \cdot 3\cos t\,dt$$

$$= -27\int_0^{\frac{\pi}{2}} (\text{sen}\,t)^2 \cdot (\cos t\,dt) + 9\int_0^{\frac{\pi}{2}} \cos^2 t\,dt + 9\int_0^{\frac{\pi}{2}} (\text{sen}\,t)^1 (\cos t\,dt)$$

$$= -27\,\frac{\text{sen}^3 t}{3}\Big|_0^{\frac{\pi}{2}} + 9\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt + 9\,\frac{\text{sen}^2 t}{2}\Big|_0^{\frac{\pi}{2}}$$

$$= -27 \cdot (1-0) + \frac{9}{2}\int_0^{\frac{\pi}{2}} dt + \frac{9}{2}\,\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2t \cdot (2\,dt) + \frac{9}{2}(1-0)$$

$$= -27 + \frac{9}{2}\left(\frac{\pi}{2} - 0\right) + \frac{9}{4} \cdot \text{sen}\,2t\Big|_0^{\frac{\pi}{2}} + \frac{9}{2}$$

$$= -27 + \frac{9\pi}{4} + \frac{9}{4}(0-0) + \frac{9}{2} =$$

$$= -27 + \frac{9}{2} + \frac{9\pi}{4}$$

$\gamma_3 : [3,1] \to \mathbb{R}^2 \quad ; \quad \gamma_3(t) = (0, t).$

$$\Rightarrow x = 0 \Rightarrow dx = 0$$

$$y = t \Rightarrow dy = dt$$

$$\int_{\gamma_3} \vec{F}\,d\vec{r} = \int_3^1 P\,dx + Q\,dy =$$

$$= \int_3^1 xy\,dx + (x+y)\,dy = \int_3^1 0 + (0+t)\,dt = \int_3^1 t\,dt =$$

$$= \left.\frac{t^2}{2}\right|_3^1 = \frac{1}{2} - \frac{9}{2} = -\frac{8}{2} = -4 \,//$$

$$\gamma_4 : \left[\frac{\pi}{2}, 0\right] \longrightarrow \mathbb{R}^2$$

$$\gamma_4(t) = (\cos t, \operatorname{sen} t)$$

$$\begin{cases} x = \cos t \rightarrow dx = -\operatorname{sen} t\,dt \\ y = \operatorname{sen} t \Rightarrow dy = \cos t\,dt \end{cases}$$

$$\Rightarrow \int_{\gamma_4} \vec{F}\,d\vec{r} = \int_{\frac{\pi}{2}}^{0} xy\,dx + (x+y)\,dy$$

$$= \int_{\frac{\pi}{2}}^{0} \cos t \cdot \operatorname{sen} t \cdot (-\operatorname{sen} t\,dt) + (\cos t + \operatorname{sen} t)\cos t\,dt$$

$$= \int_{\frac{\pi}{2}}^{0} -(\operatorname{sen} t)^2 \cos t\,dt + \int_{\frac{\pi}{2}}^{0} \cos^2 t\,dt + \int_{\frac{\pi}{2}}^{0} (\operatorname{sen} t)^2 \cos t\,dt$$

$$- \left.\frac{\operatorname{sen}^3 t}{3}\right|_{\frac{\pi}{2}}^{0} + \int_{\frac{\pi}{2}}^{0} \frac{1+\cos 2t}{2}\,dt + \left.\frac{\operatorname{sen}^2 t}{2}\right|_{\frac{\pi}{2}}^{0} =$$

$$-\frac{1}{3}\left(\operatorname{sen}0-\operatorname{sen}^3\frac{\pi}{2}\right)+\frac{1}{2}\,t\Big|_{\frac{\pi}{2}}^{0}+\frac{1}{2}\cdot\frac{1}{2}\int_{\frac{\pi}{2}}^{0}\cos 2t\cdot(2\,dt)+\frac{1}{2}(\operatorname{sen}0-1)$$

$$+\frac{1}{3}+\frac{1}{2}\left(0-\frac{\pi}{2}\right)+\underbrace{\frac{1}{4}\operatorname{sen}2t}_{0}\Big|_{\frac{\pi}{2}}^{0}-\frac{1}{2}$$

$$\frac{1}{3}-\frac{\pi}{4}-\frac{1}{2}$$

$$\Rightarrow \oint_{\gamma}\vec{F}\,d\vec{r}=\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}+\int_{\gamma_4}$$

$$=0-27+\frac{9}{2}+\frac{9\pi}{4}-4+\frac{1}{3}-\frac{\pi}{4}-\frac{1}{2}$$

$$=-\frac{26}{3}+\cancel{4}-\cancel{4}+\frac{8\pi}{4}=-\frac{26}{3}+2\pi$$

b) Pelo T. de Green:

$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$Q = x+y \Rightarrow \frac{\partial Q}{\partial x} = 1$$

$$P = xy \Rightarrow \frac{\partial P}{\partial y} = x$$

$$= \iint_\Omega (1 - y)\, dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=1}^{\rho=3} (1 - \rho\cos\theta)\cdot \rho\, d\rho\, d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\int_{\rho=1}^{\rho=3} (\rho - \rho^2 \cos\theta)\, d\rho \right) d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{\rho^2}{2} - \frac{\rho^3}{3}\cos\theta\right)\Bigg|_1^3 d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{9}{2} - 9\cos\theta - \frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(4 - \frac{26}{3}\cos\theta\right) d\theta$$

$$4\theta \Big|_0^{\frac{\pi}{2}} - \frac{26}{3}\,\mathrm{sen}\,\theta \Big|_0^{\frac{\pi}{2}} =$$

$$4\cdot\frac{\pi}{2} - \frac{26}{3}\cdot\Big(\underbrace{\mathrm{sen}\,\frac{\pi}{2} - \mathrm{sen}\,0}_{=1}\Big) = 2\pi - \frac{26}{3}$$

