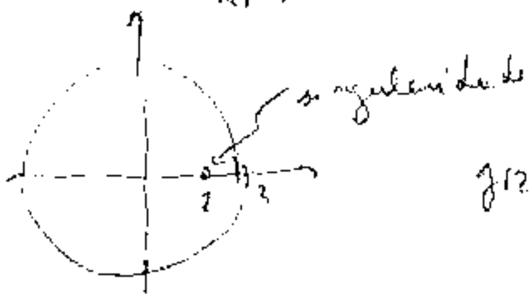
$\boxed{L10}$

02) $\displaystyle\int_{|z|=3} \frac{z^2+1}{z+2}\,dz$



$$g(-2) = \frac{1}{2\pi i}\int_{|z|=3} \frac{g(z)\,dz}{z-(-2)}$$

$g(z) = z^2+1$

$$\Rightarrow \int \frac{z^2+1}{z+2}\,dz = 2\pi i\cdot g(-2)$$

$$= 2\pi i\cdot\left((-2)^2+1\right) = 10\pi i$$

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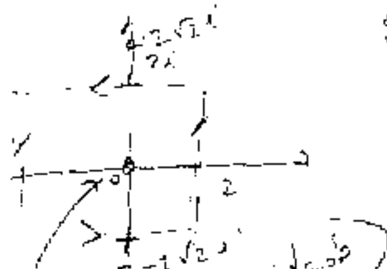
03) (1) $\displaystyle\int_\gamma \frac{\cos z\,dz}{z(z^2+8)}$



$z=0$ é singularidade

$z^2+8=0$
$\Downarrow$
$z = \pm 2\sqrt{2}\,i$
$\notin \operatorname{int}(\gamma)$

Escreva.

$$f(z) = \frac{\cos z}{z^2+8}$$

Então, pelo T. Cauchy:

$$f(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-(0)} =$$

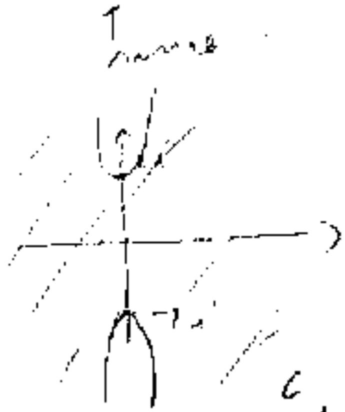
$$= f(0) = \frac{1}{2\pi i}\int_\gamma \frac{\overbrace{\frac{\cos z}{z^2+8}}^{f(z)}}{z-0}\,dz$$

$$\Rightarrow \int_\gamma \frac{\cos z}{(z^2+8)\cdot z}\,dz = 2\pi i\cdot f(0)$$

$$= 2\pi i\cdot\frac{\cos 0}{0^2+8} = \frac{\pi}{4}i$$

05) $\displaystyle\int_C \frac{\sqrt{z^2+4}}{4z^2+4z-3}\,dz$

$\sqrt{4} = -2$



$4z^2+4z-3=0$

$z = \dfrac{-4 \pm \sqrt{16+48}}{8}$

$z = \dfrac{-4 \pm \sqrt{64}}{8}$

$z = \dfrac{-4+8}{8} = \dfrac{1}{2}$

$z = \dfrac{-4-8}{8} = -\dfrac{3}{2}$

[illegible] C.

$$\int_C \frac{\sqrt{z^2+4}\,dz}{4z^2+4z-3} = \int_C \frac{\sqrt{z^2+4}\,dz}{4\left[(z-\frac{1}{2})\cdot(z+\frac{3}{2})\right]}$$

$$= 2\pi i\,\frac{1}{2\pi i}\int_C \frac{\overbrace{\frac{\sqrt{z^2+4}}{4(z+\frac{3}{2})}}^{g(z)}}{z-\frac{1}{2}}\,dz = 2\pi i\, g\left(\tfrac{1}{2}\right)$$

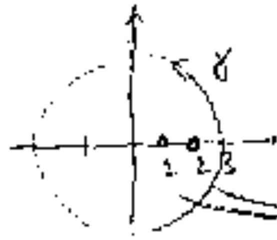
$$= 2\pi i\,\frac{\sqrt{(\frac{1}{2})^2+4}}{4(\frac{1}{2}+\frac{3}{2})}$$

$$= \cancel{2}\pi i\,\frac{\sqrt{\frac{17}{4}}}{4\cdot\frac{4}{2}} = 2\pi i\,\frac{\frac{\sqrt{17}}{\cancel{2}}}{\frac{8}{\cancel{2}}}$$

$$= \frac{\pi\sqrt{17}}{4}\,i$$

02

07) $\gamma : |z| = 3.$



$z = 1$ e $z = 2$ são singularidades no int(γ).

