

Introductio in analysin
infinitorum. Tomus 1 /
auctore Leonhardo Eulero,...

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1967



INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,
*Professore Regio BEROLINENSI, & Academiae Im-
perialis Scientiarum PETROPOLITANÆ
Socio.*

TOMUS PRIMUS.



LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

1

2

1



A Lausanne et Geneve, chez MARC-MICHEL BOUSQUET et Comp^e 1748.



ILLUSTRISSIMO VIRO
JOHANNI JACOBO
DORTOUS DE MAIRAN,
UNI EX XLVIRIS
ACADEMIÆ GALLICÆ,
REGIÆ ETIAM SCIENTIARUM
PARISIENSIS,
IN QUA SECRETARII PERPETUI MUNUS NUPER
ABDICAVIT,

*

NEC

NEC NON
ALIARUM BENE MULTARUM,
LONDINENSIS, PETROPOLITANÆ,
&c.
SOCIETATUM, ACADEMIARUMVE
SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

*V*IR ILLUSTRISIME,



*Atronos Euleriano
scripto quærere necesse
neutiquam esse, Mathe-
maticarum Disciplina-
rum cultoribus satis
constat. Sciunt utique
illi, varias earum partes novis eum lumi-
nibus sic illustrasse, ut inde meritò clarif-
simi*

simi rerum in his abstrusissimarum interpretis locum sit consequutus. Quem quin egregiè tueatur, immò tollat se altiùs quoque opere isthoc, nemo dubitabit, certior hisce factus, indulgissè Te mihi, ut illustrissimo nomini Tuo dicatum publicè prodiret. Pertinere autem hoc in me collatum beneficium ad Auctoris decus probe intelligens, Ipse, ut eo uterer, lubens concessit; & cum in rem meam faciat omnimodò, quì neglexissem?

Ab his equidem, quibus Libros inscribunt, sibi nescio quid ideò deberi, plerique tacitè constituunt; acceptaque beneficia quodammodo remunerari, ut sese ferè nexu liberent omni. Ego verò secus sentio. Mihi certè merum est beneficium patroni, quòd scriptoris aut excusoris

*opera id genus honore condecorari patiantur. Hac mente utique Tibi, VIR ILLUSTRIS-
SIME, animi gratissimi sum-
mæque observantiæ professionem hisce pu-
blicam excipias, rogo.*

*Paratum promptumque semper juvan-
dis litterarum studiis qui Te novit, &
notus vel hoc nomine es cuicumque in Re-
publica doctorem Europæ totius non hos-
piti, plurimis officiis meæ etiam conditio-
nis homines à Te affectos fuisse statuat
necesse est. Nempe, tanquam Tibi uni
esset injunctum curare, ut floreat huma-
num ingenium illustrantes scientiæ omnes,
hominumque in usus adinventæ artes,
ad singulis inservientium artifices etiam
Te demittere dignaris, vel ab illa subli-
mum rerum perscrutatione, Cælive ip-
sius*

DEDICATORIA.

v

suis Tibi tam nota regione, ut quæ hucusque mentes hominum metu complebant Phænomena minus intellecta, per Te jam grato tantum admirationis sensu contemplantur, earumque causas habeant perspectas.

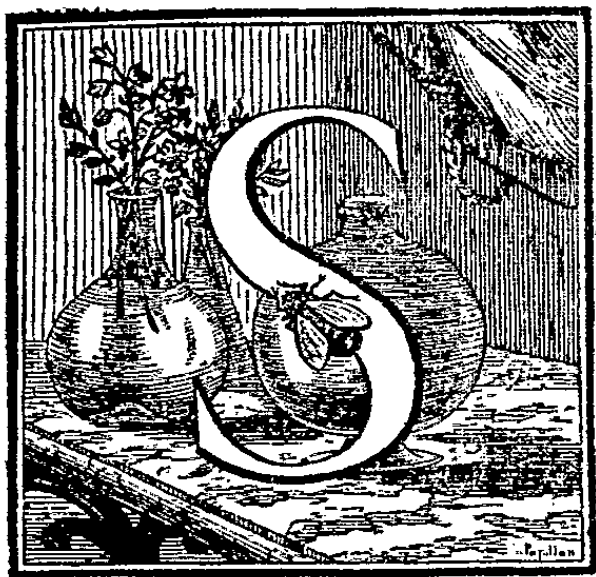
*Hinc ille veluti ex condicto Academia-
rum Orbis eruditi concursus, ut adlectum
Te cætui suo consequerentur, ornamento
aliàs carituro insigni, quo cæteras nol-
lent præ se frui. Hinc inprimis Illustris-
simæ Parisiensis de Te iudicium, cum
ageretur de successore sufficiendo in locum
emeriti Fontenellii, Viri, cujus ex ore
calamoque fluere Scientiarum Artiumque
omnium exquisitiores divitiæ, elegantia-
que universæ perpetuo visæ sunt, & vi-
debuntur dum sani sensus quicquam hu-*

mano ingenio erit. Tibi, scilicet, Commentariorum Academiae conscribendorum provincia, cui praefectus ille erat, demandabatur continuo; quam, ut ornare diutius voluisses, docti omnes optabant: hoc uno minus dolentes Te aliter censuisse, quod aliis Tibi magis placitum, profuturisque nihilominus litteris in universum eruditionis ingenive thesauros impenderes. Quod ut ad ultimas usque metas hominum vitae positas incolumis, florens, atque beatus praestes, omni votorum contentione precor. Vale!

Dabam Lausannae die 1. Aprilis
Anni Æræ Dionys. 1748.



P R Æ F A T I O.



Æpenumero animadverti, maximam difficultatum partem, quas Matheseos cultores in addiscenda Analyfi infinitorum offendere solent, inde oriri; quod, Algebra communi vix apprehensa, animum ad illam sublimiorem artem appellant; quo fit, ut non solum quasi in limine subsistant, sed etiam perversas ideas illius infiniti, cujus notio in subsidium vocatur, sibi forment. Quanquam autem Analyfis infinitorum non perfectam Algebrae communis, omniumque artificiorum adhuc inventorum cognitionem requirit; tamen plurimæ extant quæstiones, quarum evolutio discentium animos ad sublimiorem scientiam præparare valet, quæ tamen in communibus Algebrae elementis, vel omittuntur, vel non satis accurate tractantur. Hanc ob rem non dubito, quin ea, quæ in his libris congeffi, hunc defectum abunde supplere queant. Non solum enim operam dedi, ut eas res, quas Analyfis
infini-

infinitorum absolute requirit, uberius atque distinctius exponerem, quam vulgo fieri solet; sed etiam satis multas quæstiones enodavi, quibus Lectores sensim & quasi præter expectationem ideam infiniti sibi familiarem reddent. Plures quoque quæstiones per præcepta communis Algebrae hic resolvi, quæ vulgo in Analyfi infinitorum tractantur: quo facilius deinceps utriusque Methodi summus consensus eluceat.

Divisi hoc Opus in duos Libros, in quorum priori, quæ ad meram Analyfin pertinent, sum complexus: in posteriori vero, quæ ex Geometria sunt scitu necessaria, explicavi, quoniam Analysis infinitorum ita quoque tradi solet, ut simul ejus applicatio ad Geometriam ostendatur. In utroque autem prima Elementa prætermisi, eaque tantum exponenda duxi, quæ alibi, vel omnino non, vel minus commode tractata, vel ex diversis principiis petita reperiuntur.

In primo igitur Libro, cum universa Analysis infinitorum circa quantitates variables earumque Functiones versetur, hoc argumentum de Functionibus imprimis fusius exposui; atque Functionum tam transformationem, quam resolutionem & evolutionem per series infinitas demonstravi. Complures enumeravi Functionum species, quarum in Analyfi sublimiori præcipue ratio est habenda. Primum eas distinxì in algebraicas & transcendentes; quarum illæ per operationes in Algebra communi usitatas ex quantitatibus variabilibus formantur, hæ vero vel per alias rationes componuntur, vel ex iisdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria subdivisio fit in rationales & irrationales, priores docui cum in partes simpliciores, tum in factores resolvere; quæ operatio in Calculo integrali maximum adjumentum affert; posteriores vero, quemadmodum idoneis substitutionibus ad formam rationalem perducì queant ostendi. Evolutio autem per series infinitas ad utrumque genus æque pertinet, atque etiam ad Functiones

Etiones transcendentes summa cum utilitate applicari solet; at quantopere doctrina de seriebus infinitis Analysin sublimiorem amplificaverit, nemo est qui ignoret. Nonnulla igitur adjunxi Capita, quibus plurimum serierum infinitarum proprietates, atque summas sum scrutatus; quarum quædam ita sunt comparatæ, ut sine subsidio Analysis infinitorum vix investigari posse videantur. Hujusmodi series sunt, quarum summæ exprimuntur, vel per Logarithmos vel Arcus circulares: quæ quantitates cum sint transcendentes, dum per quadraturam Hyperbolæ & Circuli exhibentur, maximam partem demum in Analysis infinitorum tractari sunt solitæ. Postquam autem a potestatibus ad quantitates exponentiales essem progressus, quæ nil aliud sunt nisi potestates, quarum exponentes sunt variabiles; ex earum conversione maxime naturalem ac fecundam Logarithmorum ideam sum adeptus: unde non solum amplissimus eorum usus sponte est consecutus, sed etiam ex ea cunctas series infinitas, quibus vulgo istæ quantitates repræsentari solent, elicere licuit: hincque adeo facillimus se prodidit modus Tabulas Logarithmorum construendi. Simili modo in contemplatione Arcuum circularium sum versatus; quod quantatum genus, etsi a Logarithmis maxime est diversum, tamen tam arcto vinculo est connexum, ut dum alterum imaginarium fieri videtur, in alterum transeat. Repetitis autem ex Geometria quæ de inventionem Sinuum & Cosinuum Arcuum multiplo- rum ac submultiplo- rum traduntur, ex Sinu vel Cosinu cujusque Arcus expressi Sinum Cosinumque Arcus minimi & quasi evanescens, quo ipso ad series infinitas sum deductus: unde, cum Arcus evanescens Sinui suo sit æqualis, Cosinus vero radio, quemvis Arcum cum suo Sinu & Cosinu ope serierum infinitarum comparavi. Tum vero tam varias expressiones cum finitas tum infinitas pro hujus generis quantitatibus obtinui, ut ad earum naturam perspicendam Calculo infinitesimali prorsus non amplius
* *
esset

esset opus. Atque quemadmodum Logarithmi peculiarem Algorithmum requirunt, cujus in universa Analyti summus extat usus, ita quantitates circulares ad certam quoque Algorithmi normam perduximus; ut in calculo æque commode ac Logarithmi & ipsæ quantitates algebraicæ tractari possent. Quantum autem hinc utilitatis ad resolutionem difficillimarum quæstionum redundet, cum nonnulla Capita hujus Libri luculenter declarant, tum ex Analyti infinitorum plurima specimina proferri possent, nisi jam satis essent cognita, & indies magis multiplicarentur. Maximum autem hæc investigatio attulit adjumentum ad Functiones fractas in factores reales resolvendas; quod argumentum, cum in Calculo integrali sit prorsus necessarium, diligentius enucleavi. Series postmodum infinitas, quæ ex hujusmodi Functionum evolutione nascuntur, & quæ recurrentium nomine innotuerunt, examini subjeci; ubi earum tam summas quam terminos generales, aliasque insignes proprietates exhibui: & quoniam ad hæc resolutio in factores manuduxit, ita vicissim, quemadmodum producta ex pluribus, imo etiam infinitis, factoribus conflata per multiplicationem in series explicentur, perpendi. Quod negotium non solum ad cognitionem innumerabilium serierum viam aperuit, sed quia hoc modo series in producta ex infinitis factoribus constantia resolvere licebat, satis commodas inveni expressiones numericas, quarum ope Logarithmi Sinuum, Cosinuum, & Tangentium facillime supputari possunt. Præterea quoque ex eodem fonte solutiones plurium quæstionum, quæ circa partitionem numerorum proponi possunt, derivavi; cujusmodi quæstiones sine hoc subsidio vires Analyseos superare videantur. Hæc tanta materiarum diversitas in plura volumina facile excrecere potuisset; sed omnia, quantum fieri potuit, tam succinctè proposui, ut ubique fundamentum clarissimè quidem explicaretur, uberius vero amplificatio industriæ Lectorum relinqueretur; quo habeant, quibus

quibus vires suas exerçant, finesque Analyſeos ulterius promoveant. Neque enim vereor profiteri, in hoc Libro non ſolum multa plane nova contineri; ſed etiam fontes eſſe detectos, unde plurima inſignia inventa adhuc hauriri queant.

Eodem inſtituto ſum uſus in altero Libro, ubi, quæ vulgo ad Geometriam ſublimiorem referri ſolent, pertractavi. Antequam autem de Sectionibus Conicis, quæ alias fere ſolæ hunc locum occupant, agerem; Theoriam Linearum Curvarum in genere ita propoſui, ut ad ſcrutationem naturæ quarumvis Linearum Curvarum cum utilitate adhiberi poſſet. Ad hoc nullum aliud ſubſidium aſſero, præter æquationem, qua cujuſque Lineæ Curvæ natura exprimitur; ex eaque cum figuram, tum primarias proprietates deducere doceo: id quod potiffimum in Sectionibus Conicis præſtitiffe mihi ſum viſus; quæ antehac vel ſecundum ſolam Geometriam vel per Analyſin quidem, ſed nimis imperſecte ac minus naturaliter, tractari ſunt ſolitæ. Ex æquatione ſcilicet generali pro Lineis ſecundi ordinis primum earum proprietates generales explicavi, tum eas in genera ſeu ſpecies ſubdiviſi; reſpiciendo utrum habeant ramos in infinitum excurrentes, an vero tota Curva finito ſpatio includatur. Priori autem caſu inſuper diſpiciendum erat, quot ſint rami in infinitum excurrentes, & cujuſ naturæ ſint ſinguli; an habeant Lineas rectas aſymptotas, an minus. Sicque obtinui tres conſuetas Sectionum Conicarum ſpecies; quarum prima eſt Ellipſis, tota in ſpatio finito contenta; ſecunda autem Hyperbola, quæ quatuor habet ramos infinitos ad duas rectas aſymptotas convergentes; tertia vero ſpecies prodiit Parabola duos habens ramos infinitos aſymptotis deſtitutos. Simili porro ratione Lineas tertii ordinis ſum perſecutus, quas, poſt expoſitas earum proprietates generales, diviſi in ſedecim genera; ad eaque omnes ſeptuaginta duas ſpecies NEWTONI revocaſi. Ipſam vero methodum ita clare deſcripſi, ut pro

quovis Linearum ordine sequente divisio in genera facillime institui queat; cuius negotii periculum quoque feci in Lineis quarti ordinis. His deinde, quæ ad ordines Linearum pertinent, expeditis, reversus sum ad generales omnium Linearum affectiones eruendas. Explicavi itaque methodum definiendi tangentes curvarum, earum normales, atque etiam ipsam curvaturam, quæ per radium osculi æstimari solet: quæ etsi nunc quidem plerumque Calculo differentiali absolvuntur, tamen idem per solam communem Algebram hic præstiti, ut deinceps transitus ab Analyfi finitorum ad Analyfin infinitorum eo facilius reddatur. Perpendi etiam curvarum puncta flexus contrarii, cuspides, puncta duplicia, ac multiplicia; modumque exposui hæc omnia ex æquationibus sine ulla difficultate definiendi. Interim tamen non nego, has quæstiones multo facilius Calculi differentialis ope enodari posse. Attigi quoque controversiam de cuspidem secundi ordinis, ubi ambo arcus in cuspidem coeuntes curvaturam in eandem partem vertunt; eamque ita composuisse mihi videor, ut nullum dubium amplius superesse possit. Denique adunxi aliquot Capita, in quibus Lineas Curvas, quæ datis proprietatibus gaudeant, invenire docui; pluraque tandem Problemata circa singulares Circuli sectiones soluta dedi. Quæ cum sint ea ex Geometria, quæ ad Analyfin infinitorum addiscendam maximum adminiculum afferre videntur, Appendicis loco ex Stereometria Theoriam solidorum eorumque superficierum per Calculum proposui, & quemadmodum cujusque superficiei natura per æquationem inter tres variables exponi queat, ostendi. Hinc, superficiibus instar linearum in ordines digestis, secundum dimensionum quas variables in æquatione constituunt numerum, in primo ordine solam superficiem planam contineri ostendi. Superficies vero secundi ordinis, ratione habita partium in infinitum expansarum, in sex genera divisi; similique modo pro ceteris ordinibus divisio institui poterit.

poterit. Contemplatus sum quoque intersectiones duarum superficierum; quæ cum plerumque sint curvæ non in eodem plano sitæ, quemadmodum æquationibus comprehendendi queant, monstravi. Tandem etiam positionem planorum tangentium, atque rectarum, quæ ad superficies sint normales, determinavi.

