LISTA 09 – algumas soluções.

01)
$\gamma : z(t) = 3e^{2it} + 2 \quad ; \quad t \in [-\pi, \pi]$

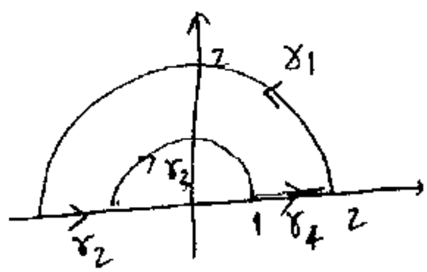
$$C(\gamma) = \int_{-\pi}^{\pi} |z'(t)|\,dt \underset{\uparrow}{=} \int_{-\pi}^{\pi} 6\,dt = 6t\Big|_{-\pi}^{\pi} = 6(\pi - (-\pi))$$

$$z'(t) = 6ie^{2it} \Rightarrow |z'(t)| = 6$$

$$= 6\cdot(2\pi)$$

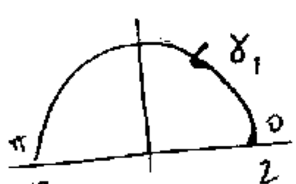
$$= 12\pi.$$

06) $f(z) = \dfrac{z}{\bar{z}}$



$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f. \qquad (*)$$

γ_1:



$$z = 2e^{i\theta} \Rightarrow dz = 2ie^{i\theta}d\theta$$

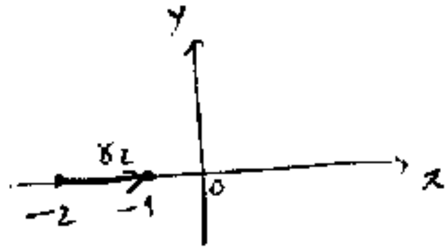
$$\bar{z} = 2e^{-i\theta} \quad \text{[verifique!]}$$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_0^{\pi} \frac{2e^{i\theta}\cdot 2ie^{i\theta}\,d\theta}{2e^{-i\theta}} = 2\int_0^{\pi} e^{3i\theta} i\,d\theta =$$

$$= \frac{2}{3}\int_0^{\pi} e^{3i\theta}(3i\,d\theta) = \frac{2}{3}\, e^{3i\theta}\Big|_0^{\pi} =$$

$$= \frac{2}{3}\left(\underbrace{\cos 3\pi}_{-1} + i\,\underbrace{\operatorname{sen} 3\pi}_{=0}\right) = -\frac{2}{3}.$$

$\underline{\gamma_2}$:



$$\gamma_2(t) = t + 0i = t, \quad t \in [-2,-1].$$

$$\int_\gamma f(z)\,dz = \int_{-2}^{-1} f(\gamma_2(t))\cdot\gamma_2'(t)\,dt = \int_{-2}^{-1} 1\cdot dt$$

$$f(\gamma_2(t)) = \frac{\gamma_2(t)}{\overline{\gamma_2(t)}} = \frac{t}{t} = 1.$$

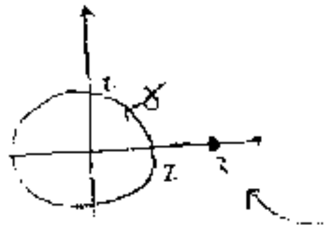
$$\gamma_2(t) = t \Rightarrow \gamma_2'(t) = 1$$

$$= t\Big|_{-2}^{-1} = -1-(-2) = 1$$

Analogamente, se calcula $\int_{\gamma_3} f$ e $\int_{\gamma_4} f$.

Depois, c.f. (*), basta somar todas as 4 integrais.

08) (a) $f(z) = \frac{z+1}{z-3}$; $\gamma : |z| = 2$



$f(z) = \frac{z+1}{z-3}$ não é holomorfa em $z = 3$.

Logo, no interior de γ ela é holomorfa. Assim, sendo γ um caminho fechado, pelo T. de Cauchy-Goursat segue que

$$\int_\gamma f(z)\,dz = 0.$$

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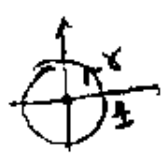
10) (a) $\displaystyle\int_{|z|=1} \frac{dz}{z} = 2\pi i$:

$$\int_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} \frac{i \cdot \cancel{e^{i\theta}}\,d\theta}{\cancel{e^{i\theta}}} = i \cdot \int_0^{2\pi} d\theta = i \cdot \theta \Big|_0^{2\pi} = \underline{\underline{2\pi i}}$$

$z = 1 \cdot e^{i\theta}$

$\Rightarrow dz = i\,e^{i\theta}\,d\theta$

$\theta \in [0, 2\pi]$



(b) $$\int_\gamma \frac{dz}{|z|} = \int_0^{2\pi} \frac{i\,e^{i\theta}\,d\theta}{1} = e^{i\theta}\Big|_0^{2\pi} = e^{2\pi i} - e^0$$

$\gamma : |z| = 1$

$z = 1 e^{i\theta}$

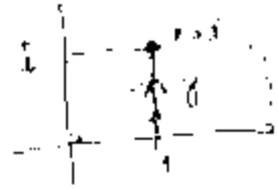
$\hookrightarrow |z| = 1$

$dz = i\,e^{i\theta}\,d\theta$

$= \cos 2\pi + i \operatorname{sen} 2\pi - 1$

$= 1 + 0i - 1 = \underline{\underline{0}}$

11) $\left|\int_\gamma \frac{z+2}{z}\,dz\right| \overset{?}{\leq} 3$. γ



$l = 1$ comprimento de γ é $l = 1$.