$$\int_\gamma \frac{\operatorname{sen}\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} \;\ldots$$

Note que $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

Daí:

$$\int_\gamma \frac{\operatorname{sen}\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = \int_\gamma \frac{\operatorname{sen}\pi z^2 + \cos\pi z^2}{z-2}\,dz - \int_\gamma \frac{\operatorname{sen}\pi z^2 + \cos\pi z^2}{z-1}\,dz$$

$$= 2\pi i \cdot \frac{1}{2\pi i}\int_\gamma \frac{g(z)\,dz}{z-2} - 2\pi i \cdot \frac{1}{2\pi i}\int_\gamma \frac{g(z)\,dz}{z-1}$$

$g(z) = \operatorname{sen}\pi z^2 + \cos\pi z^2$

$$= 2\pi i\, g(2) - 2\pi i\, g(1) =$$

$$= 2\pi i(\operatorname{sen}4\pi + \cos 4\pi) - 2\pi i(\operatorname{sen}\pi i + \cos\pi i)$$

$$= 2\pi i(0 + i) - 2\pi i(0 - i) = -2\pi - 2\pi = -4\pi$$

$$\int_\gamma \frac{e^{2z}\,dz}{(z+1)^4}$$

$z=-1$ no interior de γ

como tem essa potência, é a fórmula geral de Cauchy.

Seja $g(z) = e^{2z}$. Então, pela fórmula geral de Cauchy $(n=3)$

$$g^{(3)}(-1) = \frac{3!}{2\pi i}\int_\gamma \frac{g(z)\,dz}{(z-(-1))^4} = \frac{6}{2\pi i}\int_\gamma \frac{e^{2z}\,dz}{(z+1)^4}$$

$$\Longrightarrow \int_\gamma \frac{e^{2z}\,dz}{(z+1)^4} = \frac{2\pi i}{6}\cdot g^{(3)}(-1) \text{ , onde}$$

$$g(z) = e^{2z} \Rightarrow g'(z) = 2e^{2z} \Rightarrow g''(z) = 2^2 e^{2z}$$

$$\Rightarrow g^{(3)}(z) = 2^3 \cdot e^{2z} = 8\,e^{2z}$$

e então:

$$\int_\gamma \frac{e^{2z}\,dz}{(z+1)^4} = \frac{\pi i}{3}\cdot g^{(3)}(-1)$$

$$= \frac{\pi i}{3}\cdot 8 \cdot e^{2\cdot(-1)} = \underline{\frac{8\pi}{3}\, e^{-2}}$$

15) $\displaystyle \frac{1}{2\pi i}\int_{|z|=n} \frac{e^z\,dz}{z^{n+1}}$

$z=0 \in \operatorname{int}(\gamma), \quad \forall n \in \mathbb{N}$

$\Rightarrow \displaystyle f^{(n)}(0) = \frac{n!}{2\pi i}\int_{|z|=n} \frac{f(z)\,dz}{(z-0)^{n+1}}, \quad f(z)=e^z$

$\Rightarrow f^{(n)}(z) = e^z$

Assim:

$$\frac{1}{2\pi i}\int_{|z|=n} \frac{e^z\,dz}{z^{n+1}} = \frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$$

Daí:

$$\frac{1}{n!} = \left|\frac{1}{n!}\right| = \left|\frac{1}{2\pi i}\int_{|z|=n} \frac{e^z\,dz}{z^{n+1}}\right| \leq \frac{1}{2\pi}\int_{|z|=n} \frac{|e^z|\cdot|dz|}{|z|^{n+1}} \underset{\text{Obs}^{(*)}}{\leq}$$

$$\leq \frac{1}{2\pi}\int_{|z|=n} \frac{e^{|z|}\cdot|dz|}{|z|^{n+1}} = \frac{1}{2\pi}\int_{|z|=n} \frac{e^n}{n^{n+1}}\cdot|dz|$$

$$\Rightarrow \frac{1}{n!} \leq \frac{1}{2\pi}\,\frac{e^n}{n^{n+1}}\cdot\int_0^{2\pi}|dz| = \frac{1}{\cancel{2\pi}}\,\frac{e^n}{n^{n+1}}\cdot\cancel{2\pi}\cdot n$$