De cetero, cum non paucæ res hic occurrant ab aliis jam tractatæ, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem genere elaborarunt, fecerim. Cum enim mihi propositum esset omnia quam brevissime pertractare, Historia cujusque Problematis magnitudinem operis non mediocriter auxisset. Interim tamen pleræque quæstiones, quæ alibi quoque solutæ reperiuntur, hic solutiones ex aliis principiis sunt nactæ; ita ut non exiguam partem mihi vindicare possem. Spero autem cum ista, tum ea potissimum, quæ prorsus nova hic proferuntur, plerisque, qui hoc studio delectantur, non ingrata esse futura.





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A P P E N D I X.

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I N T R O-

INTRODUCTIO
I N
ANALYSIN INFINITORUM.
LIBER PRIMUS,

Continens

Explicationem de Functionibus quantitatum variabilium ; earum resolutione in Factores, atque evolutione per Series infinitas : una cum doctrina de Logarithmis, Arcubus circularibus, eorumque Sinubus & Tangentibus; pluribusque aliis rebus, quibus Analysis infinitorum non mediocriter adjuvatur.

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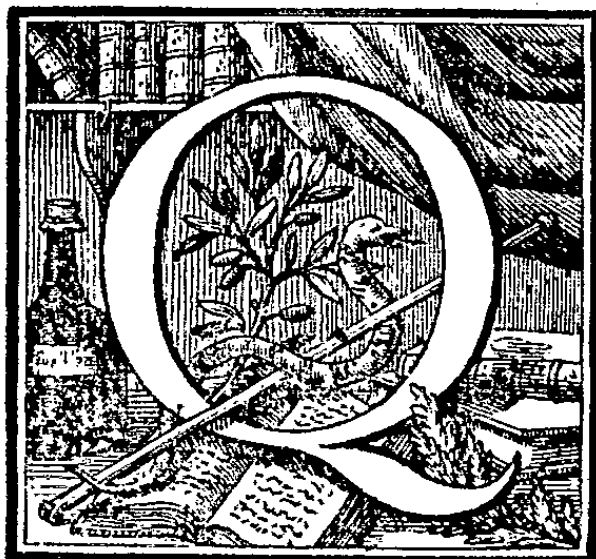


LIBER PRIMUS.

CAPUT PRIMUM.

DE FUNCTIONIBUS IN GENERE.

I.



Uantitas constans est quantitas determinata, perpetuo eundem valorem servans.

Ejusmodi quantitates sunt numeri cujusvis generis, quippe qui eundem, quem semel obtinuerunt, valorem constanter conseruant: atque si hujusmodi quantitates constantes per characteres indicare convenit, adhibentur lit-

teræ Alphabethi initiales *a, b, c*, &c. In Analyfi quidem communi, ubi tantum quantitates determinatæ considerantur, hæ litteræ Alphabethi priores quantitates cognitæ denotare solent, posteriores vero quantitates incognitæ; at in Analyfi sublimiori hoc discrimen non tantopere spectatur, cum hic ad illud quantitatum discrimen præcipue respiciatur, quo aliæ constantes, aliæ vero variabiles statuuntur.

LIB. I. 2. *Quantitas variabilis est quantitas indeterminata seu universalis, quæ omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cujusvis generis involvit. Quemadmodum scilicet ex ideis individuorum formantur idæ specierum & generum; ita quantitas variabilis est genus, sub quo omnes quantitates determinatæ continentur. Hujusmodi autem quantitates variabiles per litteras Alphabeti postremas z, y, x , &c. repræsentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari potest, cum omnes omnino numeros ejus loco substituere liceat. Neque significatus quantitatis variabilis exhaustitur, nisi omnes valores determinati ejus loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significato quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, & numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit Functio ipsius z : Sic $a + 3z$; $az - 4zz$; $az + b\sqrt{aa - zz}$; c^z ; &c. sunt Functiones ipsius z .

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc Functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem Functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic etsi hæc Functio $\sqrt{9 - zz}$, numeris realibus loco z substituendis, nunquam valorem ternario majorem recipere potest; tamen ipsi z valores imaginarios tribuendo

tribuendo ut $5\sqrt{-1}$, nullus assignari poterit valor determinatus C A P. I.
 quin ex formula $\sqrt{(9 - zz)}$ elici queat. Occurrunt autem non-
 nunquam Functiones tantum apparentes, quæ, utcunque quan-
 titas variabilis varietur, tamen usque eundem valorem retinent,
 ut z^0 ; 1^z ; $\frac{aa - az}{a - z}$, quæ, etsi speciem Functionis mentiuntur,
 tamen revera sunt quantitates constantes.

6. *Precipuum Functionum discrimen in modo compositionis, quo
 ex quantitate variabili & quantitatibus constantibus formantur, po-
 situm est.*

Pendet ergo ab Operationibus quibus quantitates inter se com-
 poni & permisceri possunt: quæ Operationes sunt Additio &
 Subtractio; Multiplicatio & Divisio: Evectio ad Potestates &
 Radicum Extractio; quo etiam Resolutio Æquationum est refe-
 renda. Præter has Operationes, quæ algebraicæ vocari solent,
 dantur complures aliæ transcendentes, ut Exponentiales, Loga-
 rithmicæ, atque innumerabiles aliæ, quas Calculus integralis
 suppeditat.

Interim species quædam Functionum notari possunt; ut mul-
 tipla $2z$; $3z$; $\frac{2}{3}z$; az ; &c. & Potestates ipsius z , ut z^2 ;
 z^3 ; $z^{\frac{1}{2}}$; z^{-1} ; &c. quæ, uti ex unica operatione sunt desum-
 ptæ, ita expressiones quæ ex operationibus quibuscunque nascun-
 tur, Functionum nomine insigniuntur.

7. *Functiones dividuntur in Algebraicas & Transcendentes; illæ
 sunt, quæ componuntur per operationes algebraicas solas, hæ vero
 in quibus operationes transcendentes insunt.*

Sunt ergo multiplex ac Potestates ipsius z Functiones algebrai-
 cæ; atque omnes omnino expressiones, quæ per operationes al-
 gebraicas ante memoratas formantur, cujusmodi est

$$\frac{a + bz^n - c\sqrt{(2z - zz)}}{aaz - 3bz^3}.$$

Quin-etiam Functiones algebraicæ
 sæpenumero nequidem explicite exhiberi possunt, cujusmodi
 Functio ipsius z est Z , si definiatur per hujusmodi æquationem;
 $Z^3 = azzZ^3 - bz^4Z^2 + cz^3Z - 1$. Quanquam enim hæc
 A 3 æquatio

LIB. I. æquatio resolvi nequit; tamen constat Z æquari expressioni cui-
 ————
 piam ex variabili z & constantibus compositæ; ac propterea fo-
 re Z Functionem quamdam ipsius z . Cæterum de Functioni-
 bus transcendentibus notandum est, eas demum fore transcen-
 dentes, si operatio transcendens non solum ingrediatur, sed
 etiam quantitatem variabilem afficiat. Si enim operationes trans-
 cendentes tantum ad quantitates constantes pertineant, Functio
 nihilominus algebraïca est censenda: uti si c denotet circumfe-
 rentiam Circuli, cujus radius sit $= 1$, erit utique c quantitas
 transcendens, verumtamen hæ expressiones $c + z$; cz^2 ; $4z^c$ &c.
 erunt Functiones algebraïcæ ipsius z . Parvi quidem est momenti
 dubium quod a quibusdam movetur, utrum ejusmodi expressio-
 nes z^c Functionibus algebraïcis annumerari jure possint, nec-
 ne; quinetiam Potestates ipsius z , quarum exponentes sint nu-
 meri irrationales, uti $z^{\sqrt{2}}$ nonnulli maluerunt Functiones inter-
 scendentes quam algebraïcas appellare.

8. *Functiones algebraïcæ subdividuntur in Rationales & Irrationa-
 les: illæ sunt, si quantitas variabilis in nulla irrationalitate invol-
 vitur; hæ vero, in quibus signa radicalia quantitatem variabilem
 afficiunt.*

In Functionibus ergo rationalibus aliæ operationes præter Ad-
 ditionem, Subtractionem, Multiplicationem, Divisionem, &
 Evectionem ad Potestates, quarum exponentes sint numeri in-
 tegri, non insunt: erunt adeo $a + z$; $a - z$; az ; $\frac{aa + zz}{a + z}$;
 $az^3 - bz^5$; &c. Functiones rationales ipsius z . At hujusmodi
 expressiones \sqrt{z} ; $a + \sqrt{aa - zz}$; $\sqrt[3]{a - 2z + zz}$;
 $\frac{aa - z\sqrt{aa + zz}}{a + z}$ erunt Functiones irrationales ipsius z .

Hæ commode distinguuntur in Explicitas & Implicitas.

Explicitæ sunt, quæ per signa radicalia sunt evolutæ, cujus-
 modi exempla modo sunt data. Implicitæ vero Functiones irra-
 tionales sunt quæ ex resolutione æquationum ortum habent.
 Sic Z erit Functio irrationalis implicita ipsius z , si per hujusmodi
 æqua-

æquationem $Z' = az Z^2 - bz^5$ definiatur; quoniam va- CAP. I.
lorem explicitum pro Z , admissis etiam signis radicalibus, ex-
hibere non licet; propterea quod Algebra communis nondum
ad hunc perfectionis gradum est evecta.

9. *Functiones rationales denuo subdividuntur in Integras & Frac-
tas.*

In illis neque z usquam habet exponentes negativos, neque
expressiones continent fractiones, in quarum denominatores
quantitas variabilis z ingrediatur: unde intelligitur Functiones
fractas esse, in quibus denominatores z continentes, vel expo-
nentes negativi ipsius z occurrant. Functionum integrarum
hæc ergo erit Formula generalis: $a + bz + cz^2 + dz^3 + ez^4$
 $+ fz^5 + \&c.$ nulla enim Functio ipsius z integra excogitari
potest, quæ non in hac expressione contineatur. Functiones
autem fractæ omnes, quia plures fractiones in unam cogi pos-
sunt, continebuntur in hac Formula:

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + fz^5 + \&c.}{\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \zeta z^5 + \&c.}$$

ubi notandum est quantitates constantes $a, b, c, d, \&c.$
 $\alpha, \beta, \gamma, \delta, \&c.$ sive sint affirmativæ, sive negativæ, sive in-
tegræ sive fractæ, sive rationales sive irrationales, sive etiam
transcendentes, naturam Functionum non mutare.

10. *Deinde potissimum tenenda est Functionum divisio in Unifor-
mes ac Multiformes.*

Functio autem uniformis est, quæ si quantitati variabili z
valor determinatus quicunque tribuatur, ipsa quoque unicum
valorem determinatum obtineat. Functio autem Multiformis
est, quæ, pro unoquoque valore determinato in locum variabi-
lis z substituto, plures valores determinatos exhibet. Sunt
igitur omnes Functiones rationales, sive integræ sive fractæ,
Functiones uniformes; quoniam ejusmodi expressiones, quicun-
que valor quantitati variabili tribuatur, non nisi unicum valorem
præbent. Functiones autem irrationales omnes sunt multiformes;
propterea quod signa radicalia sunt ambigua, & geminum valorem
involvunt. Dantur autem quoque inter Functiones transcenden-
tes,

LIB. I. tes, & uniformes, & multiformes : quin-etiam habentur Functiones infinitiformes; cujusmodi est Arcus Circuli Sinui z respondens; dantur enim Arcus circulares innumerabiles qui omnes eundem habeant Sinum. Denotent autem hæ litteræ P , Q , R , S , T &c. singulæ Functiones uniformes ipsius z .

11. *Functio biformis ipsius z est ejusmodi Functio, quæ pro quovis ipsius z valore determinato, geminum valorem præbeat.*

Hujusmodi Functiones radices quadratæ exhibent, ut $\sqrt{(2z + zz)}$: quicunque enim valor pro z statuatur expressio $\sqrt{(2z + zz)}$ duplicem habet significatum, vel affirmativum vel negativum. Generatim vero Z erit Functio biformis ipsius z , si determinetur per æquationem quadraticam $Z^2 - PZ + Q = 0$: si quidem P & Q fuerint Functiones uniformes ipsius z . Erit namque $Z = \frac{1}{2}P \pm \sqrt{(\frac{1}{4}P^2 - Q)}$; ex quo patet cuique valori determinato ipsius z duplicem valorem determinatum ipsius Z respondere. Hic autem notandum est, vel utrumque valorem Functionis Z esse realem, vel utrumque imaginarium. Tum vero erit semper, uti constat ex natura æquationum, binorum valorum ipsius Z summa $= P$, ac productum $= Q$.

12. *Functio triformis ipsius z est, quæ pro quovis ipsius z valore, tres valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum cubicarum originem trahunt. Si enim fuerint P , Q , & R Functiones uniformes, sitque $Z^3 - PZ^2 + QZ - R = 0$, erit Z Functio triformis ipsius z ; quia pro quolibet valore determinato ipsius z triplicem valorem obtinet. Tres isti ipsius Z valores unicuique valori ipsius z respondentes, vel erunt omnes reales, vel unus erit realis, dum bini reliqui sunt imaginarii. Cæterum constat horum trium valorum summam perpetuo esse $= P$; summam factorum ex binis esse $= Q$, & productum ex omnibus tribus esse $= R$.

13. *Functio quadriformis ipsius z est, quæ pro quovis ipsius z valore quatuor valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum biquadraticarum

ticarum nascuntur. Quod si enim $P, Q, R, \& S$ denotent CAP. I.
 Functiones uniformes ipsius z , fueritque $Z^4 - PZ^3 + QZ^2 - RZ + S = 0$, erit Z Functio quadriformis ipsius z ; eo
 quod cuique valori ipsius z quadruplex valor ipsius Z respon-
 det. Quatuor horum valorum ergo, vel omnes erunt reales,
 vel duo reales duoque imaginarii, vel omnes quatuor erunt
 imaginarii. Ceterum perpetuo summa horum quatuor valorum
 ipsius Z est $= P$, summa factorum ex binis $= Q$, summa
 factorum ex ternis $= R$, ac productum omnium $= S$. Si-
 mili autem modo comparata est ratio Functionum quinquefor-
 mium & sequentium.

14. Erit ergo Z Functio multiformis ipsius z , quæ, pro quovis
 valore ipsius z , tot exhibet valores quot numerus n continet unitates;
 si Z definiatur per hanc æquationem $Z^n - PZ^{n-1} +$
 $QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \&c. = 0$.

Ubi quidem notandum est n esse oportere numerum inte-
 grum; atque perpetuo, ut dijudicari possit quam multiformis sit
 Functio Z ipsius z , æquatio, per quam Z definitur, reduci
 debet ad rationalitatem; quo facto exponens maximæ potesta-
 tis ipsius Z indicabit quæsitum valorum numerum cuique ipsius
 z valori respondentium. Deinde quoque tenendum est litte-
 ras $P, Q, R, S, \&c.$ denotare debere Functiones uniformes
 ipsius z : si enim aliqua earum jam esset Functio multiformis,
 tum Functio Z multo plures præbitura esset valores unicuique
 valori ipsius z respondentes, quam quidem numerus dimensio-
 num ipsius Z indicaret. Semper autem, si qui valores ipsius
 fuerint imaginarii, eorum numerus erit par; unde intelligitur,
 si fuerit n numerus impar, perpetuo unum ad minimum valorem
 ipsius Z fore realem: contra autem fieri posse, si numerus n
 fuerit par, ut nullus prorsus valor ipsius Z sit realis.

