Resolução de Exercícios da Lista 03, feita pelos alunos das turmas M1 e M2.

Exercício 1. Utilizando do algoritmo para [illegible] de matrizes, encontre a inversa A' de A em cada caso, se A for inversível.

(a) $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}$

Solução: (A : I)

$$\left(\begin{array}{ccc|ccc} 3 & 4 & 1 & 1 & 0 & 0 \\ 2 & -7 & -1 & 0 & 1 & 0 \\ 8 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_1 \to \frac{L_1}{3}} \left(\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -7 & -1 & 0 & 1 & 0 \\ 8 & 1 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_2 \to L_2 - 2L_1} \left(\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{29}{3} & -\frac{5}{3} & -\frac{2}{3} & 1 & 0 \\ 8 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_3 \to L_3 - 8L_1} \left(\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{29}{3} & -\frac{5}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{29}{3} & \frac{7}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_2 \to \frac{3L_2}{29}} \left(\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{5}{29} & \frac{2}{29} & -\frac{3}{29} & 0 \\ 0 & -\frac{29}{3} & \frac{7}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right) \xrightarrow{L_1 \to \frac{-4L_2}{3} + L_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{29} & \frac{7}{29} & \frac{4}{29} & 0 \\ 0 & 1 & \frac{5}{29} & \frac{2}{29} & -\frac{3}{29} & 0 \\ 0 & -\frac{29}{3} & \frac{7}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \to \frac{29L_2}{3} + L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{29} & \frac{7}{29} & \frac{4}{29} & 0 \\ 0 & 1 & \frac{5}{29} & \frac{2}{29} & -\frac{3}{29} & 0 \\ 0 & 0 & 4 & -2 & -1 & 1 \end{array}\right) \xrightarrow{L_3 \to \frac{L_3}{4}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{29} & \frac{7}{29} & \frac{4}{29} & 0 \\ 0 & 1 & \frac{5}{29} & \frac{2}{29} & -\frac{3}{29} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{array}\right)$$

$$\xrightarrow{L_1 \to \frac{-3L_3}{29} + L_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{17}{58} & \frac{19}{116} & -\frac{3}{116} \\ 0 & 1 & \frac{5}{29} & \frac{2}{29} & -\frac{3}{29} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{array}\right) \xrightarrow{L_2 \to \frac{5L_3}{29} + L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{17}{58} & \frac{19}{116} & -\frac{3}{116} \\ 0 & 1 & 0 & \frac{9}{58} & -\frac{7}{116} & -\frac{5}{116} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{array}\right)$$

$$\qquad\qquad\qquad\qquad I \qquad\qquad A^{-1}$$

Com isso, temos que a inversa de A será:

$$A^{-1} = \begin{pmatrix} \frac{17}{58} & \frac{19}{116} & \frac{-3}{116} \\ \frac{9}{58} & \frac{-7}{116} & -\frac{5}{116} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

ere

10,0

① b) $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & 4 \end{bmatrix}$; $A^{-1} = ?$

A matriz aumentada será: $(A|I) = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{array}\right]$

Então,

$$\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{L_1 \leftrightarrow L_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{L_2 \leftarrow L_2 - 3L_1}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{L_3 \leftarrow L_3 - 2L_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -2 & 0 & -2 & 1 \end{array}\right] \xrightarrow{L_2 \leftarrow \frac{L_2}{4}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 1/4 & -3/4 & 0 \\ 0 & 5 & -2 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{L_3 \leftarrow L_3 - 5L_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 21/2 & -5/4 & 7/4 & 1 \end{array}\right] \xrightarrow{L_3 \leftarrow \frac{2}{21}L_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -5/42 & 1/6 & \frac{2}{21} \end{array}\right] \xrightarrow{L_1 \leftarrow L_1 - 3L_3}$$

OK!

