LISTA 01

9. **(Sel. Mestr. UFRGS 2010/1)** Prove que se $\lim_{n\to\infty} a_n = 0$ e $(b_n)_{n\geq 0}$ é uma sequência limitada, então $\lim_{n\to\infty} a_n b_n = 0$. Mostre com um contra-exemplo que a conclusão acima não é válida se retirarmos a hipótese de $(b_n)_{n\geq 0}$ ser limitada.

Dado $\varepsilon > 0$.

Como (b_n) é limitada, então, $\exists\, K > 0$ tal que

$$|b_n| \leq K, \quad \forall n \in \mathbb{N}. \qquad (*)$$

Como $\lim_{n\to\infty} a_n = 0$, então, $\exists\, n_0 \in \mathbb{N}$, tal que

$$\forall n \geq n_0 \Rightarrow |a_n - 0| = |a_n| < \frac{\varepsilon}{K} \qquad (**)$$

Então, $\forall n \geq n_0$, valem $(*)$ e $(**)$.

Logo,

$$|a_n \cdot b_n - 0| = |a_n \cdot b_n| = \underbrace{|a_n|}_{<\frac{\varepsilon}{K}} \cdot \underbrace{|b_n|}_{<K} < \frac{\varepsilon}{\cancel{K}} \cdot \cancel{K} = \varepsilon$$

Portanto, para o $\varepsilon > 0$ dado encontramos $n_0 \in \mathbb{N}$, tal que, $\forall n \geq n_0 \Rightarrow |a_n b_n - 0| < \varepsilon$; i.e.

$$a_n \cdot b_n \to 0$$

Para a 2ª parte do exercício: tome

$$a_n = \frac{1}{n}, \text{ e então } a_n \to 0$$

e

$$b_n = n, \text{ e então } b_n \text{ não é limitada } (b_n \to +\infty)$$

Mas: $a_n \cdot b_n = \frac{1}{\cancel{n}} \cdot \cancel{n} = 1$, e daí

$$a_n \cdot b_n = 1 \not\to 0 \quad (n \to +\infty)$$

L1.

12. Mostre que $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$.

Sugestão: Escreva inicialmente $1-\frac{1}{n} = \frac{n-1}{n} = \frac{1}{\frac{n}{n-1}} = \frac{1}{1+\frac{1}{n-1}}$.

sabemos: $\lim_{n\to\pm\infty}\left(1+\frac{1}{n}\right)^n = e$

$$1 - \frac{1}{n} = \frac{n-1}{n} = \frac{1}{\frac{n}{n-1}} = \frac{1}{\frac{n-1+1}{n-1}} = \frac{1}{\underbrace{\frac{n-1}{n-1}}_{1} + \frac{1}{n-1}} = \frac{1}{1+\frac{1}{n-1}}$$

Então: $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n-1}\right)^n} =$

$$\frac{1}{\underbrace{\left[\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{1}}\right]}_{e}^{\frac{1}{n-1}\cdot n}} = \frac{1}{e^1} = e^{-1}.$$

L3.

12) b) $\int x^2 \cdot \ln(1+x)\,dx$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n = \sum_{n=0}^{+\infty} (-1)^n \cdot x^n.$$

Como $\int_0^x \frac{dt}{1+t} = \ln(1+x)$, então:

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \sum_{n=0}^{+\infty} (-1)^n \cdot t^n \cdot dt$$

$$= \sum_{n=0}^{+\infty} (-1)^n \cdot \int_0^x t^n \cdot dt = \sum_{n=0}^{+\infty} (-1)^n \cdot \frac{t^{n+1}}{n+1}\Bigg|_0^x$$

$$= \sum_{n=0}^{+\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1}$$

Portanto:

$$\int x^2 \cdot \ln(1+x)\,dx = \int x^2 \cdot \sum_{n=0}^{+\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} \cdot dx$$

$$= \int \sum_{n=0}^{+\infty} (-1)^n \cdot \frac{x^{n+3}}{n+1}\,dx = \sum_{n=0}^{+\infty} (-1)^n \cdot \frac{1}{n+1} \cdot \int x^{n+3}\,dx =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} \cdot \frac{x^{n+4}}{n+4}$$

$$R = \frac{1}{\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| =$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)(n+4)} \cdot \frac{(n+2)(n+5)}{1}$$

$$= \lim_{n\to+\infty} \frac{n^2}{n^2} = 1.$$

L3 - 06) - C -

$f(x) = e^{x-2}$ em $x = 2$ (S. TAYLOR)

$f'(x) = e^{x-2} = f''(x) = f'''(x) = \dots \qquad (e^{u})' = e^{u}\cdot u'$

$f(2) = e^0 = 1 = f'(2) = f''(2) = \dots = f^{(n)}(2) = \dots$

$$f(x) = f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}\cdot(x-2)^3 + \dots$$

$$e^{x-2} = 1 + 1\cdot(x-2) + \frac{1}{2!}(x-2)^2 + \frac{1}{3!}(x-2)^3 + \dots$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!}\cdot(x-2)^n$$

L3 ; 08 (a) :

$f(x) = e^{-x}\cdot \cos x$

- $e^{-x} = \sum_{n=0}^{+\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{+\infty} \frac{(-1)^n\cdot x^n}{n!}$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

- $\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n\cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$

Assim.

$$(e^{-x}\cdot\cos x) = \left(\underset{a_0}{1} - \underset{a_1}{1x} + \underset{a_2}{\frac{1x^2}{2!}} - \underset{a_3}{\frac{1x^3}{3!}} + \underset{a_4}{\frac{1x^4}{4!}} - \frac{x^5}{5!} + \dots\right)\cdot$$

$$\cdot\left(\underset{b_0}{1} + \underset{b_1}{0x} \underset{b_2}{- \frac{1x^2}{2!}} + \underset{b_3}{0x^3} + \underset{b_4}{\frac{1x^4}{4!}} + \underset{b_5}{0x^5} \underset{b_6}{- \frac{1x^6}{6!}} + \dots\right)$$

$$a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3)x^3 +$$

$$+ (a_4 b_0 + a_3 b_1 + a_2 b_2 + a_1 b_3 + a_0 b_4)x^4 + \dots$$

$$= 1\cdot 1 + ((-1)\cdot 1 + 1\cdot 0)x + \underbrace{\left(\frac{1}{2!}\cdot 1 + (-1)\cdot(0) + 1\cdot\left(-\frac{1}{2!}\right)\right)}_{=0}x^2 +$$

$$+ \left(-\frac{1}{3!}\cdot 1 + \frac{1}{2!}\cdot 0 + (-1)\cdot\left(-\frac{1}{2!}\right) + 1\cdot 0\right)\cdot x^3 + \dots$$

$$= 1 - x + \frac{1}{3}x^3 - \dots$$