15. Si Z ejusmodi fuerit Functio multiformis ipsius z ut perpetuo
 nonnisi unicum valorem exhibeat realem; tum Z Functionem uni-
 formem ipsius z mentietur, ac plerumque loco Functionis uniformis
 usurpari poterit.

Euleri *Introduct. in Anal. infin. parv.*

B

Ejus-

LIB. I. Eiusmodi Functiones erunt $\sqrt[n]{P}$, $\sqrt[n]{P}$, $\sqrt[n]{P}$, &c. quippe quæ perpetuo nonnisi unicum valorem realem præbent, reliquis omnibus existentibus imaginariis, dummodo P fuerit Functio uni-

formis ipsius z . Hanc ob rem huiusmodi expressio $P^{\frac{m}{n}}$, quoties n fuerit numerus impar, Functionibus uniformibus annumerari poterit; siue m fuerit numerus par siue impar. Quod si

autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem, vel duos; ex quo eiusmodi expressiones $P^{\frac{m}{n}}$, existente n numero pari, eodem jure Functionibus bifor-
mibus accenseri poterunt: siquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. Si fuerit y Functio quaecunque ipsius z ; tum vicissim z erit Functio ipsius y .

Cum enim y sit Functio ipsius z , siue uniformis siue multiformis; dabitur æquatio, qua y per z & constantes quantitates definitur. Ex eadem vero æquatione vicissim z per y & constantes definiri poterit; unde quoniam y est quantitas variabilis, z æquabitur expressioni ex y & constantibus compositæ, eritque adeo Functio ipsius y . Hinc quoque patebit quam multiformis Functio futura sit z ipsius y : fierique potest ut, etiamsi y fuerit Functio uniformis ipsius z , tamen z futura sit Functio multiformis ipsius y . Sic si y ex hac æquatione per z definiatur; $y^3 = ayz - bzz$; erit utique y Functio trifor-
mis ipsius z , contra vero z Functio tantum bifor-
mis ipsius y .

17. Si fuerint y & x Functiones ipsius z , erit quoque y Functio ipsius x , & vicissim x Functio ipsius y .

Cum enim sit y Functio ipsius z , erit quoque Functio ipsius y : similique modo erit etiam z Functio ipsius x . Hanc ob rem Functio ipsius y æqualis erit Functioni ipsius x ; ex qua æquatione & y per x & viceversa x per y definiri poterit: quocirca manifestum est esse y Functionem ipsius x , atque x Functionem
ipsius

ipſius y . Sæpiſſime quidem has Functiones explicite exhibere CAP. I.
non licet ob defectum Algebræ; interim tamen nihilo minus,
quaſi omnes æquationes reſolvi poſſent, hæc Functionum reci-
procatio perſpicitur. Ceterum per methodum in Algebra tra-
ditam, ex datis binis æquationibus, quarum altera continet y
& z , altera vero x & z , per eliminationem quantitatis z for-
mabitur una æquatio relationem inter x & y exprimens.

18 *Species denique quadam Functionum peculiareſ ſunt notandæ;
ſic Functio par ipſius z eſt, quæ eundem dat valorem, ſive pro z
ponatur valor determinatus $+k$ ſive $-k$.*

Huiusmodi ergo Functio par ipſius z erit $z z$; ſive enim po-
natur $z = +k$, ſive $z = -k$, eundem valorem præbebit
expreſſio $z z$, nempe $z z = +k k$. Simili modo Functiones
pares ipſius z erunt hæ ipſius z poteſtates z^4 , z^6 , z^8 , & ge-
neratim omnis poteſtas z^m , ſi fuerit m numerus par, ſive af-

firmativus ſive negativus. Quin etiam cum $z^{\frac{m}{n}}$ mentiatur
Functionem ipſius z uniformem, ſi n ſit numerus impar, per-

ſpicuum eſt $z^{\frac{m}{n}}$ fore Functionem parem ipſius z , ſi m fuerit
numerus par, n vero numerus impar. Hanc ob rem, expreſ-
ſiones ex huiusmodi poteſtatibus utcunque compoſitæ præbebunt
Functiones pares ipſius z ; ſic Z erit Functio par ipſius z , ſi
fuerit $Z = a + b z^2 + c z^4 + d z^6 + \&c.$ item ſi fuerit Z
 $= \frac{a + b z^2 + c z^4 + d z^6 + \&c.}{a + e z^2 + \gamma z^4 + \delta z^6 + \&c.}$; Similique modo exponen-
tes fractos ipſius z introducendo, erit Z Functio par ipſius z ſi
fuerit $Z = a + b z^{\frac{2}{3}} + c z^{\frac{4}{3}} + d z^{\frac{6}{3}} + \&c.$ vel $Z = a +$
 $b z^{-\frac{2}{3}} + c z^{-\frac{4}{3}} + d z^{-\frac{6}{3}} + \&c.$ vel $Z =$
 $\frac{a + b z^{\frac{2}{7}} + c z^{-\frac{4}{7}} + d z^{\frac{8}{7}}}{a + e z^{\frac{2}{7}} + \gamma z^{-\frac{4}{7}} + \delta z^{\frac{8}{7}}}$. Cuiusmodi expreſſiones, cum om-
nes ſint Functiones uniformes ipſius z , appellari poterunt Fun-
ctiones pares uniformes ipſius z .

LIB. I. 19. *Functio multiformis par ipsius z est, quæ etiam si pro quovis valore ipsius z plures exhibeat valores determinatos, tamen eosdem valores præbet, siue ponatur $z = +k$, siue $z = -k$.*

Sit Z ejusmodi Functio multiformis par ipsius z ; quoniam natura Functionis multiformis exprimitur per æquationem inter Z & z , in qua Z tot habeat dimensiones, quot varios valores complectatur; manifestum est Z fore Functionem multiformem parem, si in æquatione naturam ipsius Z exprimente quantitas variabilis z ubique pares habeat dimensiones. Sic, si fuerit $Z^2 = azZ^4 + bz^2$, erit Z Functio biformis par ipsius z ; sin autem sit $Z^3 = az^2Z^2 + bz^4Z - cz^3 = 0$, erit Z Functio triformis par ipsius z ; atque generatim, si P, Q, R, S &c. denotent Functiones uniformes pares ipsius z , erit Z Functio biformis par ipsius z si sit $Z^2 = PZ + Q = 0$. At Z erit Functio triformis par ipsius z si sit $Z^3 = PZ^2 + QZ - R = 0$, & ita porro.

20 *Functio ergo, siue uniformis siue multiformis, par ipsius z erit ejusmodi expressio ex quantitate variabili z & constantibus constata in qua ubique numerus dimensionum ipsius z sit par.*

Hujusmodi ergo Functiones, præter uniformes quarum exempla ante sunt allata, erunt hæ expressiones $a + \sqrt{bb - zz}$; $az + \sqrt[3]{(a^3z^4 - bz^2)}$ item $az^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri posse, ut dicantur esse Functiones ipsius zz .

Si enim ponatur $y = zz$, fueritque Z Functio quæcunque ipsius y ; restituto ubique zz loco y , erit Z ejusmodi Functio ipsius z , in qua z ubique parem habeat dimensionum numerum. Excipiendi tamen sunt ii casus, quibus in expressione ipsius Z occurrunt \sqrt{y} : ac hujusmodi aliæ formæ, quæ, facto $y = zz$ signa radicalia amittunt. Quamvis enim sit $y + \sqrt{ay}$ Functio ipsius y , tamen posito $y = zz$, eadem expressio non erit Functio par ipsius z ; cum fiat $y + \sqrt{ay} = zz + z\sqrt{a}$. Exclusis ergo his casibus, definitio ultima Functionum

num parium erit bona, atque ad ejusmodi Functiones forman- CAP. I.
das idonea.

21. *Functio impar ipsius z est ejusmodi Functio, cujus valor, si loco z ponatur $-z$, fit quoque negativus.*

Hujusmodi Functiones ergo impares erunt omnes potestates ipsius z , quarum exponentes sunt numeri impares, ut z^1 , z^3 , z^5 , z^7 ; &c. item z^{-1} , z^{-3} , z^{-5} ; &c. tum vero

etiam $z^{\frac{m}{n}}$ erit Functio impar, si ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expressio ex hujusmodi potestatibus composita erit Functio impar ipsius z ; cujusmodi sunt, $az + bz^3 : az + az^{-1}$; item $z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{5}{7}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perspicietur.

22. *Si Functio par ipsius z multiplicetur per z vel per ejusdem Functionem imparem quamcunque, productum erit Functio impar ipsius z .*

Sit P Functio par ipsius z , quæ idcirco manet eadem si loco z ponatur $-z$; quod si ergo in producto Pz , ponatur $-z$ loco z , prodibit $-Pz$; unde Pz erit Functio impar ipsius z . Sit jam P Functio par ipsius z , & Q functio impar ipsius z ; atque ex Definitione patet si loco z ponatur $-z$, valorem ipsius P manere eundem, at valorem ipsius Q abire in sui negativum $-Q$; quare productum PQ , posito $-z$ loco z , abibit in $-PQ$, hoc est in sui negativum; eritque ideo PQ Functio impar ipsius z . Sic cum sit $a + \sqrt{aa + zz}$ functio par, & z^3 Functio impar ipsius z , erit productum $az^3 + z^3 \sqrt{aa + zz}$ Functio impar ipsius z ; similique modo $z \times \frac{a + bzz}{a + 6zz} = \frac{az + bz^3}{a + 6zz}$ Functio impar ipsius z . Ex his vero etiam intelligitur, si duarum Functionum P & Q , quarum altera P est par, altera Q , impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipsius z .

LIB. I. 23. Si Functio impar per Functionem imparem vel multiplicetur, vel dividatur; quod resultat erit Functio par.

Sint Q & S Functiones impares ipsius z ; ita ut, posito $-z$ loco z , Q abeat in $-Q$, & S in $-S$; atque perspicuum est tam productum $Q S$, quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro z ponatur $-z$; ideoque esse utrumque Functionem paralem ipsius z . Manifestum itaque porro est cujusque Functionis imparis quadratum esse Functionem paralem; cubum vero Functionem imparem; biquadratum iterum Functionem paralem, atque ita porro.

24. Si fuerit y Functio impar ipsius z ; erit vicissim z Functio impar ipsius y .

Cum enim sit y Functio impar ipsius z ; si ponatur $-z$ loco z , abibit y in $-y$. Quod si ergo z per y definiatur, necesse est ut posito $-y$ loco y , quoque z abeat in $-z$; eritque ideo z Functio impar ipsius y . Sic quia, posito $y = z^3$, est y Functio impar ipsius z ; erit quoque, ex æquatione $z^3 = y$ seu $z = y^{\frac{1}{3}}$, z Functio impar ipsius y . Et quia si fuerit $y = az + bz^3$, est y Functio impar ipsius z , erit vicissim, ex æquatione $bz^3 + az = y$, valor ipsius z per y expressus Functio impar ipsius y .

25. Si natura Functionis y per ejusmodi æquationem definiatur, in cujus singulis terminis numerus dimensionum, quas y & z occupant conjunctim, sit vel par ubique, vel impar; tum erit y Functio impar ipsius z .

Quod si enim in ejusmodi æquatione ubique loco z scribatur $-z$; simulque $-y$ loco y ; omnes æquationis termini vel manebunt iidem, vel fient negativi, utroque vero casu æquatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur; & hanc ob rem, si loco z ponatur $-z$, valor ipsius y abibit in $-y$, seu y erit Functio impar ipsius z . Sic si fuerit vel $yy = ayz + bzz + c$; vel $y^3 + ayyz = byzz + cy + dz$, ex utraque æquatione y erit Functio impar ipsius z .

26. Si

26. Si Z fuerit Functio ipsius z , & Y Functio ipsius y , at- CAP. I.
que Y eodem modo definitur per variabilem y & constantes, quo
 Z definitur per variabilem z & constantes; tum hæ Functioes
 Y et Z vocantur Functioes similes ipsarum y & z .

Si scilicet fuerit $Z = a + bz + cz^2$, & $Y = a + by + cy^2$, erunt
 Z & Y Functioes similes ipsarum z & y , similique modo in mul-
tiformibus, si fuerit $Z^3 = azzZ + b$ & $Y^3 = ayyY + b$; erunt
 Z & Y Functioes similes ipsarum z & y . Hinc sequitur, si
 Y & Z fuerint huiusmodi Functioes similes ipsarum y & z ; tum
si loco z scribatur y , Functioem Z abituram esse in Functioem
 Y . Solet hæc similitudo etiam hoc modo verbis exprimi, ut
 Y talis Functio dicatur ipsius y , qualis Functio sit Z ipsius z .
Hæ locutiones perinde occurrent, siue quantitates variables z
& y a se invicem pendeant, siue secus: sic qualis Functio est
 $ay + by^3$ ipsius y , talis Functio erit $a(y + n) + b(y + n)^3$
ipsius $y + n$, existente scilicet $z = y + n$: tum qualis Functio
est $\frac{a + bz + cz^2}{a + cz + yzz}$ ipsius z , talis Functio erit $\frac{azz + bz + c}{azz + cz + y}$
ipsius $\frac{1}{z}$; posito $y = \frac{1}{z}$. Atque ex his luculenter perspi-
citur ratio similitudinis Functioem, cujus per universam Analy-
sin sublimiorem uberrimus est usus. Ceterum hæc in genere
de natura Functioem unius variabilis sufficere possunt; cum
plenior expositio in applicatione sequente tradatur.

C A P U T I I.

De transformatione Functioem.

27. **F**unctioes in alias formas transmutantur, vel loco quan-
titatis variabilis aliam introducendo, vel eandem quan-
titem variabilem retinendo.

Quod si eadem quantitas variabilis servatur, Functio pro-
prie mutari non potest. Sed omnis transformatio consistit in
alio

LIB. I.

alio modo eandem Functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse. Hujusmodi transformationes sunt, si loco hujus Functionis $2 - 3z + zz$ ponatur $(1 - z)(2 - z)$, vel $(a + z)^3$ loco $a^3 + 3aa z + 3a z z + z^3$, vel $\frac{a}{a - z} + \frac{a}{a + z}$ loco $\frac{2aa}{aa - zz}$; vel $\sqrt{(1 + zz)} + z$ loco $\frac{1}{\sqrt{(1 + zz)} - z}$; quæ expressiones, etsi forma differunt, tamen revera congruunt. Sæpe numero autem harum plurium formarum idem significantium una aptior est ad propositum efficiendum quam reliquæ, & hanc ob rem formam commodissimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quæ quidem ad z datam teneat relationem, per substitutionem fieri dicitur; hocque modo ita uti convenit, ut Functio proposita succinctius & commodius exprimatur, uti si ista proposita fuerit ipsius z Functio, $a^4 - 4a^3 z + 6aa z z - 4a z^3 + z^4$; si loco $a - z$ ponatur y , prodibit ista multo simplicior ipsius y Functio y^4 : & si habeatur hæc Functio irrationalis $\sqrt{(aa + zz)}$ ipsius z , si ponatur $z = \frac{aa - yy}{2y}$, ista Functio per y expressa fiet rationalis $= \frac{aa + yy}{2y}$. Hunc autem transformationis modum in sequens

Caput differam, hoc Capite illum, qui sine substitutione procedit, expositurus.