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/14 & 1/2 & -2/7 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -5/42 & 1/6 & 2/21 \end{array}\right] \xrightarrow{L_2 \leftarrow L_2 + \frac{5}{2}L_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/14 & 1/2 & -2/7 \\ 0 & 1 & 0 & -1/21 & -1/3 & 5/21 \\ 0 & 0 & 1 & -5/42 & 1/6 & 2/21 \end{array}\right]$$

Logo, $A^{-1} = \begin{bmatrix} 5/14 & 1/2 & -2/7 \\ -1/21 & -1/3 & 5/21 \\ -5/42 & 1/6 & 2/21 \end{bmatrix}$

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c) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$, $A^{-1} = ?$

$$(A \,\vdots\, I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \; l_3 \leftarrow 1/3\, l_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right) \; l_2 \leftarrow l_2 - 3l_3 \; \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right) \; l_1 \leftarrow l_1 + l_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right) \; l_2 \leftarrow 1/2\, l_2 \; \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right) \; l_1 \leftarrow l_1 - 3l_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right), \quad A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/3 \end{pmatrix} //$$

02. Obtenha a inversa de $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, se existir.

$$\left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 1&1&0&1&0&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 1&1&1&1&0&0&0&1 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 - L_1} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 1&1&1&1&0&0&0&1 \end{array}\right) \xrightarrow{L_4 \leftarrow L_4 - L_1} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 0&1&0&1&-1&0&0&1 \end{array}\right) \xrightarrow{L_4 \leftarrow L_4 - L_2}$$

$$\left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 0&0&1&0&0&-1&0&1 \end{array}\right) \xrightarrow{L_4 \leftarrow L_4 - L_3} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 0&0&0&-1&0&-1&-1&1 \end{array}\right) \xrightarrow{L_4 \leftarrow -L_4} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 0&0&0&1&0&1&1&-1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&1&0&0&1&0 \\ 0&0&0&1&0&1&1&-1 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 - L_4} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&1&-1&1&0&0 \\ 0&0&1&0&0&-1&0&1 \\ 0&0&0&1&0&1&1&-1 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 - L_4} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&0&-1&0&-1&1 \\ 0&0&1&0&0&-1&0&1 \\ 0&0&0&1&0&1&1&-1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&-1&0&-1&0&-1&1 \\ 0&0&1&0&0&-1&0&1 \\ 0&0&0&1&0&1&1&-1 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 + L_3} \left(\begin{array}{cccc|cccc} 1&0&1&0&1&0&0&0 \\ 0&1&0&0&-1&-1&-1&2 \\ 0&0&1&0&0&-1&0&1 \\ 0&0&0&1&0&1&1&-1 \end{array}\right) \xrightarrow{L_1 \leftarrow L_1 - L_3}$$

$$\left(\begin{array}{cccc|cccc} 1&0&0&0&1&1&0&-1 \\ 0&1&0&0&-1&-1&-1&2 \\ 0&0&1&0&0&-1&0&1 \\ 0&0&0&1&0&1&1&-1 \end{array}\right)$$

A matriz A foi transformada numa matriz I, na parte esquerda, logo a matriz ~~da parte~~ do bloco da direita representa A^{-1}.

3. Considere o sistema linear a seguir:

$$\begin{cases} x + y + z = 6 \\ x - y + 2z = 2 \\ 2x + y - z = 3 \end{cases}$$

a) Resolva-o aplicando operações elementares sobre linhas.

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{pmatrix} \ell_2 \hookrightarrow \ell_2 - 1\ell_1 \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 2 & 1 & -1 & 3 \end{pmatrix} \ell_3 \hookrightarrow \ell_3 - 2\ell_1$$