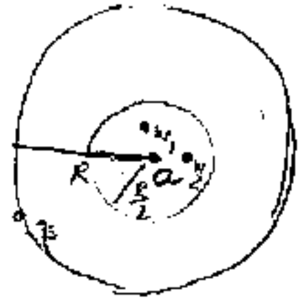
$$\Rightarrow n! \geq \frac{n^n}{e^n} \Rightarrow \boxed{n! \geq n^n\cdot e^{-n}}$$

(*) Obs.: $|e^z| = |e^{x+iy}| = e^x$ e $e^{|z|} = e^{\sqrt{x^2+y^2}}$, como $\sqrt{x^2+y^2} \geq x$ e a exponencial real é crescente, então $|e^z| \leq e^{|z|}$.

1b) $D_R(a) = \{ z \in \mathbb{C} : |z-a| \le R \} \subset \Omega$

Suponha $|f(z)| \le M$, $\forall z \in \partial D_R(a)$

Dados $w_1, w_2 \in D_{\frac{R}{2}}(a)$



$\forall z \in \partial D_R(a)$, temos:

$$|f(w_1) - f(w_2)| = \left| \frac{1}{2\pi i} \int_{\partial D_R(a)} \frac{f(z)}{z-w_1} dz - \frac{1}{2\pi i} \int_{\partial D_R(a)} \frac{f(z)\, dz}{z-w_2} \right|$$

$$= \left| \frac{1}{2\pi i} \int_{\partial D_R(a)} \frac{z-w_2-(z-w_1)}{(z-w_1)(z-w_2)} \cdot f(z)\, dz \right|$$

$$\le \frac{1}{2\pi} \int_{\partial D_R(a)} \frac{|w_2-w_1| \cdot \overbrace{|f(z)|}^{\le M} \cdot |dz|}{|z-w_1| \cdot |z-w_2|} \le$$

$$|z-w_1| \ge \frac{R}{2} \Rightarrow \frac{1}{|z-w_1|} \le \frac{2}{R}$$

$$|z-w_2| \ge \frac{R}{2} \Rightarrow \frac{1}{|z-w_2|} \le \frac{2}{R}$$

$$\le \frac{1}{2\pi} M \cdot \frac{2}{R} \cdot \frac{2}{R} |w_2-w_1| \int_{\partial D_R(a)} |dz|$$

$$= \frac{2M}{\pi R^2} \cdot |w_2-w_1| \cdot 2\pi R = \frac{4M}{R} \cdot |w_2-w_1|$$

$$\Rightarrow \boxed{|f(w_1) - f(w_2)| \le \frac{4M}{R} |w_2-w_1|}$$

06

18)

$f:\Omega\to\mathbb{C}$, $f\in C^{\omega}(\Omega)$

$\overline{D_5(0)}\subset\Omega$. $|f(z)|\le 7$, $\forall z\in\partial D_5(0)$.

Obter $M>0$ tal que $|f'''(2i)|\le M$

Pela fórmula geral de Cauchy:

$$f'''(2i) = \frac{3!}{2\pi i}\int_{\partial D_5(0)}\frac{f(z)\,dz}{(z-2i)^4}$$

e então

$$|f'''(2i)| = \left|\frac{6}{2\pi i}\int_{\partial D_5(0)}\frac{f(z)\,dz}{(z-2i)^4}\right| \le$$

$$\le \frac{3}{\pi}\int_{\partial D_5(0)}\frac{|f(z)|\cdot|dz|}{|z-2i|^4} \le$$

$z\in\partial D_5(0)$

$\Rightarrow |z-2i|\geq 3$

$\Rightarrow \frac{1}{|z-2i|}\le\frac{1}{3}$

$$\le \frac{3}{\pi}\int_{\partial D_5(0)}\frac{7\cdot|dz|}{\left(\frac{1}{3}\right)^4}$$

$$= \frac{3}{\pi}\cdot\frac{1}{(3)^3}\cdot\int_{\partial D_5(0)}|dz|$$

$$= \frac{7}{27\pi}\cdot 2\pi\cdot 5 = \frac{70}{27} =: M$$

Temos $M = \frac{70}{27}$

(07)

26) Não, pois $f(z) = \operatorname{sen} z$ é uma função inteira, e se tivéssemos $|f(z)| = |\operatorname{sen} z| \leq 1$, $\forall z \in \mathbb{C}$, então, além de inteira, f seria limitada. Logo, pelo T. de Liouville seguiria que $\operatorname{sen} z$ seria uma constante, o que é um absurdo.