28. *Functio integra ipsius z sæpe numero commode in suos factores resolvitur, sicque in productum transformatur.*

Quando Functio integra hoc pacto in factores resolvitur, ejus natura multo facilius perspicitur; casus enim statim innotescunt, quibus Functionis valor fit $= 0$. Sic hæc ipsius z Functio $6 - 7z + z^3$ transformatur in hoc productum $(1 - z)(2 - z)(3 + z)$ ex quo statim liquet Functionem propositam tribus casibus fieri $= 0$; scilicet si $z = 1$, & $z = 2$, & $z = -3$, quæ proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Istiusmodi Factores, in quibus variabiles z
nulla

nulla occurrit potestas, vocantur Factores simplices, ut distin- CAP. II.
guantur a Factoribus compositis, in quibus ipsius z inest qua-
dratum vel cubus, vel alia potestas altior. Erit ergo in genere
 $f + gz$ forma Factorum simplicium, $f + gz + hzz$ forma
Factorum duplicium; $f + gz + hzz + iz^3$ forma Factorum
triplicium, & ita porro. Perspicuum autem est Factorem dupli-
cem duos complecti valores simplices, Factorem triplicem tres
simplices, & ita porro. Hinc Functio ipsius z integra, in qua
exponens summæ potestatis ipsius z est $= n$, continebit n Fac-
tores simplices; ex quo simul, si qui Factores fuerint vel dupli-
ces vel triplices, &c. numerus Factorum cognoscetur.

29. *Factores simplices Functionis cujuscunque integra Z ipsius z reperiuntur, si Functio Z nihilo aqualis ponatur, atque ex hac æquatione omnes ipsius z radices investigentur: singula enim ipsius z radices dabunt totidem Factores simplices Functionis Z .*

Quod si enim ex æquatione $Z = 0$, fuerit quæpiam radix
 $z = f$, erit $z - f$ divisor, ac proinde Factor Functionis Z , sic
igitur investigandis omnibus radicibus æquationis $Z = 0$, quæ
sint $z = f$, $z = g$, $z = h$; &c., Functio Z resolvetur in
suos Factores simplices, atque transformabitur in productum Z
 $= (z - f)(z - g)(z - h)$ &c.: ubi quidem notandum
est si summæ potestatis ipsius z in Z non fuerit coefficientis $=$
 $+1$, tum productum $(z - f)(z - g)$ &c. insuper per illum
coefficientem multiplicari debere. Sic si fuerit $Z = Az^n$
 $+ Bz^{n-1} + Cz^{n-2} + \&c.$ erit $Z = A.(z - f).(z - g).$
 $(z - h).$ &c. At si fuerit $Z = A + Bz + Cz^2 + Dz^3 + Ez^4$
 $+ \&c.$ atque æquationis $Z = 0$ radices z repertæ sint; $f; g; h; i;$
&c. erit $Z = A(1 - \frac{z}{f})(1 - \frac{z}{g})(1 - \frac{z}{h}).$ &c. Ex his autem
vicissim intelligitur, si Functionis Z Factor fuerit $z - f$, seu
 $1 - \frac{z}{f}$; tum valorem Functionis in nihilum abire, si lo-
co z ponatur f . Facto enim $z = f$, unus Factor $z - f$, seu
 $1 - \frac{z}{f}$, Functionis Z , ideoque ipsa Functio Z evanescere debet.

Euleri *Introduct. in Anal. infin.*

C

30. *Facto-*

LIB. I. 30. *Factores simplices ergo erunt vel reales, vel imaginarii; & si Functio Z habeat Factores imaginarios eorum numerus semper erit par.*

Cum enim Factores simplices nascantur ex radicibus æquationis $Z = 0$, radices reales præbunt Factores reales, & imaginariæ imaginarios; in omni autem æquatione numerus radicum imaginariarum semper est par: quamobrem Functio Z, vel nullos habebit Factores imaginarios, vel duos, vel quatuor, vel sex, &c. Quod si Functio Z duos tantum habeat Factores imaginarios, eorum productum erit reale, ideoque præbebit Factorem duplicem realem. Sit enim $P =$ producto ex omnibus Factoribus realibus, erit productum duorum Factorum imaginariarum $= \frac{Z}{P}$; hincque reale. Simili modo si Functio Z habeat quatuor, vel sex, vel octo &c. Factores imaginarios; erit eorum productum semper reale: nempe æquale quoto, qui oritur, si Functio Z dividatur per productum omnium Factorum realium.

31. *Si fuerit Q productum reale ex quatuor Factoribus simplicibus imaginariis, tum idem hoc productum Q resolvi poterit in duos Factores duplices reales.*

Habebit enim Q ejusmodi formam $z^4 + Az^3 + Bz^2 + Cz + D$; quæ si negetur in duos Factores duplices reales resolvi posse, resolubilis erit statuenda in duos Factores duplices imaginarios; qui hujusmodi formam habebunt $zz - 2(p + q\sqrt{-1})z + r + s\sqrt{-1}$. & $zz - 2(p - q\sqrt{-1})z + r - s\sqrt{-1}$; aliæ enim formæ imaginariæ concipi non possunt, quarum productum fiat reale, nempe $= z^4 + Az^3 + Bz^2 + Cz + D$. Ex his autem Factoribus imaginariis duplicibus sequentes emergent quatuor Factores simplices imaginarii ipsius Q,

- I. $z - (p + q\sqrt{-1}) + \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
- II. $z - (p + q\sqrt{-1}) - \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
- III. $z - (p - q\sqrt{-1}) + \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$
- IV. $z - (p - q\sqrt{-1}) - \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

Horum

Horum Factorum multiplicentur primus ac tertius in se invicem, CAP. II.
 posito brevitatis gratia, $t = pp - qq - r$, & $u = 2pq - s$; erit-
 que horum Factorum productum $= z^2 - (2p - \sqrt{2t + 2\sqrt{(tt + uu)}})z$
 $+ pp + qq - p\sqrt{2t + 2\sqrt{(tt + uu)}} + \sqrt{(tt + uu)}$; quod uti-
 $+ q\sqrt{-2t + 2\sqrt{(tt + uu)}}$
 que est reale. Simili autem modo productum ex Factoribus se-
 cundo & quarto erit reale nempe $= z^2 - (2p + \sqrt{2t + 2\sqrt{(tt + uu)}})z$
 $+ pp + qq + p\sqrt{2t + 2\sqrt{(tt + uu)}} + \sqrt{(tt + uu)}$.
 $+ q\sqrt{-2t + 2\sqrt{(tt + uu)}}$

Quocirca productum propositum Q , quod in duos Factores
 duplices reales resolvi posse negabatur, nihilo minus actu in duos
 Factores duplices reales est resolutum.

32. Si Functio integra Z ipsius z quotcunque habeat Factores sim-
 plices imaginarios, bini semper ita conjungi possunt, ut eorum pro-
 ductum fiat reale.

Quoniam numerus radicum imaginariarum semper est par,
 sit is $= 2n$; ac primo quidem patet productum harum radicum
 imaginariarum omnium esse reale. Quod si ergo duæ tantum
 radices imaginariæ habeantur, erit earum productum utique rea-
 le; sin autem quatuor habeantur Factores imaginarii, tum, uti
 vidimus, eorum productum resolvi potest in duos Factores du-
 plices reales formæ $fxz + gz + h$. Quanquam autem eundem
 demonstrandi modum ad altiores potestates extendere non licet,
 tamen extra dubium videtur esse positum eandem proprietatem
 in quotcunque Factores imaginarios competere; ita ut semper
 loco $2n$ Factorum simplicium imaginariarum induci queant n Fac-
 tores duplices reales. Hinc omnis Functio integra ipsius z re-
 solvi poterit in Factores reales vel simplices vel duplices. Quod
 quamvis non summo rigore sit demonstratum, tamen ejus veri-
 tas in sequentibus magis corroborabitur, ubi hujus generis Func-
 tiones $a + bz^n$; $a + bz^n + cz^{2n}$; $a + bz^n + cz^{2n} + dz^{3n}$ &c.
 actu in istiusmodi Factores duplices reales resolventur.

LIB. I. 33. Si Functio integra Z , posito $z = a$ induat valorem A , & posito $z = b$, induat valorem B ; tum, loco z valores medios inter a & b ponendo, Functio Z quosvis valores medios inter A & B accipere potest.

Cum enim Z sit Functio uniformis ipsius z , quicunque valor realis ipsi z tribuatur, Functio quoque Z hinc valorem realem obtinebit. Cum igitur Z , priore casu $z = a$, nanciscatur valorem A ; posteriore casu $z = b$, autem, valorem B ; ab A ad B transire non poterit, nisi per omnes valores medios transiendo. Quod si ergo æquatio $Z - A = 0$ habeat radicem realem, simulque $Z - B = 0$ radicem realem suppeditet; tum æquatio quoque $Z - C = 0$ radicem habebit realem; si quidem C intra valores A & B contineatur. Hinc si expressiones $Z - A$ & $Z - B$ habeant Factorem simplicem realem, tum expressio quæcunque $Z - C$ Factorem simplicem habebit realem, dummodo C intra valores A & B contineatur.

34. Si in Functione integra Z exponens maxima ipsius z potestatis fuerit numerus impar $2n + 1$, tum ea Functio Z unicum ad minimum habebit Factorem simplicem realem.

Habebit scilicet Z hujusmodi formam $z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} + \&c.$ in qua si ponatur $z = \infty$, quia valores singulorum terminorum præ primo evanescent, fiet $Z = (\infty)^{2n+1} = \infty$; ideoque $Z - \infty$ Factorem simplicem habebit realem nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet $Z = (-\infty)^{2n+1} = -\infty$, ideoque habebit $Z + \infty$ Factorem simplicem realem $z + \infty$. Cum igitur tam $Z - \infty$, quam $Z + \infty$ habeat Factorem simplicem realem; sequitur etiam $Z + C$ habiturum esse Factorem simplicem realem, siquidem C contineatur intra limites $+\infty$ & $-\infty$; hoc est si C fuerit numerus realis quicunque, sive affirmativus, sive negativus. Hanc ob rem, facto $C = 0$, habebit quoque ipsa Functio Z Factorem simplicem realem $z - c$; atque quantitas c contine-

continebitur intra limites $+\infty$ & $-\infty$, eritque idcirco vel CAP. II.
quantitas affirmativa, vel negativa, vel nihil.

35. *Functio igitur integra Z, in qua exponens maxima potestatis ipsius z est numerus impar, vel unum habebit Factorem simplicem realem, vel tres, vel quinque, vel septem &c.*

Cum enim demonstratum sit Functionem Z certo unum habere Factorem simplicem realem $z - c$; ponamus eam prætere a unum Factorem habere $z - d$, atque dividatur Functio Z, in qua maxima ipsius z potestas sit z^{2n+1} , per $(z - c) \cdot (z - d)$, erit quoti maxima potestas $= z^{2n-1}$, cujus exponens, cum sit numerus impar, indicat denuo ipsius Z dari Factorem simplicem realem. Si ergo Z plures uno habeat Factores simplices reales, habebit vel tres, vel (quoniam eodem modo progredi licet) quinque, vel septem, &c. Erit scilicet numerus Factorum simplicium realium impar, & quia numerus omnium Factorum simplicium est $= 2n + 1$, erit numerus Factorum imaginariorum par.

36. *Functio integra Z, in qua exponens maxima potestatis ipsius z est numerus par $2n$, vel duos habebit Factores simplices reales vel quatuor, vel sex, vel &c.*

Ponamus ipsius Z constare Factorum simplicium realium numerum imparem $2m + 1$; si ergo per horum omnium productum dividatur Functio Z, quoti maxima potestas erit $= z^{2n - 2m - 1}$, ejusque ideo exponens numerus impar; habebit ergo Functio Z præterea unum certo Factorem simplicem realem, ex quo numerus omnium Factorum simplicium realium ad minimum erit $= 2m + 2$, ideoque par; ac numerus Factorum imaginariorum pariter par. Omnis ergo Functionis integræ Factores simplices imaginarii sunt numero pares; quemadmodum quidem jam ante statuimus.

37. *Si in Functione integra Z exponens maxima potestatis ipsius z fuerit numerus par, atque terminus absolutus, seu constans, signo — affectus, tum Functio Z ad minimum duos habet Factores simplices reales.*

LIB. I. Functio ergo Z , de qua hic sermo est; hujusmodi formam
 habebit $z^{2n} \pm az^{2n-1} \pm bz^{2n-2} \pm \dots \pm$
 $\pm z - A$. Si jam ponatur $z = \infty$, fiet, uti supra vidimus,
 $Z = \infty$; atque, si ponatur $z = 0$, fiet $Z = -A$. Habebit
 ergo $Z - \infty$ Factorem realem $z - \infty$, & $Z + A$ Factorem
 $z - 0$: unde cum 0 contineatur intra limites $-\infty$ & $+A$,
 sequitur $Z + 0$ habere Factorem simplicem realem $z - c$, ita
 ut c contineatur intra limites 0 & ∞ . Deinde, cum posito
 $z = -\infty$, fiat $Z = \infty$, ideoque $Z - \infty$ Factorem habe-
 at $z + \infty$, & $Z + A$ Factorem $z + 0$, sequitur quoque Z
 $+ 0$ Factorem simplicem realem habere $z + d$; ita ut d intra
 limites 0 & ∞ contineatur; unde constat propositum. Ex
 his igitur perspicitur si Z talis fuerit Functio, qualis hic est de-
 scripta, æquationem $Z = 0$, duas ad minimum habere debere
 radices reales, alteram affirmativam, alteram negativam. Sic
 æquatio hæc $z^4 + az^3 + bz^2 + \gamma z - aa = 0$, duas habet
 radices reales, alteram affirmativam, alteram negativam.

38. Si in Functione fracta, quantitas variabilis z tot vel plures
 habeat dimensiones in numeratore, quam in denominatore; tum ista
 Functio resolvi poterit in duas partes, quarum altera est Functio
 integra, altera fracta; in cujus numeratore quantitas variabilis z
 pauciores habeat dimensiones quam in denominatore.

Si enim exponens maximæ potestatis ipsius z minor fuerit
 in denominatore quam in numeratore; tum numerator per de-
 nominatorem dividatur more solito, donec in quoto ad expo-
 nentes negativos ipsius z perveniatur; hoc ergo loco abrupta
 divisionis operatione quotus constabit ex parte integra atque
 fractione, in cujus numeratore minor erit dimensionum nume-
 rus ipsius z quam in denominatore; hic autem quotus Functioni
 propositæ est æqualis. Sic, si hæc proposita fuerit Functio
 fracta $\frac{1+z^4}{1+zz}$, ea per divisionem ita resolvetur.

$zz + 1$)

$$2z+1) z^4+1 (2z-1 + \frac{2}{1+2z}$$

$$\begin{array}{r} z^4+2z \\ -2z+1 \\ \hline 2z-1 \\ +2 \end{array}$$

eritque $\frac{1+z^4}{1+2z} = 2z-1 + \frac{2}{1+2z}$. Hujusmodi Functiones fractæ, in quibus quantitas variabilis z tot vel plures habet dimensiones in numeratore quam in denominatore, ad similitudinem Arithmeticæ vocari possunt fractiones spuria, vel Functiones fractæ spuria, quo distinguantur a Functionibus fractis genuinis, in quarum numeratore quantitas variabilis z pauciores habet dimensiones quam in denominatore. Functio itaque fracta spuria resolvi poterit in Functionem integram, & Functionem fractam genuinam; hæcque resolutio per vulgarem divisionis operationem absolvetur.

39. Si denominator Functionis fractæ duos habeat Factores inter se primos; tum ipsa Functio fracta resolvetur in duas fractiones, quarum denominatores sint illis binis Factoribus respectively æquales.