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{pmatrix} \ell_2 \hookrightarrow \ell_2 - 3\ell_3 \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 10 & 23 \\ 0 & -1 & -3 & -9 \end{pmatrix} \ell_3 \hookrightarrow \ell_3 + 1\ell_2 \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 10 & 23 \\ 0 & 0 & 7 & 14 \end{pmatrix}$$

$$\ell_3 \hookrightarrow \frac{1}{7}\ell_3 \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 10 & 23 \\ 0 & 0 & 1 & 2 \end{pmatrix} \Big\}$$ É equivalente ao sistema linear

$$\begin{cases} x+y+z=6 \\ \quad y+10z=23 \\ \qquad z=2 \end{cases}$$

$y+10z=23$

$y+10(2)=23$

$y=23-20$

$y=3$

$\to x+3+2=6$

$x=6-5$

$x=1$

$(1,3,2)$

3b) Reescrevendo o sistema (1) na notação matricial

$$Ax=b$$

podemos encontrar o valor da matriz-solução x, multiplicando (2) à esquerda por A^{-1}, se esta inversa existir. Encontre A^{-1} e obtenha o valor de x desta forma. Compare sua resposta com a obtida no item anterior.

* Reescrevendo o sistema no formato $Ax=b$, temos

$$\underset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}} \underset{x}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{b}{\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}}$$

que seria igual ao sistema anterior dado.

* Multiplicando à esquerda por A^{-1}, temos:

$$Ax=b$$

$$A^{-1}(Ax)=A^{-1}.b$$

$$\underbrace{(A^{-1}.A)}.x=A^{-1}b$$

$$I.x=A^{-1}b$$

$$x=A^{-1}.b$$

* Precisamos achar a inversa de A para fazer este cálculo

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \quad A^{-1}?$$

$\ell_2 \mapsto \ell_2 - \ell_1$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \ell_2 - 1\ell_1 \to \ell_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \ell_3 - 2\ell_1 \to \ell_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \frac{\ell_2}{(-2)} \to \ell_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \ell_3 - (-1).\ell_2 \to \ell_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -7/2 & -3/2 & -1/2 & 1 \end{array}\right) \; \ell_3 \cdot \left(\frac{-2}{7}\right) \to \ell_3 \; \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 3/7 & 1/7 & -2/7 \end{array}\right)$$

$$\ell_2 - \left(\frac{-1}{2}\right) \cdot \ell_3 \to \ell_2 \; \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5/7 & -3/7 & -1/7 \\ 0 & 0 & 1 & 3/7 & 1/7 & -2/7 \end{array}\right) \; \ell_1 - 1 \cdot \ell_3 \to \ell_1 \; \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 4/7 & -1/7 & 2/7 \\ 0 & 1 & 0 & 5/7 & -3/7 & -1/7 \\ 0 & 0 & 1 & 3/7 & 1/7 & -2/7 \end{array}\right)$$

$$\ell_1 - 1\ell_2 \to \ell_1 \; \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/7 & 2/7 & 3/7 \\ 0 & 1 & 0 & 5/7 & -3/7 & -1/7 \\ 0 & 0 & 1 & 3/7 & 1/7 & -2/7 \end{array}\right) \therefore$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}^{(-1)} = \begin{pmatrix} -1/7 & 2/7 & 3/7 \\ 5/7 & -3/7 & -1/7 \\ 3/7 & 1/7 & -2/7 \end{pmatrix}$$

$$x = A^{-1} \cdot b$$

$$x = \begin{pmatrix} -1/7 & 2/7 & 3/7 \\ 5/7 & -3/7 & -1/7 \\ 3/7 & 1/7 & -2/7 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6/7 + 4/7 + 9/7 \\ 30/7 + (-6/7) + (-3/7) \\ 18/7 + (2/7) + (-6/7) \end{pmatrix} \Rightarrow$$

$$x = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$\therefore$ Comparando os resultados obtidos ao multiplicar a inversa de A por b, percebemos que obtemos a mesma resposta de quando resolvemos o sistema linear através de operações elementares nas linhas.

