SÉRIE BINOMIAL.

Vamos descrever um procedimento para representar a série de Taylor para funções do tipo

$$f(x) = \sqrt[p]{h(x)} = h(x)^{\frac{1}{p}}.$$

Ex.: para $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$.

Vamos obter a série de Taylor em $x=0$.

$$\boxed{(r^k)' = k.r^{k-1}.r'}$$

$$f(0) = \sqrt{1} = 1.$$

$$f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \Rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = +\frac{1}{4}(1-x)^{-\frac{3}{2}}.(-1) \Rightarrow f''(0) = -\frac{1}{4} = \frac{-1}{2^2}$$

$$f'''(x) = -\frac{3}{8}.(1-x)^{-\frac{5}{2}} \Rightarrow f'''(0) = -\frac{3}{8} = \frac{-3}{2^3}$$

$$f^{(4)}(x) = +\frac{3.5}{16}.(1-x)^{-\frac{7}{2}}.(-1) \Rightarrow f^{(4)}(0) = -\frac{3.5}{16} = \frac{-3.5}{2^4}$$

$$f^{(5)}(x) = \frac{3.5.7}{32}.(1-x)^{-\frac{9}{2}}.(-1) \Rightarrow f^{(5)}(0) = \frac{-3.5.7}{2^5}$$

$$\vdots$$

Assim:

$$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}\cdot x^3 + \cdots$$

$$f(x) = 1 + \left(-\frac{1}{2}\right)\cdot x + \left(-\frac{1}{2^2}\right)\cdot\frac{1}{2!}x^2 + \left(-\frac{3}{2^3}\right)\frac{1}{3!}\cdot x^3 + \left(-\frac{3\cdot 5}{2^4}\right)\cdot\frac{1}{4!}\cdot x^4 + \cdots$$

$$= 1 - \sum_{n=1}^{+\infty} \frac{?}{2^n\cdot n!}\cdot x^n$$

PARA AJUDAR A ENCONTRAR UM PADRÃO, VAMOS PROCURAR DESENVOLVER UMA SÉRIE COM UM EXEMPLO MAIS GERAL.

02) $f(x) = (1-x)^m$, $m \in \mathbb{Q}\setminus\mathbb{N}$.

Assim, desenvolvendo em $x=0$, temos:

$$\boxed{f(0) = 1.}$$

$$f'(x) = m(1-x)^{m-1}\cdot(-1) = -m(1-x)^{m-1}$$

$$\Rightarrow \boxed{f'(0) = -m}$$

$$f''(x) = -m\cdot(m-1)\cdot(1-x)^{m-2}\cdot(-1)$$

$$\Rightarrow f''(x) = m(m-1)\cdot(1-x)^{m-2}$$

$$\Rightarrow \boxed{f''(0) = m(m-1)}$$

$$f'''(x) = m.(m-1).(m-2).(1-x)^{m-3}.(-1)$$

$$= -m(m-1)(m-2).(1-x)^{m-3}$$

$$\Rightarrow \boxed{f'''(0) = -m(m-1)(m-2)}$$

$$f^{(4)}(x) = -m(m-1)(m-2).(m-3).(1-x)^{m-4}.(-1)$$

$$\Rightarrow f^{(4)}(x) = +m(m-1)(m-2)(m-3)(1-x)^{m-4}$$

$$\Rightarrow \boxed{f^{(4)}(0) = m(m-1)(m-2)(m-3)}$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^k.\ m(m-1).(m-2)\ldots(m-(k-1))$$

$$\Rightarrow \boxed{f^{(k)}(0) = (-1)^k\ m(m-1)(m-2)\ldots(m-k+1)}$$

Assim, a série de Taylor para f será:

$$f(x) = f(0) + f'(0).x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$f(x) = 1 - m.x + \frac{m(m-1)}{2!}x^2 - \frac{m(m-1)(m-2)}{3!}x^3 + \ldots$$

$$\boxed{f(x) = 1 + \sum_{n=1}^{+\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} \cdot (-1)^n \cdot x^n}$$

SÉRIE BINOMIAL PARA $f(x) = (1-x)^m$ EM $x=0$.

Qual é o seu raio de convergência?

$$R = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} =$$

$$= \frac{1}{\lim\limits_{n\to+\infty} \left| \frac{\cancel{m(m-1)\cdots}(m-n+2)}{(n+1)\cancel{!}} \times \frac{\cancel{n!}}{\cancel{m(m-1)\cdots(m-n+1)}} \right|}$$

$$= \frac{1}{\lim\limits_{n\to+\infty} \left|\frac{m-n+2}{n+1}\right|} = 1. \Rightarrow \boxed{R=1}$$

Voltando ao exemplo anterior, para achar um desenvolvimento para $\sqrt{1-x} = (1-x)^{\frac{1}{2}}$

Neste caso, $m = \frac{1}{2}$

Assim.

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \sum_{n=1}^{+\infty} \frac{+\,\frac{1}{2}\cdot\overbrace{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}^{n-1 \text{ SINAIS NEGATIVOS}}}{n!}(-1)^n x^n =$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\overbrace{\left(\frac{3}{2}-n\right)}^{\frac{3-2n}{2}}}{n!}(-1)^n\cdot x^n\cdot(-1)^{n-1}$$

$$= 1 + \sum_{n=1}^{+\infty} \underbrace{(-1)^{2n-1}}_{=\,-1}\cdot\frac{1\cdot3\cdot5\cdots(2(n+1)+1)}{2^n\cdot n!}\,x^n$$

$$= 1 - \sum_{n=1}^{+\infty} \frac{1\cdot1\cdot3\cdot5\cdots(2(n+1)-1)}{2^n\, n!}\,x^n$$

$$1 - \left[\frac{1}{2}x + \frac{1\cdot1}{2^2\cdot2!}x^2 + \frac{1\cdot3}{2^3\cdot3!}x^3 + \frac{1\cdot3\cdot5}{2^4\cdot4!}x^4 + \cdots\right]$$

Ex. 102

Para $f(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$ temos

$$(1-x^2)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{+\infty} \frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}^{n \text{ números negativos}}(-1)^n(x^2)^n}{n!}$$

$$= \frac{-2n+2-1}{2}$$

$$\frac{-2n+1}{2}$$

$$= 1 + \sum_{n=1}^{+\infty} \overbrace{(-1)^{2n}}^{1} \cdot \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ \ldots\ \left(\frac{2n-1}{2}\right)}{n!}\, x^{2n}$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot n!}\, x^{2n}.$$

Vamos aproveitar e encontrar outra série de potências:
Como

$$\operatorname{arcsen} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}\ ; \quad \text{então} ;$$

$$\operatorname{arcsen} x = \int_0^x \left(1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot n!}\cdot t^{2n}\right) dt$$

$$t\Big|_0^x + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot n!}\cdot \int_0^x t^{2n}\, dt$$

$$= x + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot n!}\, \frac{t^{2n+1}}{2n+1}\Big|_0^x$$

$$= x + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{(2n+1)\cdot 2^n\cdot n!}\cdot x^{2n+1} \quad ; \quad \boxed{R = 1}$$