Quoniam hæc resolutio ad Functiones fractas spurias æque pertinet atque ad genuinas, tamen eam ad genuinas potissimum accomodabimus. Resoluto autem denominatore hujusmodi Functionis fractæ in duos Factores inter se primos, ipsa Functio resolvetur in duas alias Functiones fractas genuinas, quarum denominatores sint illis binis Factoribus respectively æquales; hæcque resolutio, si quidem fractiones sint genuinae, unico modo fieri potest; cujus rei veritas ex exemplo clarius quam per ratiocinium perspicietur. Sit ergo proposita hæc Functio

fracta $\frac{1-2z+3zz-4z^3}{1+4z^4}$, cujus denominator $1+4z^4$ cum sit

æqualis huic producto $(1+2z+2zz)(1-2z+2zz)$, fractio proposita in duas fractiones resolvetur, quarum alterius denominator erit $1+2z+2zz$, alterius $1-2z+2zz$: ad quas inveniendas, quia sunt genuinae, statuantur numeratores illius $= a+6z$, hujus $= \gamma+\delta z$, eritque per hypothelin

1 —

LIB. I. $\frac{1 - 2z + 3z^2 - 4z^3}{1 + 4z^4} = \frac{\alpha + 6z}{1 + 2z + 2z^2} + \frac{\gamma + \delta z}{1 - 2z + 2z^2}$: addantur actu hæ duæ fractiones, eritque summa

Numerator	Denominator
$ \begin{aligned} &+ \alpha - 2\alpha z + 2\alpha z^2 \\ &\quad + 6z - 26z^2 + 26z^3 \\ &+ \gamma + 2\gamma z + 2\gamma z^2 \\ &\quad + \delta z + 2\delta z^2 + 2\delta z^3 \end{aligned} $	$1 + 4z^4$

Cum ergo denominator æqualis sit denominatori fractionis propositæ, numeratores quoque æquales reddi debent: quod, ob tot litteras incognitas $\alpha, 6, \gamma, \delta$, quot sunt termini æquales efficiendi, utique fieri, idque unico modo poterit: nanciscimur scilicet has quatuor æquationes

$$\begin{array}{ll}
 \text{I.} & \alpha + \gamma = 1 \\
 \text{II.} & -2\alpha + 6 + 2\gamma + \delta = -2 \\
 \text{III.} & 2\alpha - 26 + 2\gamma + 2\delta = 3 \\
 \text{IV.} & 26 + 2\delta = -4.
 \end{array}$$

Hinc ob $\alpha + \gamma = 1$, & $6 + \delta = -2$; æquationes II. & III. dabunt $\alpha - \gamma = 0$ & $\delta - 6 = \frac{1}{2}$; ex quibus fit

$\alpha = \frac{1}{2}$; $\gamma = \frac{1}{2}$; $6 = -\frac{5}{4}$; $\delta = -\frac{3}{4}$; ideoque fractio proposita $\frac{1 - 2z + 3z^2 - 4z^3}{1 + 4z^4}$, transformatur in has duas

$\frac{\frac{1}{2} - \frac{5}{4}z}{1 + 2z + 2z^2} + \frac{\frac{1}{2} - \frac{3}{4}z}{1 - 2z + 2z^2}$. Simili autem modo facile perspicietur resolutionem semper succedere debere: quoniam semper tot litteræ incognitæ introducuntur, quot opus est ad numeratorem propositum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc resolutionem succedere non posse, nisi isti denominatoris Factores fuerint inter se primi.

40. Functio igitur fracta $\frac{M}{N}$ in tot fractiones simplices forma $\frac{A}{p - qz}$ resolvi poterit, quot Factores simplices habet denominator N inter se inæquales.

Repræ-

Repræsentat hic fractio $\frac{M}{N}$ Functionem quamcunque fractam genuinam, ita ut M & N sint Functiones integræ ipsius z , atque summa potestas ipsius z in M minor sit quam in N . Quod si ergo denominator N in suos Factores simplices resolvatur, hi- que inter se fuerint inæquales, expressio $\frac{M}{N}$ in tot fractiones resolvetur, quot Factores simplices in denominatore N continentur; propterea quod quisque Factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit Factor ipsius N , is erit denominator fractionis cujusdam partialis, & cum in numeratore hujus fractionis numerus dimensionum ipsius z minor esse debeat quam in denominatore $p - qz$, numerator necessario erit quantitas constans. Hinc ex unoquoque Factore simplici $p - qz$ denominatoris N nascetur fractio simplex $\frac{A}{p - qz}$; ita ut summa omnium harum fractionum sit æqualis fractioni propositæ $\frac{M}{N}$.

E X E M P L U M.

Sit, exempli causa, proposita hæc Functio fracta $\frac{1 + zz}{z - z^3}$; quia Factores simplices denominatoris sunt z , $1 - z$, & $1 + z$, ista Functio resolvetur in has tres fractiones simplices $\frac{A}{z} + \frac{B}{1 - z} + \frac{C}{1 + z} = \frac{1 + zz}{z - z^3}$; ubi numeratores constantes A , B , & C definire oportet. Reducantur hæ fractiones ad communem denominatorem, qui erit $z - z^3$; atque numeratorum summa æquari debebit ipsi $1 + zz$, unde ista æquatio oritur:

$$\begin{aligned} A + Bz - Azz &= 1 + zz = 1 + 0z + zz. \\ + Cz + Bzz & \\ - Czz & \end{aligned}$$

Euleri *Introduct. in Anal. infra. parv.*

D

quæ

LIB. I. quæ totidem comparationes præbet, quot sunt litteræ incognitæ A, B, C ; erit scilicet,

$$\text{I}^{\circ}. A = 1.$$

$$\text{II}^{\circ}. B + C = 0.$$

$$\text{III}^{\circ}. -A + B - C = 1:$$

Hinc fit $B - C = 2$; & porro $A = 1$; $B = 1$ & $C =$

— 1. Functio ergo proposita $\frac{1+z}{z-1}$ resolvitur in hanc for-

mam $\frac{1}{z} + \frac{1}{1-z} - \frac{1}{1+z}$. Simili autem modo intelligitur,

quotcunque habuerit denominator N Factores simplices inter se inæquales, semper fractionem $\frac{M}{N}$ in totidem fractiones simpli-

ces resolvi. Sin autem aliquot Factores fuerint æquales inter se, tum alio modo post-explicando resolutio institui debet.

41. Cum igitur quilibet Factor simplex denominatoris N suppediet fractionem simplicem pro resolutione Functionis propositæ $\frac{M}{N}$; ostendendum est quomodo ex Factore simplice denominatoris N cognito, fractio simplex respondens reperiatur.

Sit $p - qz$ Factor simplex ipsius N , ita ut sit $N = (p - qz)S$; atque S Functio integra ipsius z ; ponatur fractio ex Factore $p - qz$ orta $= \frac{A}{p - qz}$, & sit fractio ex altero Factore denominatoris S oriunda $= \frac{P}{S}$, ita ut, secundum §. 39., futurum

fit $\frac{M}{N} = \frac{A}{p - qz} + \frac{P}{S} = \frac{M}{(p - qz)S}$; hinc erit $\frac{P}{S} = \frac{M - AS}{(p - qz)S}$; quæ fractiones cum congruere debeant, necesse

est ut $M - AS$ sit divisibile per $p - qz$; quoniam Functio integra P ipsi quoto æquatur. Quando vero $p - qz$ Divisor

existit ipsius $M - AS$, hæc expressio posito $z = \frac{p}{q}$ evanes-

cit. Ponatur ergo ubique loco z hic valor constans $\frac{p}{q}$ in M

&

& S , erit $M - AS = 0$, ex quo fiet $A = \frac{M}{S}$; hocque ergo C A P. II.
 modo reperitur numerator A fractionis quæsitæ $\frac{A}{p - qz}$; atque si
 ex singulis denominatoris N Factoribus simplicibus, dummodo
 sint inter se inæquales, hujusmodi fractiones simplices formen-
 tur, harum fractionum simplicium omnium summa erit æqualis
 Functioni propositæ $\frac{M}{N}$.

E X E M P L U M.

Sic, si in Exemplo præcedente $\frac{1 + zz}{z - z^3}$, ubi est $M = 1 + zz$,
 & $N = z - z^3$, sumatur z pro Factore simplice, erit $S =$
 $1 - zz$, atque fractionis simplicis $\frac{A}{z}$ hinc ortæ erit numerator
 $A = \frac{1 + zz}{1 - zz} = 1$ posito $z = 0$, quem valorem z obtinet si
 ipse hic Factor simplex z nihilo æqualis ponatur. Simili modo
 si pro denominatoris Factore sumatur $1 - z$, ut sit $S = z + zz$
 erit $A = \frac{1 + zz}{z + zz}$, facto $1 - z = 0$, unde erit $A = 1$, &
 ex Factore $1 - z$ nascitur fractio $\frac{1}{1 - z}$. Tertius denique Fa-
 ctor $1 + z$, ob $S = z - zz$, & $A = \frac{1 + zz}{z - zz}$, posito $1 + z$
 $= 0$, seu $z = -1$, dabit $A = -1$, & fractionem sim-
 plicem $= \frac{-1}{1 + z}$. Quare per hanc regulam reperitur $\frac{1 + zz}{z - z^3}$
 $= \frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$, ut ante.

42. *Functio fracta hujus formæ $\frac{P}{(p - qz)^n}$, cujus numerator*
 P non tantam ipsius z potestatem involvit quantam denominator
 $(p - qz)^n$, transmutari potest in hujusmodi fractiones partiales
D 2 A

LIB. I.
$$\frac{A}{(p - qz)^n} + \frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \dots + \frac{K}{p - qz};$$

quarum omnium numeratores sint quantitates constantes.

Quoniam maxima potestas ipsius z in P minor est quam z^n , erit z^{n-1} , ideoque P hujusmodi habebit formam:

$$\alpha + \epsilon z + \gamma z^2 + \delta z^3 + \dots + x z^{n-1}$$

existente terminorum numero $= n$, cui æquari debet numerator summæ omnium fractionum partialium, postquam singulæ ad eundem denominatorem $(p - qz)^n$ fuerint perductæ: qui numerator propterea erit $= A + B(p - qz) + C(p - qz)^2 + D(p - qz)^3 + \dots + K(p - qz)^{n-1}$. Hujus maxima ipsius z potestas est, ut ibi, z^{n-1} , atque tot habentur litteræ incognitæ A, B, C, \dots, K , (quarum numerus est $= n$,) quot sunt termini congruentes reddendi. Quamobrem litteræ constantes A, B, C , &c. ita definiri poterunt, ut fiat Functio fracta genuina $\frac{P}{(p - qz)^n} = \frac{A}{(p - qz)^n} + \frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \frac{D}{(p - qz)^{n-3}} + \dots + \frac{K}{p - qz}$. Ipsa autem horum numeratorum inventio mox facilis aperietur.

43. Si Functionis fracta $\frac{M}{N}$ denominator N Factorem habeat $(p - qz)^2$, sequenti modo fractiones partiales ex hoc Factore oriunda reperientur.

Cujusmodi fractiones partiales ex singulis Factoribus denominatoris simplicibus, qui alios sibi æquales non habeant, oriuntur, ante est ostensum: nunc igitur ponamus duos Factores inter se esse æquales, seu, iis conjunctis, denominatoris N Factorem esse $(p - qz)^2$. Ex hoc ergo Factore per §. præced. duæ nascentur fractiones partiales hæc $\frac{A}{(p - qz)^2} + \frac{B}{p - qz}$. Sit autem

tem

tem $N = (p - qz)^2 S$, eritque $\frac{M}{N} = \frac{M}{(p - qz)^2 S} = \frac{A}{(p - qz)^2} + \frac{B}{p - qz} + \frac{P}{S}$, denotante $\frac{P}{S}$ omnes fractiones simplices junctim sumptas ex denominatoris Factore S ortas. Hinc erit $\frac{P}{S} = \frac{M - AS - B(p - qz)S}{(p - qz)^2 S}$, & $P = \frac{M - AS - B(p - qz)S}{(p - qz)^2}$ Functioni integræ. Debet ergo $M - AS - B(p - qz)S$ divisibile esse per $(p - qz)^2$: sit primum divisibile per $p - qz$, atque tota expressio $M - AS - B(p - qz)S$ evanescet, posito $p - qz = 0$, seu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco z , eritque $M - AS = 0$, ideoque $A = \frac{M}{S}$; scilicet fractio $\frac{M}{S}$, si loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipsius A constantem. Hoc invento quantitas $M - AS - B(p - qz)S$ etiam per $(p - qz)^2$ divisibilis esse debet, seu $\frac{M - AS}{p - qz} - BS$ denuo per $p - qz$ divisibile esse debet. Posito ergo ubique $z = \frac{p}{q}$ erit $\frac{M - AS}{p - qz} = BS$, ideoque $B = \frac{M - AS}{(p - qz)S} = \frac{1}{p - qz} \left(\frac{M}{S} - A \right)$, ubi notandum est, cum $M - AS$ divisibile sit per $p - qz$, hanc divisionem prius institui debere, quam loco z substituat $\frac{p}{q}$. Vel ponatur $\frac{M - AS}{p - qz} = T$, eritque $B = \frac{T}{S}$ posito $z = \frac{p}{q}$; inventis ergo numeratoribus A & B , erunt fractiones partiales ex denominatoris N Factore $(p - qz)^2$ ortæ hæc $\frac{A}{(p - qz)^2} + \frac{B}{p - qz}$.

E X E M P L U M I.

Sit hæc proposita Functio fracta $\frac{1 - zz}{zz(1 + zz)}$ erit, ob denomi-
D 3 nomina-

LIB. I. nominatoris Factorem quadratum zz ; $S = 1 + zz$ & $M = 1 - zz$. Sint fractiones partiales ex zz ortæ $\frac{A}{zz} + \frac{B}{z}$, erit $A = \frac{M}{S} = \frac{1 - zz}{1 + zz}$, posito Factore $z = 0$; hincque $A = 1$. Tum erit $M - AS = -2zz$ quod divisum per Factorem simplicem z , dabit $T = -2z$, hincque $B = \frac{T}{S} = \frac{-2z}{1 + zz}$, posito $z = 0$; unde erit $B = 0$; atque ex Factore denominatoris zz orietur unica hæc fractio partialis $\frac{1}{zz}$.

EXEMPLUM II.

Sit hæc proposita Functio fracta $\frac{z^3}{(1-z)^2(1+z^4)}$, cujus, ob denominatoris Factorem quadratum $(1-z)^2$, fractiones partiales sint $\frac{A}{(1-z)^2} + \frac{B}{1-z}$. Erit ergo $M = z^3$ & $S = 1 + z^4$; ideoque $A = \frac{M}{S} = \frac{z^3}{1 + z^4}$, posito $1 - z = 0$, seu $z = 1$: unde fit $A = \frac{1}{2}$. Prodibit ergo $M - AS = z^3 - \frac{1}{2} - \frac{1}{2}z^4 = -\frac{1}{2} + z^3 - \frac{1}{2}z^4$, quod per $1 - z$ divisum dat $T = -\frac{1}{2} - \frac{1}{2}z - \frac{1}{2}zz + \frac{1}{2}z^3$; ideoque $B = \frac{T}{S} = \frac{-1 - z - zz + z^3}{2 + 2z^4}$, posito $z = 1$; ita ut fit $B = -\frac{1}{2}$; fractiones ergo partiales quæsitæ sunt $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

44. Si Functionis fractæ $\frac{M}{N}$ denominator N Factorem habeat $(p - qz)^3$ sequenti modo fractiones partiales ex hoc Factore oriunda $\frac{A}{(p - qz)^3} + \frac{B}{(p - qz)^2} + \frac{C}{p - qz}$ reperientur.

Ponatur

Ponatur $N = (p - qz)^3 S$, sitque fractio ex Factore S orta CAP. II.
 $= \frac{P}{S}$, erit $P = \frac{M - AS - B(p - qz)S - C(p - qz)^2 S}{(p - qz)^3}$
 $=$ Functioni integræ. Numerator ergo $M - AS - B(p - qz)S - C(p - qz)^2 S$ ante omnia divisibilis esse debet per $(p - qz)$; unde is, posito $p - qz = 0$, seu $z = \frac{p}{q}$, evanescere debet, eritque adeo $M - AS = 0$, ideoque $A = \frac{M}{S}$, posito $z = \frac{p}{q}$. Invento hoc pacto A erit $M - AS$ divisibile per $p - qz$ ponatur ergo $\frac{M - AS}{p - qz} = T$, atque $T - BS - C(p - qz)S$ adhuc per $(p - qz)^2$ erit divisibile; fiet ergo $= 0$, posito $p - qz = 0$; ex quo prodit $B = \frac{T}{S}$ posito $z = \frac{p}{q}$. Sic autem invento B erit $T - BS$ divisibile per $p - qz$. Hanc ob rem, posito $\frac{T - BS}{p - qz} = V$, superest ut $V - CS$ divisibile sit per $p - qz$; eritque ergo $V - CS = 0$, posito $p - qz = 0$, atque $C = \frac{V}{S}$, posito $z = \frac{p}{q}$. Inventis ergo hoc modo numeratoribus A, B, C , fractiones partiales ex denominatoris N Factore $(p - qz)^3$ ortæ erunt $\frac{A}{(p - qz)^3} + \frac{B}{(p - qz)^2} + \frac{C}{p - qz}$.