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06) Pelo T. da divergência, temos:

$$(*) \quad \oint_\gamma \vec{F}\cdot\vec{n}\,ds = \iint_D \mathrm{div}\,\vec{F}\cdot dA.$$

No nosso caso, temos:

$$\oint_\gamma g\cdot\frac{\partial g}{\partial \vec{n}}\cdot ds = \oint_\gamma g\cdot\nabla g\cdot\vec{n}\,ds,$$

e então, confrontando com (*), concluímos que devemos ter

$$\vec{F} = g\cdot\nabla g = g\cdot\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$$

$$= \left(g\cdot\frac{\partial g}{\partial x}, g\cdot\frac{\partial g}{\partial y}\right)$$

Então: $\mathrm{div}\,\vec{F} = \mathrm{div}\left(g\frac{\partial g}{\partial x}, g\cdot\frac{\partial g}{\partial y}\right) =$

$$= \frac{\partial}{\partial x}\Big(\underbrace{g \cdot \frac{\partial g}{\partial x}}_{u \cdot v}\Big) + \frac{\partial}{\partial y}\Big(\underbrace{g \cdot \frac{\partial g}{\partial y}}_{u \cdot v}\Big) =$$

$$= g \cdot \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x} \cdot \frac{\partial g}{\partial x} + g \cdot \frac{\partial^2 g}{\partial y^2} + \frac{\partial g}{\partial y} \cdot \frac{\partial g}{\partial y} =$$

$$= g \cdot \underbrace{\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right)}_{\Delta g} + \underbrace{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}_{\|\nabla g\|^2}$$

$$= g \cdot \Delta g + \|\nabla g\|^2 .$$

Ou seja, $\operatorname{div} \vec{F} = g \cdot \Delta g + \|\nabla g\|^2$.

Disso, pelo T. da divergência, segue que

$$\underbrace{\oint_\gamma \vec{F} \cdot \vec{n}\, ds}_{\oint_\gamma g \cdot \frac{\partial g}{\partial \vec{n}}\, ds} = \iint_D \underbrace{\operatorname{div} \vec{F}}_{g \cdot \Delta g + \|\nabla g\|^2} \, dA \, ;$$

ou seja,

$$\oint_\gamma g \cdot \frac{\partial g}{\partial \vec{n}} \, ds = \iint_D (g \cdot \Delta g + \|\nabla g\|^2) \, dA$$

Agora, sendo $g(x,y) = x^2 + y^2$ e $\gamma : x^2+y^2 = 1$;

teremos

$$\frac{\partial g}{\partial x} = 2x \; ; \; \frac{\partial g}{\partial y} = 2y \; . \quad \text{Assim:}$$

$$\nabla g = (2x, 2y) \Rightarrow \|\nabla g\| = \sqrt{(2x)^2 + (2y)^2}$$

$$\Rightarrow \|\nabla g\| = \sqrt{4x^2 + 4y^2} \Rightarrow \|\nabla g\|^2 = 4(x^2+y^2)$$

Ainda: $\dfrac{\partial^2 g}{\partial x^2} = 2$ e $\dfrac{\partial^2 g}{\partial y^2} = 2$, e daí

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 2+2 = 4.$$

Então:

$$\oint_\gamma g \cdot \frac{\partial g}{\partial \vec{n}} \, ds = \iint_{x^2+y^2 \leq 1} (g \cdot \Delta g + \|\nabla g\|^2) \, dA =$$

$$= \iint_{x^2+y^2 \leq 1} \left[(x^2+y^2) \cdot 4 + 4(x^2+y^2) \right] dA =$$

$$= 8 \iint_{x^2+y^2\le 1} (x^2+y^2)\, dA = 8\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} \rho^2 \cdot \rho \, d\rho\, d\theta$$

$$= 8\, \theta\Big|_0^{2\pi} \cdot \frac{\rho^4}{4}\Big|_0^1 = \cancel{8}\cdot(2\pi)\cdot\frac{1}{\cancel{4}}(1-0) = 4\pi .$$

$$\Rightarrow \boxed{\oint_\gamma g \cdot \frac{\partial g}{\partial \vec{n}}\, ds = 4\pi}$$

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07) $\vec{F}(x,y,z) = (\overbrace{x^2z^2}^{F_1}, \overbrace{2xy}^{F_2}, \overbrace{y^2-z^2}^{F_3})$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$$

$$= 2xz^2 + 2y + (-2z) = 2(xz^2+y-z)$$

$$\text{rot}\,\vec{F} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2z^2 & 2xy & y^2-z^2 & x^2z^2 & 2xy \end{array}\right.$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$= 2y\,\vec{i} + 2x^2z\,\vec{j} + 2y\,\vec{k} - 0\vec{k} - 0\vec{i} - 0\vec{j}$$

$$= (2y, 2x^2z, 2y)$$