$\forall z \in \gamma(t)$, $|z| \geq 1 \Rightarrow \frac{1}{|z|} \leq 1$.

Note que:

$$\left|\frac{z+2}{z}\right| \leq \frac{|z|+2}{|z|} = 1 + \frac{2}{|z|} \leq 1 + \frac{2}{1} = 3$$

Assim:

$$\left|\int_\gamma \frac{z+2}{z}\,dz\right| \leq \int_\gamma \underbrace{\left|\frac{z+2}{z}\right|}_{\leq 3}\cdot|dz| \leq 3\underbrace{\int_\gamma |dz|}_{\text{comprim. de } \gamma} = 3\cdot 1 = 3$$

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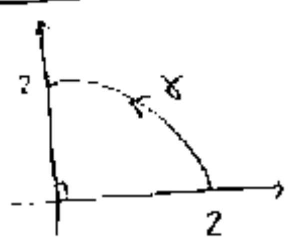
13)



1º q. Mostrar:

$$\left|\int_\gamma \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}$$

Note que, como

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|}, \text{ e como}$$

$$|z^2+1| \geq |z|^2 - 1 = (2)^2 - 1 = 3 \Rightarrow \frac{1}{|z^2+1|} \leq \frac{1}{3}.$$

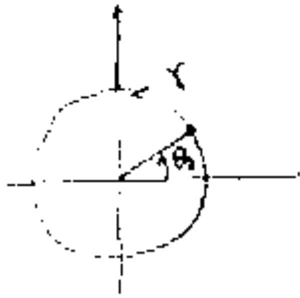
$|z| = 2$ em γ

Então:

$$\left|\int_\gamma \frac{dz}{z^2+1}\right| \leq \int_\gamma \left|\frac{1}{z^2+1}\right|\cdot|dz| \leq \frac{1}{3}\underbrace{\int_\gamma |dz|}_{\text{comprim. de } \gamma} = \frac{1}{3}\,\underbrace{\frac{2\pi\cdot \overset{\text{RAIO}}{2}}{4}}_{\frac{1}{4} \text{ parte da circunf.}}$$

$$= \frac{\pi}{3} \Rightarrow \left|\int_\gamma \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}.$$

14)



$$z_0 = r e^{i\theta_0}$$

$$z = C(\theta) = r e^{i\theta}, \quad \theta \in [\theta_0, \theta_0 + 2\pi]$$

$$\int_\gamma z^a \, dz = \int_{\theta_0}^{\theta_0+2\pi} (r e^{i\theta})^a \cdot r\, i\, e^{i\theta} \, d\theta = \int_{\theta_0}^{\theta_0+2\pi} r^a e^{ia\theta} \cdot r\, i\, e^{i\theta} \, d\theta =$$

$z = r e^{i\theta}$
$dz = r\, i\, e^{i\theta} d\theta$

$$= r^{a+1} \int_{\theta_0}^{\theta_0+2\pi} e^{i\theta(a+1)} \cdot i \, d\theta = \frac{r^{a+1}}{a+1} \int_{\theta_0}^{\theta_0+2\pi} e^{i\theta(a+1)} \cdot i(a+1) \, d\theta$$

$\int e^w dw$
$w = i\theta(a+1)$
$\Rightarrow dw = i(a+1) d\theta$

$$= \frac{r^{a+1}}{a+1} \left. e^{i\theta(a+1)} \right|_{\theta_0}^{\theta_0+2\pi} = \frac{r^{a+1}}{a+1} \left[e^{i(\theta_0+2\pi)(a+1)} - e^{i\theta_0(a+1)} \right]$$

$$= \frac{r^{a+1}}{a+1} \left[e^{i\theta_0(a+1)} \cdot e^{2\pi i(a+1)} - e^{i\theta_0(a+1)} \right] =$$

$$= \frac{r^{a+1}}{a+1} \cdot e^{i\theta_0(a+1)} \left[e^{2\pi i(a+1)} - 1 \right]$$

$$= \frac{(r e^{i\theta_0})^{a+1}}{a+1} \cdot \left[e^{2\pi i(a+1)} - 1 \right] =$$

$z_0 = r e^{i\theta_0}$

$$= \frac{z_0^{a+1}}{a+1} \cdot \left[e^{2\pi i(a+1)} - 1 \right].$$