E X E M P L U M.

Sit proposita hæc fracta Functio $\frac{zz}{(1 - z)^3(1 + zz)}$, ex cuius denominatoris Factore cubico $(1 - z)^3$ oriantur hæ fractiones partiales: $\frac{A}{(1 - z)^3} + \frac{B}{(1 - z)^2} + \frac{C}{1 - z}$. Erit ergo $M = zz$ & $S = 1 + zz$; unde fit primum $A = \frac{zz}{1 + zz}$
 posito

L I E. I. posito $1 - z = 0$ seu $z = 1$; ex quo prodit $A = \frac{1}{2}$. Jam

ponatur $T = \frac{M - AS}{1 - z}$, erit $T = \frac{\frac{1}{2} z z - \frac{1}{2}}{1 - z} = -\frac{1}{2} -$

$\frac{1}{2} z$; unde oritur $B = \frac{-\frac{1}{2} - \frac{1}{2} z}{1 + z z}$, posito $z = 1$, ita ut sit

$B = -\frac{1}{2}$. Ponatur porro $V = \frac{T - BS}{1 - z} = \frac{T + \frac{1}{2} S}{1 - z}$; erit

$V = \frac{-\frac{1}{2} z + \frac{1}{2} z z}{1 - z} = -\frac{1}{2} z$; unde fit $C = \frac{V}{S} = \frac{-\frac{1}{2} z}{1 + z z}$

posito $z = 1$, ita ut sit $C = -\frac{1}{4}$. Quo circa fractiones
partiales ex denominatoris Factore $(1 - z)^3$ ortæ erunt

$$\frac{1}{2(1 - z)^3} - \frac{1}{2(1 - z)^2} - \frac{1}{4(1 - z)}.$$

45. Si Functionis fractæ $\frac{M}{N}$ denominator N Factorem habeat

$(p - qz)^n$; fractiones partiales hinc ortæ $\frac{A}{(p - qz)^n} +$

$\frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \dots + \frac{K}{p - qz}$ sequenti
modo invenientur.

Ponatur denominator $N = (p - qz)^n Z$, atque, ratiocinium ut ante instituendo, reperietur ut sequitur:

Primo $A = \frac{M}{Z}$, posito $z = \frac{p}{q}$. Ponatur $P = \frac{M - AZ}{p - qz}$

Secundo $B = \frac{P}{Z}$, posito $z = \frac{p}{q}$. Ponatur $Q = \frac{P - BZ}{p - qz}$

Tertio $C = \frac{Q}{Z}$, posito $z = \frac{p}{q}$. Ponatur $R = \frac{Q - CZ}{p - qz}$

Quarto $D = \frac{R}{Z}$, posito $z = \frac{p}{q}$. Ponatur $S = \frac{R - DZ}{p - qz}$

Quinto $E = \frac{S}{Z}$, posito $z = \frac{p}{q}$. &c.

Hoc ergo modo si definiantur singuli numeratores constantes

tes $A, B, C, D, &c.$ invenientur omnes fractiones partiales, CAP. II.
quæ ex denominatoris N Factore $(p - qz)^n$ nascuntur.

E X E M P L U M.

Sit proposita ista Functio fracta $\frac{1 + zz}{z^5 (1 + z^3)}$ ex cujus denominatoris Factore z^5 nascantur hæ fractiones partiales $\frac{A}{z^5} + \frac{B}{z^4} + \frac{C}{z^3} + \frac{D}{z^2} + \frac{E}{z}$. Ad quarum numeratores constantes inveniendos, erit $M = 1 + zz$ atque $Z = 1 + z^3$; & $\frac{p}{q} = 0$. Sequens ergo calculus ineatur.

Primum est $A = \frac{M}{Z} = \frac{1 + zz}{1 + z^3}$, posito $z = 0$; ergo $A = 1$.

Ponatur $P = \frac{M - AZ}{z} = \frac{zz - z^3}{z} = z - zz$. Eritque secundo $B = \frac{P}{Z} = \frac{z - zz}{1 + z^3}$, posito $z = 0$; ergo $B = 0$.

Ponatur $Q = \frac{P - BZ}{z} = \frac{z - zz}{z} = 1 - z$; eritque tertio $C = \frac{Q}{Z} = \frac{1 - z}{1 + z^3}$, posito $z = 0$: ergo $C = 1$.

Ponatur $R = \frac{Q - CZ}{z} = \frac{-z - z^3}{z} = -1 - zz$; erit quarto $D = \frac{R}{Z} = \frac{-1 - zz}{1 + z^3}$, posito $z = 0$; ex quo fit $D = -1$.

Ponatur $S = \frac{R - DZ}{z} = \frac{-zz + z^3}{z} = -z + zz$; erit quinto $E = \frac{S}{Z} = \frac{-z + zz}{1 + z^3}$, posito $z = 0$; unde fit $E = 0$.

Quo circa fractiones partiales quæsitæ erunt hæ:

$$\frac{1}{z^5} + \frac{0}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{0}{z}.$$

LIB. I. 46. Quaecunque ergo proposita fuerit Functio rationalis fracta $\frac{M}{N}$, ea sequenti modo in partes resolvetur, atque in formam simplicissimam transmutabitur.

Quærantur denominatoris N omnes Factores simplices sive reales sive imaginarii; quorum qui sibi pares non habeant, seorsim tractentur & ex unoquoque per §. 41, fractio partialis eruat. Quod si idem Factor simplex bis vel pluries occurrat, ii conjunctim sumantur atque ex eorum producto, quod erit potestas formæ $(p - qz)^n$, quærantur fractiones partiales convenientes per §. 45. Hocque modo cum ex singulis Factoribus simplicibus denominatoris erutæ fuerint fractiones partiales, tum harum omnium aggregatum æquabitur Functioni propositæ $\frac{M}{N}$, nisi fuerit spuria; si enim fuerit spuria, pars integra insuper extrahi atque ad istas fractiones partiales inventas adjici debet, quo prodeat valor Functionis $\frac{M}{N}$ in forma simplicissima expressus. Perinde autem est sive fractiones partiales ante extractionem partis integræ, sive post quærantur. Eadem enim ex singulis denominatoris N Factoribus prodeunt fractiones partiales, sive adhibeatur ipse numerator M , sive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM.

Quærat. valor Functionis $\frac{1}{z^3(1-z)^2(1+z)}$ in forma simplicissima expressus. Sumatur primum Factor denominatoris solitarius $1+z$, qui dat $\frac{p}{q} = -1$. erit $M = 1$ & $Z = z^3 - 2z^4 + z^5$. Hinc ad fractionem $\frac{A}{1+z}$ inveniendam, erit $A = \frac{1}{z^3 - 2z^4 + z^5}$, posito $z = -1$; ideoque fit $A =$

$-\frac{1}{4}$, atque ex Factore $1+z$ oritur hæc fractio partialis CAP. II.

$-\frac{1}{4(1+z)}$. Jam sumatur Factor quadratus $(1-z)^2$ qui dat $\frac{p}{q} = 1$. $M = 1$, & $Z = z^3 + z^4$; positis ergo fractionibus

partialibus hinc ortis $\frac{A}{(1-z)^2} + \frac{B}{1-z}$, erit $A =$

$\frac{1}{z^3 + z^4}$, posito $z = 1$; ergo $A = \frac{1}{2}$; fiat $P = \frac{M - \frac{1}{2}Z}{1-z}$

$= \frac{1 - \frac{1}{2}z^3 - \frac{1}{2}z^4}{1-z} = 1 + z + zz + \frac{1}{2}z^3$; eritque $B =$

$\frac{P}{Z} = \frac{1 + z + zz + \frac{1}{2}z^3}{z^3 + z^4}$, posito $z = 1$; ergo $B =$

$\frac{7}{4}$ & fractiones partiales quæsitæ $\frac{1}{2(1-z)^2} + \frac{7}{4(1-z)}$.

Denique tertius Factor cubicus z^3 dat $\frac{p}{q} = 0$; $M = 1$; &

$Z = 1 - z - zz + z^3$. Positis ergo fractionibus partialibus his $\frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z}$; erit primum. $A = \frac{M}{N} = \frac{1}{1-z-zz+z^3}$

posito $z = 0$; ergo $A = 1$. Ponatur $P = \frac{M - Z}{z} = 1 +$

$z - zz$, erit $B = \frac{P}{Z}$, posito $z = 0$; ergo $B = 1$. Pona-

tur $Q = \frac{P - Z}{z} = 2 - zz$; erit $C = \frac{Q}{Z}$, posito $z = 0$;

ergo $C = 2$. Hanc ob rem Functio proposita $\frac{1}{z^3(1-z)^2(1+z)}$

in hanc formam resolvitur $\frac{1}{z^3} + \frac{1}{z^2} + \frac{2}{z} + \frac{1}{2(1-z)^2}$

$+ \frac{7}{4(1-z)} - \frac{1}{4(1+z)}$: nulla enim pars integra insuper accedit, quia fractio proposita non est spuria.

CAPUT III.

De transformatione Functionum per substitutionem.

46. **S**i fuerit y Functio quaecunque ipsius z , atque z definiatur per novam variabilem x , tum quoque y per x definiri poterit.

Cum ergo antea y fuisset Functio ipsius z , nunc nova quantitas variabilis x inducitur, per quam utraque priorum y & z definiatur. Sic, si fuerit $y = \frac{1-zz}{1+zz}$, atque ponatur $z = \frac{1-x}{1+x}$;

hoc valore loco z substituto, erit $y = \frac{2x}{1+xx}$. Sumpto

ergo pro x valore quocunque determinato, ex eo reperientur valores determinati pro z & y , sicque invenitur valor ipsius y respondens illi valori ipsius z qui simul prodiit. Uti si

fit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$, & $y = \frac{4}{5}$; reperitur autem quoque

$y = \frac{4}{5}$, si in $\frac{1-zz}{1+zz}$, cui expressioni y æquatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem hæc novæ variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expressio ipsius y per z data laborat, tollitur; vel quando ob æquationem altioris gradus, qua relatio inter y & z exprimitur, non licet Functionem explicitam ipsius z ipsi y æqualem exhibere, nova variabilis x inducitur, ex qua utraque y & z commode definiri queat: unde insignis substitutionum usus jam satis elucet, ex sequentibus vero multo clarius perspicitur.

47. Si fuerit $y = \sqrt{a+bz}$; nova variabilis x per quam utraque z & y rationaliter exprimatur, sequenti modo invenietur.

Quoniam tam z quam y debet esse Functio rationalis ipsius x ; perspicuum est hoc obtineri si ponatur $\sqrt{a+bz} = bx$: Fiet enim primo $y = bx$; & $a+bz = b^2xx$; hincque $z = bxx - \frac{a}{b}$.

Quocirca utraque quantitas y & z per Functionem rationalem ipsius x exprimitur; scilicet cum sit $y = \sqrt{a+bz}$ fiat $z = bxx - \frac{a}{b}$; erit $y = bx$.

48. S.

48. Si fuerit $y = (a + bz)^{m:n}$; nova variabilis x , per quam tam y quam z rationaliter exprimatur, sic reperietur.

Ponatur $y = x^m$, fietque $(a + bz)^{m:n} = x^m$ ideoque $(a + bz)^{1:n} = x$: ergo $a + bz = x^n$ & $z = \frac{x^n - a}{b}$. Sic ergo utraque quantitas y & z rationaliter per x definietur, ope scilicet substitutionis $z = \frac{x^n - a}{b}$, quæ præbet $y = x^m$. Quamvis igitur neque y per z , neque vicissim z per y rationaliter exprimi possit; tamen utraque reddita est Functio rationalis novæ quantitatis variabilis x per substitutionem introductæ, scopo substitutionis omnino convenienter.

49. Si fuerit $y = \left(\frac{a + bz}{f + gz}\right)^{m:n}$; requiritur nova quantitas variabilis x per quam utraque y & z rationaliter exprimatur.

Manifestum primo est si ponatur $y = x^m$, quæsito satisfieri; erit enim $\left(\frac{a + bz}{f + gz}\right)^{m:n} = x^m$, ideoque $\frac{a + bz}{f + gz} = x^n$; ex qua æquatione elicitur $z = \frac{a - fx^n}{gx^n - b}$; quæ substitutio præbet $y = x^m$.

Hinc quoque intelligitur si fuerit $\left(\frac{a + by}{c + dy}\right)^n = \left(\frac{a + bz}{f + gz}\right)^m$, tam y quam z rationaliter per x expressum iri, si utraque formulâ ponatur $= x^{mn}$; reperietur enim $y = \frac{a - \gamma x^n}{\delta x^m - \epsilon}$ & $z = \frac{a - fx^n}{gx^n - b}$.

qui casus nil habent difficultatis.

50. Si fuerit $y = \sqrt{(a + bz)(c + dz)}$; substitutio idonea invenietur, qua y & z rationaliter exprimuntur, hoc modo.

Ponatur $\sqrt{(a + bz)(c + dz)} = (a + bz)x$, facile enim perspicitur hinc valorem rationalem pro z esse proditurum; quia valor ipsius z per æquationem simplicem determinatur. Erit

LIB. I. ergo $c + dz = (a + bz)xx$, hincque $z = \frac{c - axx}{bxx - d}$. Quare porro fiet $a + bz = \frac{bc - ad}{bxx - d}$; & ob $y = \sqrt{(a + bz)(c + dz)}$ $= (a + bz)x$ habebitur $y = \frac{(bc - ad)x}{bxx - d}$. Functio ergo irrationalis $y = \sqrt{(a + bz)(c + dz)}$ ad rationalitatem perducitur ope substitutionis $z = \frac{c - axx}{bxx - d}$, quippe quæ dabit $y = \frac{(bc - ad)x}{bxx - d}$. Sic, si fuerit $y = \sqrt{(aa - zz)} = \sqrt{(a + z)(a - z)}$; ob $b = +1$; $c = a$, $d = -1$, ponatur $z = \frac{a - axx}{1 + xx}$, eritque $y = \frac{2ax}{1 + xx}$. Quoties ergo quantitas post signum $\sqrt{}$ habuerit duos Factores simplices reales, hoc modo reductio ad rationalitatem absolvetur; sin autem Factores bini simplices fuerint imaginarii, sequenti modo uti præstabit.

51. Sit $y = \sqrt{(p + qz + rzz)}$; atque requiritur substitutio idonea pro z facienda, ut valor ipsius y fiat rationalis.

Pluribus modis hoc fieri potest, prout p & q fuerint quantitates vel affirmativæ vel negativæ. Sit primo p quantitas affirmativa, ac ponatur aa pro p ; etiamsi enim p non sit quadratum, tamen irrationalitas quantitatum constantium præsens negotium non turbat. Sit igitur

I. $y = \sqrt{(aa + bz + czz)}$; ac ponatur $\sqrt{(aa + bz + czz)} = a + xz$; erit $b + cz = 2ax + xxz$; unde fit $z = \frac{b - 2ax}{xx - c}$; tum vero erit $y = a + xz = \frac{bx - axx - ac}{xx - c}$; ubi z & y sunt Functiones rationales ipsius x . Sit jam

II. $y = \sqrt{(aazz + bz + c)}$; ac ponatur $\sqrt{(aazz + bz + c)} = az + x$; erit $bz + c = 2axz + xx$, & $z = \frac{xx - c}{b - 2ax}$. Tum autem fit $y = az + x = \frac{-ac + bx - axx}{b - 2ax}$.