Questão 4) Resolva o sistema linear $AX = B$, isolando a variável X da equação dada, onde

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 1 & 4 & 1 \end{pmatrix} \quad e \quad B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$A \cdot x = B$$
$$A^{-1} \cdot (A \cdot x) = A^{-1} \cdot B$$
$$(A^{-1}A) \cdot x = A^{-1} \cdot B$$
$$I \cdot x = A^{-1} \cdot B$$
$$\boxed{x = A^{-1} \cdot B}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{l_3 \leftarrow l_3 - l_1} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{l_2 \leftarrow \frac{1}{2} l_2} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{l_3 \leftarrow l_3 - 2 l_2} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 - \frac{1}{2} l_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 - (-1) l_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 - 2 l_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) //$$

$$\boxed{A^{-1} = \begin{pmatrix} -1 & -3 & 2 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & -1 & 1 \end{pmatrix}}$$

$$X = A^{-1} \cdot B$$

$$X = \begin{pmatrix} -1 & -3 & 2 \\ 1/2 & 1 & -3/2 \\ -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 + (-3) + 6 \\ 1 + 1 + (-3/2) \\ -2 + (-1) + 3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} //$$

5) Em que condições a matriz diagonal $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & a_n \end{pmatrix}$ é invertível e qual é sua inversa?

Existe A^{-1} se, e somente se, $A_n \sim I_n$

ou seja, a forma escalonada por linhas de A_n é I_n, e isso ocorre se, e só se, $a_i \neq 0, \forall i \in \{1, 2, \ldots, n\}$

$$(A|I) = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & a_n & 0 & 0 & \cdots & 1 \end{pmatrix} =$$

mudamos todas as linhas individualmente somente fazendo $l_1 \leftrightarrow \frac{1}{a_1} \cdot l_1$, $l_2 \leftrightarrow \frac{1}{a_2} \cdot l_2$, $l_n \leftrightarrow \frac{1}{a_n} \cdot l_n$, até encontrarmos

a inversa da matriz A seria,

$$A^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \frac{1}{a_n} \end{pmatrix} //$$

6 a) Mostre que uma matriz ortogonal M cumpre a propriedade $M.M^T = M^T.M = I$.

Seja M uma matriz ortogonal, assim $M^T = M^{-1}$. Então, multiplicando à esquerda por M;

$M.M^T = M.M^{-1} = I$, e também, à direita;

$M^T.M = M^{-1}.M = I$

Logo, $M.M^T = M^T.M = I$. □

b) Verificar se $M = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$ é ortogonal:

Se M é ortogonal, então $M^T = M^{-1}$.

Vejamos, então, a inversa de M, utilizando o algoritmo: $(M|I) = \left[\begin{array}{cc|cc} 1/2 & \sqrt{3}/2 & 1 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|cc} 1/2 & \sqrt{3}/2 & 1 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 1 \end{array}\right] \xrightarrow{L_1 \leftrightarrow 2L_1} \left[\begin{array}{cc|cc} 1 & \sqrt{3} & 2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 1 \end{array}\right] \xrightarrow{L_2 \leftrightarrow -\frac{\sqrt{3}}{2}L_1} \left[\begin{array}{cc|cc} 1 & \sqrt{3} & 2 & 0 \\ 0 & -2 & -\sqrt{3} & 1 \end{array}\right]$$

$$\xrightarrow{L_2 \leftrightarrow -\frac{L_2}{2}} \left[\begin{array}{cc|cc} 1 & \sqrt{3} & 2 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow{L_1 \leftrightarrow -\sqrt{3}L_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & \sqrt{3}/2 \\ 0 & 1 & \sqrt{3}/2 & -1/2 \end{array}\right]$$

Logo, $M^{-1} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$.

Também, temos que: $M^T = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$ e, portanto: $M^{-1} = M^T$

Logo, M é ortogonal.

Mostre que $A = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ é ortogonal, usando o item a).

Se A é ortogonal, então $M^T = M^{-1}$ e $M.M^T = M^T.M = I$, assim, seja $A^T = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, segue que:

$$A.A^T = \begin{bmatrix} \cos^2\theta + \operatorname{sen}^2\theta & \cos\theta\operatorname{sen}\theta - \operatorname{sen}\theta\cos\theta & 0 \\ \operatorname{sen}\theta\cos\theta - \cos\theta\operatorname{sen}\theta & \operatorname{sen}^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Analogamente mostramos que $A^T.A = I$.

Logo, $A.A^T = A^T.A = I$, ou seja, A é ortogonal. □