III. Si fuerint p & r quantitates negativæ; tum, nisi sit $qq > 4pr$, valor ipsius y semper erit imaginarius. Quod si autem fuerit $qq > 4pr$; expressio $p + qz + rzz$ in duos Factores resolvi

resolvi poterit, qui casus ad §. præced. reducitur. Sæpenumero CAP. III.

autem commodius ad hanc formam reducitur, $y = \sqrt{(aa + (b + cz)(d + ez))}$; pro qua ad rationalitatem perducenda ponatur $y = a + (b + cz)x$, eritque $d + ez = 2ax + bxx + cxxz$; unde fit $z = \frac{d - 2ax - bxx}{c - e}$, & $y = \frac{ae + (cd - be)x - acxx}{c - e}$. Interdum commodius fieri potest reductio ad hanc formam, $y = \sqrt{(aaz + (b + cz)(d + ez))}$. Tum ponatur $y = az + (b + cz)x$; erit $d + ez = 2axz + bxx + cxxz$ & $z = \frac{bxx - d}{e - 2ax - cxx}$, atque $y = \frac{ad + (be - cd)x - abxx}{e - 2ax - cxx}$.

EXEMPLUM.

Si habeatur ista ipsius z Functio irrationalis $y = \sqrt{(-1 + 3z - 2z^2)}$; quæ cum reduci queat ad hanc formam $y = \sqrt{(1 - 2 + 3z - 2z^2)} = \sqrt{(1 - (1 - z)(2 - z))}$; ponatur $y = 1 - (1 - z)x$, erit $-2 + z = -2x + xx - xxz$ & $z = \frac{2 - 2x + xx}{1 + xx}$. Deinde est $1 - z = \frac{1 + 2x}{1 + xx}$ & $y = 1 - (1 - z)x = \frac{1 + x - xx}{1 + xx}$. Atque hi sunt fere casus, quos Algebra indeterminata, seu methodus *Diophantæ*, suppeditat; neque alios casus in his tractatis non comprehensos per substitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum substitutionis usum monstrandum progredior.

52. Si y ejusmodi fuerit Functio ipsius z ut sit $ay^\alpha + bz^\epsilon + cy^\gamma z^\delta = 0$, invenire novam variabilem x , per quam valores ipsarum y & z explicite assignari queant.

Quoniam resolutio æquationum generalis non habetur, ex æquatione proposita $ay^\alpha + bz^\epsilon + cy^\gamma z^\delta = 0$ neque y per z neque

LIB. I. & neque vicissim z per y exhiberi potest. Quo igitur huic incom-
modo remedium afferatur; ponatur $y = x^m z^n$, eritque $ax^{\alpha m}$
 $z^{\alpha n} + bz^{\zeta} + cx^{\gamma m} z^{\gamma n + \delta} = 0$. Determinetur nunc ex-
ponens n ita ut ex hac æquatione valor ipsius z definiri queat:
quod tribus modis præstari potest.

I. Sit $\alpha n = \zeta$; ideoque $n = \frac{\zeta}{\alpha}$; erit, æquatione per $z^{\alpha n}$
 $= z^{\zeta}$ divisa, $ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \zeta + \delta} = 0$; unde
oritur $z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}} \right)^{\frac{1}{\gamma n - \zeta + \delta}}$, five

$$z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}} \right)^{\frac{\alpha}{\zeta \gamma - \alpha \zeta + \alpha \delta}}, \text{ \&}$$

$$y = x^m \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}} \right)^{\frac{\zeta}{\zeta \gamma - \alpha \zeta + \alpha \delta}}$$

II. Sit $\zeta = \gamma n + \delta$ seu $n = \frac{\zeta - \delta}{\gamma}$; erit, æquatione per
 z^{ζ} divisa, $ax^{\alpha m} z^{\alpha n - \zeta} + b + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}} \right)^{\frac{1}{\alpha n - \zeta}} = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}} \right)^{\frac{\gamma}{\alpha \zeta - \alpha \delta - \zeta \gamma}},$$

$$\text{atque } y = x^m \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}} \right)^{\frac{\zeta - \delta}{\alpha \zeta - \alpha \delta - \zeta \gamma}}.$$

III. Sit $\alpha n = \gamma n + \delta$, seu $n = \frac{\delta}{\alpha - \gamma}$; erit, æquatione
per $z^{\alpha n}$ divisa, $ax^{\alpha m} + bz^{\zeta - \alpha n} + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b} \right)^{\frac{1}{\zeta - \alpha n}} =$$

$$\left(\frac{ax^{\alpha m} - cx^{\gamma m}}{b} \right)^{\frac{\alpha - \gamma}{\alpha \delta - \delta \gamma - \alpha \delta}}; \text{ atque}$$

$$y = x^m \left(\frac{ax^{\alpha m} - cx^{\gamma m}}{b} \right)^{\frac{\delta}{\alpha \delta - \delta \gamma - \alpha \delta}}.$$

Tribus igitur diversis modis erutæ sunt Functiones ipsius x , quæ ipsis z & y sunt æquales. Præterea vero pro m numerum pro lubitu substituere licet cyphra excepta; sicque formulæ ad commodissimam expressionem reduci poterunt.

EXEMPLUM.

Exprimatur natura Functionis y per hanc æquationem $y^3 + z^3 - cyz = 0$; atque quarantur Functiones ipsius x ipsis y & z æquales. Erit ergo $a = -1$; $b = -1$; $\alpha = 3$; $\delta = 3$; $\gamma = 1$; & $\delta = 1$. Hinc primus modus dabit, posito $m = 1$, $z = \left(\frac{x^3 + 1}{cx} \right)^{-1}$ & $y = x \left(\frac{x^3 + 1}{cx} \right)^{-1}$, five $z = \frac{cx}{1 + x^3}$ & $y = \frac{cx}{1 + x^3}$; quarum expressionum utraque adeo est rationalis.

Secundus modus vero dabit hos valores:

$$z = \left(\frac{cx - 1}{x^3} \right)^{1/3}, \text{ & } y = x \left(\frac{cx - 1}{x^3} \right)^{2/3}, \text{ five}$$

$$z = \frac{1}{x} \sqrt[3]{(cx - 1)}, \text{ & } y = \frac{1}{x} \sqrt[3]{(cx - 1)^2}.$$

Tertius modus ita rem expediet ut fit

$$z = (cx - x^3)^{2/3}, \text{ & } y = x(cx - x^3)^{1/3}.$$

53. Hinc a posteriori intelligitur cujusmodi æquationes, quibus valor Functionis y per z determinatur, hoc modo novam variabilem x introducendo resolvæ queant.

Ponamus enim resolutione jam instituta prodiisse has determinatæ
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LIB. I.

minationes $z = \left(\frac{ax^a + bx^b + cx^c + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{p:r}$, atque $y = x$

$\left(\frac{ax^a + bx^b + cx^c + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{q:r}$; eritque $y^p = x^p z^q$; & hinc

$x = yz^{-q:p}$. Cum igitur sit $z^{r:p} = \frac{ax^a + bx^b + cx^c + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}$, si

loco x ejus valorem $yz^{-q:p}$ substituamus; prodibit ista æquatio $z^{r:p} = \frac{ay^a z^{-aq:p} + by^b z^{-bq:p} + cy^c z^{-cq:p} + \&c.}{A + By^\mu z^{-\mu q:p} + Cy^\nu z^{-\nu q:p} + \&c.}$;

quæ reducitur ad hanc $Az^{r:p} + By^\mu z^{(r-\mu q):p} + Cy^\nu z^{(r-\nu q):p} + \&c. = ay^a z^{-aq:p} + by^b z^{-bq:p} + cy^c z^{-cq:p} + \&c.$ quæ multiplicata per $z^{aq:p}$ transibit in hanc: $Az^{(aq+r):p} + By^\mu z^{(aq-\mu q+r):p} + Cy^\nu z^{(aq-\nu q+r):p} + \&c. = ay^a + by^b z^{(aq-bq):p} + cy^c z^{(aq-cq):p} + \&c.$

Ponatur $\frac{aq+r}{p} = m$ & $\frac{aq-bq}{p} = n$: fiet $p = a - b$; $q = n$, & $r = am - bn$: atque nascetur ista æquatio: $Az^m + By^\mu z^{m-\mu n(\alpha-\beta)} + Cy^\nu z^{m-\nu n(\alpha-\beta)} + \&c. = ay^a + by^b z^n + cy^c z^{(a-\gamma)n:(a-\beta)} + \&c.$ quæ propterea ita resolvetur ut sit:

$$z = \left(\frac{ax^a + bx^b + cx^c + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{\frac{a-b}{am-bn-\alpha n}} \&$$

$$y = x \left(\frac{ax^a + bx^b + cx^c + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{\frac{n}{am-bn-\alpha n}}$$

Vel ponatur $\frac{aq+r}{p} = m$, & $\frac{aq-\mu q+r}{p} = n$, erit $m - n$

=

$= \frac{\mu q}{p}$; & $\frac{q}{p} = \frac{m-n}{\mu}$, atque $\frac{r}{p} = m - \frac{am+an}{\mu}$. Hinc CAP. III.
 fit $p = \mu$; $q = m - n$; & $r = \mu m - am + an$; atque
 hæc æquatio resultabit:

$$\begin{aligned}
 & Az^m + By^\mu z^n + Cy^\nu z^{\mu m - \nu(m-n):\mu} + \&c. = ay^a \\
 & + by^b z^{(a-b)(m-n):\mu} + cy^\gamma z^{(a-\gamma)(m-n):\mu} \\
 & + \&c. \text{ quæ ita resolvetur ut fit:}
 \end{aligned}$$

$$z = \left(\frac{ax^a + bx^b + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{\frac{\mu}{\mu m - am + an}} \&c.$$

$$y = x \left(\frac{ax^a + bx^b + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{\frac{m-n}{\mu m - am + an}}$$

54. Si y ita pendeat a z ut sit $ayy + byz + czz + dy + ez = 0$, sequenti modo tam y quam z rationaliter per novam variabilem x exprimetur.

Ponatur $y = xz$, erit divisione per z facta:
 $axxz + bxz + cz + dx + e = 0$, ex qua reperitur
 $z = \frac{-dx - e}{axx + bx + c}$, & $y = \frac{-dxx - ex}{axx + bx + c}$.

At vero ad formam propositam reduci potest hæc æquatio inter y & z : $ayy + byz + czz + dy + ez + f = 0$ diminuendo vel augendo utramque variabilem certa quadam quantitate constante, unde & hæc æquatio per novam variabilem x rationaliter explicari potest.

55. Si y ita pendeat a z , ut sit $ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0$; sequenti modo tam y quam z rationaliter per novam variabilem x exprimi poterit.

Ponatur $y = xz$, & facta substitutione tota æquatio per z dividi poterit: prodibit autem $ax^3z + bxxz + cxz + dz + exx + fx + g = 0$. Unde oritur $z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d}$
 ex quo erit $y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$.

LIB. I. Ex his casibus facile intelligitur quemadmodum æquationes altiorum graduum, quibus y per z definitur, comparatæ esse debeant, ut hujusmodi resolutio locum habere queat. Ceterum hi casus in superioribus formulis §. 53. continentur: at, quia formulæ generales non tam facile ad hujusmodi casus sæpius occurrentes accommodantur, visum est horum aliquos seorsim evolvere.

56. Si y ita pendeat a z ut sit $ayy + byz + czz = d$ hoc modo utraque quantitas y & z per novam variabilem x exprimetur.

Ponatur $y = xz$, eritque $(axx + bx + c)zz = d$, ideoque $z = \sqrt{\frac{d}{axx + bx + c}}$ & $y = x \sqrt{\frac{d}{axx + bx + c}}$.

Simili modo si fuerit, $ay^3 + by^2z + cyz^2 + dz^3 = ey + fz$; posito $y = xz$, tota æquatio per z divisa dabit $(ax^3 + bxx + cx + d)zz = ex + f$; unde oritur $z = \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$; & $y = x \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$. Hi autem casus alique similes resolutiones admittentes comprehenduntur in sequente paragrapho.

57. Si y ita pendeat a z ut sit $ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \&c. = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \&c.$ Sequenti modo tam z quam y commode per novam variabilem x exprimetur.

Sit $y = xz$, atque facta substitutione tota æquatio dividi poterit per z^n , siquidem exponens m sit major quam n ; eritque $(ax^m + bx^{m-1} + cx^{m-2} + \&c.)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \&c.$ unde obtinebitur

$z =$

$$z = \left(\frac{ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \&c.}{ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.} \right)^{1:(m-n)} \&$$

$$y = x \left(\frac{ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \&c.}{ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.} \right)^{1:(m-n)}$$

Hæc scilicet resolutio locum habet, si in æquatione naturam Functionis y per z exprimente, duplex tantum ubique occurrit dimensionum ab y & z sumptarum numerus; uti in casu tractato in singulis terminis numerus dimensionum vel est m vel n .

58. Si in æquatione inter y & z triplicis generis dimensiones occurrant, quarum summa tantum superet mediam, quantum hæc media infimam, ope resolutionis æquationis quadratæ variables y & z per novam x exprimi poterunt.

Si enim ponatur $y = xz$, divisione per minimam ipsius z potestatem facta, valor ipsius z per x , ope extractionis radicis quadratæ exhibebitur, id quod ex sequentibus exemplis erit manifestum.

EXEMPLUM I.

Sit $ay^3 + by^2z + cyzz + dz^3 = 2eyy + 2fyz + 2gzx + hy + iz$; ac ponatur $y = xz$: erit, divisione per z facta, $(ax^3 + bxx + cx + d)zz = 2(exx + fx + g)z + hx + i$; ex qua sequens ipsius z obtinebitur valor:

$$z = \frac{exx + fx + g \pm \sqrt{((exx + fx + g)^2 + (ax^3 + bxx + cx + d)(hx + i))}}{ax^3 + bxx + cx + d}$$

quo invento erit $y = xz$.

EXEMPLUM II.

Sit $y^5 = 2az^3 + by + cz$; ac, posito $y = xz$, erit $x^5z^4 = 2azz + bx + c$; ex qua reperitur $zz = \frac{a \pm \sqrt{(aa + bx^6 + cx^5)}}{x^5}$; &

$$z = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{xx\sqrt{x}} \& y = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{x\sqrt{x}}$$

EXEMPLUM III.

Sit $y^{10} = 2ayz^6 + byz^3 + cz^4$, in qua cum dimensiones sint 10, 7, & 4, ponatur $y = xz$; atque æquatio per z^4 divisa abibit in hanc: $x^{10} z^6 = 2axz^3 + bx + c$ seu $z^6 = \frac{2axz^3 + bx + c}{x^{10}}$; unde invenitur $z^3 = \frac{ax \pm x\sqrt{(aa + bx^2 + cx^3)}}{x^{10}}$; ideoque erit $z = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^2 + cx^3))}}}{x^3}$; atque $y = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^2 + cx^3))}}}{x^2}$. Ex quibus exemplis usus hujusmodi substitutionum abunde perspicitur.

CAPUT IV.

De explicatione Functionum per series infinitas.

59. **C**UM Functiones fractæ atque irrationales ipsius z non in forma integra $A + Bz + Cz^2 + Dz^3 + \&c.$ continentur, ita ut terminorum numerus sit finitus, quæri solent hujusmodi expressiones in infinitum excurrentes, quæ valorem cujusvis Functionis sive fractæ sive irrationalis exhibeant. Quin etiam natura Functionum transcendentium melius intelligi censetur, si per ejusmodi formam, etsi infinitam, exprimantur. Cum enim natura Functionis integræ optime perspiciatur, si secundum diversas potestates ipsius z explicetur, atque adeo ad formam $A + Bz + Cz^2 + Dz^3 + \&c.$ reducatur, ita eadem forma aptissima videtur ad reliquarum Functionum omnium indolem menti repræsentandam, etiamsi terminorum numerus sit revera infinitus. Perspicuum autem est nullam Functionem non integram ipsius z per numerum hujusmodi terminorum $A + Bz + Cz^2 + \&c.$ finitum exponi posse; eo ipso enim
Functio

Functio foret integra; num vero per hujusmodi terminorum se- CAP. IV.
riem infinitam exhiberi possit, si quis dubitet, hoc dubium
per ipsam evolutionem cujusque Functionis tollitur. Quo au-
tem hæc explicatio latius pateat, præter potestates ipsius z ex-
ponentes integros affirmativos habentes, admitti debent pote-
states quæcunque. Sic dubium erit nullum quin omnis Functio
ipsius z in hujusmodi expressionem infinitam transmutari possit:

$Az^a + Bz^b + Cz^c + Dz^d + \&c.$ denotantibus exponen-
tibus $a, b, c, d, \&c.$ numeros quoscunque.

60. Per divisionem autem continuam intelligitur fractionem
 $\frac{a}{a + bz}$ resolvi in hanc seriem infinitam $\frac{a}{a} - \frac{abz}{a^2} + \frac{a^2bz^2}{a^3}$
 $- \frac{a^3bz^3}{a^4} + \frac{a^4bz^4}{a^5} - \&c.$, quæ, cum quilibet terminus ad se-
quentem habeat rationem constantem $1: \frac{bz}{a}$, vocatur series geo-
metrica.

Potest vero quoque hæc series ita inveniri, ut ipsa initio
pro incognita habeatur: ponatur enim $\frac{a}{a + bz} = A + Bz +$
 $Cz^2 + Dz^3 + Ez^4 + \&c.$ atque ad æqualitatem producendam
quærantur coëfficientes $A, B, C, D, \&c.$ Erit ergo $a =$
 $(a + bz)(A + Bz + Cz^2 + Dz^3 + \&c.)$, & multiplica-
tione actu peracta fiet

$$a = aA + aBz + aCz^2 + aDz^3 + aEz^4 + \&c.$$

$$+ bAz + bBz^2 + bCz^3 + bDz^4 + \&c.$$

Quamobrem esse debet $a = aA$, ideoque $A = \frac{a}{a}$, & coëf-
ficientium uniuscujusque potestatis ipsius z summa nihilo æqualis
est ponenda: unde prodibunt hæ æquationes,

$$\begin{aligned} aB + bA &= 0 && \text{cognito ergo quovis coëfficiente} \\ aC + bB &= 0 && \text{facile reperitur sequens; si enim} \\ aD + bC &= 0 && \text{fuerit coëfficiens termini cujusque} = P \\ aE + bD &= 0 && \text{\& sequens} = Q \text{ erit } aQ + bP = 0 \\ &&& \&c. \qquad \qquad \text{five } Q = \frac{bP}{a}. \end{aligned}$$

Cum

LIB. I. Cum igitur terminus primus A sit determinatus $= \frac{a}{\alpha}$ ex eo sequentes litteræ $B, C, D, \&c.$ definiuntur eodem modo, quo ex divisione sunt orti. Ceterum ex inspectione perspicitur in serie infinita pro $\frac{a}{\alpha + \zeta z}$ inventa potestatis z^n coefficientem fore $= \pm \frac{a \zeta^n}{\alpha^{n+1}}$, ubi signum $+$ locum habet si n sit numerus par, signum $-$ autem si n sit numerus impar: seu coefficientis erit $= \frac{a}{\alpha} \left(\frac{-\zeta}{\alpha} \right)^n$.

61. Simili modo ope divisionis continuata hac Functio fracta $\frac{a + bz}{\alpha + \zeta z + \gamma zz}$ in seriem infinitam converti potest.

Cum autem divisio sit tædiosa, neque tam facile naturam seriei infinitæ ostendat, commodius erit seriem quæsitam fingere, atque modo ante tradito determinare. Sit igitur

$$\frac{a + bz}{\alpha + \zeta z + \gamma zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$$

multiplicetur utrinque per $\alpha + \zeta z + \gamma zz$, atque fiet

$$\begin{aligned} a + bz &= \alpha A + \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \zeta Az + \zeta Bz^2 + \zeta Cz^3 + \zeta Dz^4 + \&c. \\ &+ \gamma Az^2 + \gamma Bz^3 + \gamma Cz^4 + \&c. \end{aligned}$$

Hinc erit $\alpha A = a$; $\alpha B + \zeta A = b$; unde reperitur $A = \frac{a}{\alpha}$ & $B = \frac{b}{\alpha} - \frac{a\zeta}{\alpha^2}$; reliquæ vero litteræ ex sequentibus æquationibus determinabuntur:

$$\begin{aligned} \alpha C + \zeta B + \gamma A &= 0 & \text{hinc ergo ex binis quibusque coëffi-} \\ \alpha D + \zeta C + \gamma B &= 0 & \text{cientibus contiguis sequens reperi-} \\ \alpha E + \zeta D + \gamma C &= 0 & \text{tur. Sic si duo coëfficientes contigui} \\ \alpha F + \zeta E + \gamma D &= 0 & \text{fuerint } P, Q \text{ \& sequens } R, \text{ erit } \alpha R \end{aligned}$$

$$\&c. \quad + \zeta Q + \gamma P = 0 \text{ seu } R = \frac{-\zeta Q - \gamma P}{\alpha}.$$

Cum igitur duæ litteræ primæ A & B jam sint inventæ sequentes C, D, E, F &c. omnes successive ex iis invenientur,

tur, sicque reperietur Series infinita $A + Bz + Cz^2 + Dz^3 + \&c.$ CAP. IV.

Functioni fractæ propositæ $\frac{a + bz}{\alpha + \epsilon z + \gamma z^2}$ æqualis.

EXEMPLUM.

Si fuerit proposita hæc fractio $\frac{1 + 2z}{1 - z - z^2}$, huicque æqualis statuatur Series $A + Bz + Cz^2 + Dz^3 + \&c.$ ob $a = 1$; $b = 2$; $\alpha = 1$; $\epsilon = -1$; $\gamma = -1$; erit $A = 1$; $B = 3$; tum vero erit

$$\begin{aligned} C &= B + A & \text{quilibet ergo coëfficiens æqualis est sum-} \\ D &= C + B & \text{mæ duorum præcedentium; quare si co-} \\ E &= D + C & \text{gniti fuerint duo coëfficientes contigui} \\ F &= E + D & P \& Q, \text{ erit sequens } R = P + Q. \\ & & \&c. \end{aligned}$$

Cum igitur duo coëfficientes primi A & B sint cogniti, fractio proposita $\frac{1 + 2z}{1 - z - z^2}$ in hanc Seriem infinitam transmutatur $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \&c.$, quæ nullo negotio quousque libuerit continuari potest.

62. Ex his jam satis intelligitur indoles Serierum infinitarum, in quas Functiones fractæ transmutantur; tenent enim ejusmodi legem, ut quilibet terminus ex aliquot præcedentibus determinari possit. Scilicet, si denominator fractionis propositæ fuerit $\alpha + \epsilon z$, atque Series infinita statuatur

$$A + Bz + Cz^2 + \dots + Pz^n + Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + \&c.;$$

quilibet coëfficiens Q ex præcedente P solo ita definietur ut sit $\alpha Q + \epsilon P = 0$. Sin denominator fuerit trinomium $\alpha + \epsilon z + \gamma z^2$, quilibet coëfficiens Seriei R ex duobus præcedentibus Q & P ita definietur ut sit $\alpha R + \epsilon Q + \gamma P = 0$: simili modo si denominator fuerit quadrinomium, ut $\alpha + \epsilon z + \gamma z^2 + \delta z^3$, quilibet coëfficiens seriei S ex tribus antecedentibus R , Q & P ita determinabitur, ut sit $\alpha S + \epsilon R + \gamma Q + \delta P = 0$,

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sicque

LIB. I. ficque de ceteris. In his ergo Seriebus quilibet terminus determinatur ex aliquot antecedentibus secundum legem quandam constantem, quæ lex ex denominatore fractionis hanc Seriem producentis sponte apparet. Vocari autem hæ Series a Celeb. MOIVRÆO, qui earum naturam maxime est scrutatus, solent *recurrentes*, propterea quod ad terminos antecedentes est recurrendum, si sequentes investigare velimus.

63. Ad harum porro Serierum formationem requiritur ut denominatoris terminus constans α non sit $= 0$: cum enim inventus sit terminus Seriei primus $A = \frac{a}{\alpha}$, tum is, tum omnes sequentes fierent infiniti, si esset $\alpha = 0$. Hoc ergo casu excluso, quem deinceps evolvam, Functio fracta in Seriem infinitam recurrentem transmutanda, hujusmodi habebit formam $\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \epsilon z^2 - \gamma z^3 - \delta z^4 - \&c.}$; ubi primum denominatoris terminum pono $= 1$, huc enim semper fractio reduci potest, nisi is sit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contempnor, ut Seriei hinc formatæ omnes termini fiant affirmativi. Quod si enim Series recurrens hinc orta ponatur $A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$ coëfficientes ita determinabuntur ut sit

$$\begin{aligned} A &= a \\ B &= \alpha A + b \\ C &= \alpha B + \epsilon A + c \\ D &= \alpha C + \epsilon B + \gamma A + d \\ E &= \alpha D + \epsilon C + \gamma B + \delta A + e \\ &\quad \&c. \end{aligned}$$

Quilibet ergo coëfficiens æqualis est aggregato ex multiplis aliquot præcedentium una cum numero quodam, quem numerator præbet. Nisi autem numerator in infinitum progredia-
tur, hæc additio mox cessabit, atque quivis terminus secundum legem constantem ex aliquot præcedentibus determinabitur. Ne ergo lex progressionis usquam turbetur conveniet
Functio-

Functionem fractam genuinam adhibere: si enim fractio spuria CAP. IV.
accipiat, tum pars integra in ea contenta ad Seriem accedet,
atque in illis terminis, quos vel auget vel minuit, legem pro-
gressionis interrumpet. Exempli gratia hæc fractio spuria

$\frac{1 + 2z - z^3}{1 - z - z^2}$, præbebit hanc Seriem $1 + 3z + 4z^2 + 6z^3$
 $+ 10z^4 + 16z^5 + 26z^6 + 42z^7 + \&c.$ ubi a lege, qua quivis
coëfficiens est summa duorum præcedentium, terminus quartus
 $6z^3$ excipitur.

64. Peculiarem contemplationem Series recurrentes meren-
tur, si denominator fractionis, unde oriuntur, fuerit potestas.

Sic, si ista fractio $\frac{a + bz}{(1 - az)^2}$ in Seriem resolvatur, prodit

$$\begin{aligned} & a + 2az + 3a^2z^2 + 4a^3z^3 + 5a^4z^4 + \&c. \\ & + b + 2ab + 3a^2b + 4a^3b \end{aligned}$$

in qua coëfficiens potestatis z^n erit $(n+1)a^n a + na^{n-1}b$. Erit
tamen hæc Series recurrens, quia quilibet terminus ex duobus
præcedentibus determinatur, cujus determinationis lex perspi-
citur ex denominatore evoluto $1 - 2az + a^2z^2$. Si pona-
tur $a = 1$ & $z = 1$, abit Series in progressionem arithmeticam
generalem $a + (2a + b) + (3a + 2b) + (4a + 3b) + \&c.$
cujus differentia sunt constantes. Omnis ergo progressio a-
rithmetica est Series recurrens: si enim sit

$A + B + C + D + E + F + \&c.$ progressio arithmetica, erit
 $C = 2B - A$; $D = 2C - B$; $E = 2D - C$, &c.

65. Deinde hæc fractio $\frac{a + bz + cz^2}{(1 - az)^3}$ ob $\frac{1}{(1 - az)^3} = (1 - az)^{-3}$
 $= 1 + 3az + 6a^2z^2 + 10a^3z^3 + 15a^4z^4 + \&c.$ trans-
mutabitur in hanc Seriem infinitam:

$$\begin{aligned} & a + 3a^2z + 6a^3z^2 + 10a^4z^3 + 15a^5z^4 + \&c. \\ & + b + 3abz + 6a^2bz^2 + 10a^3bz^3 \\ & + c + 3acz + 6a^2cz^2 \end{aligned}$$

LIB. I. in qua potestas z^n coefficientem habebit $\frac{(n+1)(n+2)}{1.2} a^n + \frac{n(n+1)}{1.2} a^{n-1} b + \frac{(n-1)n}{1.2} a^{n-2} c$. Quod si autem ponatur $a = 1$ & $z = 1$, Series hæc abibit in progressionem generalem secundi ordinis, cujus differentia secunda sunt constantes. Designet $A + B + C + D + E + \&c.$ hujusmodi progressionem, erit ea simul Series recurrens, cujus quilibet terminus ex tribus antecedentibus ita determinatur ut sit $D = 3C - 3B + A$; $E = 3D - 3C + B$; $F = 3E - 3D + C$ &c. Cum igitur terminorum in progressionem arithmetica procedentium secunda differentia quoque sint æquales, nempe $= 0$, hæc proprietas quoque ad progressionem arithmeticas extenditur.

66. Simili modo hæc fractio $\frac{a + bz + cz^2 + dz^3}{(1 - az)^4}$ dabit

Seriem infinitam, in qua potestas ipsius z quæcunque z^n hunc habebit coefficientem $\frac{(n+1)(n+2)(n+3)}{1.2.3} a^n + \frac{n(n+1)(n+2)}{1.2.3} a^{n-1} b + \frac{(n-1)n(n+1)}{1.2.3} a^{n-2} c + \frac{(n-2)(n-1)n}{1.2.3} a^{n-3} d$: posito ergo $a = 1$ & $z = 1$;

hæc Series in se complectetur omnes progressionem algebraicas tertii ordinis, quarum differentia tertia sunt constantes: omnes ergo hujus ordinis progressionem, cujusmodi sit $A + B + C + D + E + F + \&c.$ erunt simul recurrentes ex denominatore $1 - 4z + 6z^2 - 4z^3 + z^4$ ortæ; unde erit $E = 4D - 6C + 4B - A$; $F = 4E - 6D + 4C - B$ &c., quæ proprietas simul in omnes progressionem inferiorum ordinum competit.

67. Hoc modo ostenduntur omnes progressionem algebraicas cujuscunque ordinis, quæ tandem ad differentias constantes deducunt, esse Series recurrentes, quarum lex definiatur ex denominatore $(1 - z)^n$, existente n numero majore quam is, qui ordinem progressionis indicat. Cum igitur $a''' + (a+b)''' + (a+2b)'''$

$(a + 2b)^m + (a + 3b)^m + \&c.$ exhibeat progressio- CAP. IV.
nem ordinis m ; erit ob naturam Serierum recurrentium

$$0 = a^m - \frac{n}{1} (a + b)^m + \frac{n(n-1)}{1 \cdot 2} (a + 2b)^m \\ - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (a + 3b)^m + \dots \pm \frac{n}{1} (a + (n-1)b)^m \mp (a + nb)^m;$$

ubi signa superiora valent si n fit numerus par, inferiora autem si n fit numerus impar. Hæc ergo æquatio semper est vera si fuerit n numerus integer major quam m . Hinc ergo intelligitur quam late pateat doctrina de Seriebus recurrentibus.

68. Si denominator fuerit potestas non binomii sed multinomii, natura Seriei quoque alio modo explicari potest. Sit nempe hæc fractio

$$\frac{1}{(1 - \alpha z - \epsilon z^2 - \gamma z^3 - \delta z^4 - \&c.)^{m+1}} \text{ pro-}$$

posita, erit Series infinita hinc nata

$$1 + \frac{(m+1)}{1} \alpha z + \frac{(m+1)(m+2)}{1 \cdot 2} \alpha^2 z^2 + \frac{(m+1)(m+2)(m+3)}{1 \cdot 2 \cdot 3} \alpha^3 z^3 \\ + \frac{(m+1)}{1} \epsilon z^2 + \frac{(m+1)(m+2)}{1 \cdot 2} 2\alpha\epsilon z^3 + \&c. \\ + \frac{(m+1)}{1} \gamma z^3$$

Ad naturam hujus Seriei penitus inspiciendam, exponatur hæc Series per litteras generales hoc modo:

$$1 + Az + Bz^2 + Cz^3 + \dots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + \\ N z^n + \&c., \text{ ac quilibet coëfficiens } N \text{ ex tot procedentibus,}$$

quot sunt litteræ $\alpha, \epsilon, \gamma, \delta, \&c.$ ita determinabitur ut fit:

$$N = \frac{m+n}{n} \alpha M + \frac{2m+n}{n} \epsilon L + \frac{3m+n}{n} \gamma K + \frac{4m+n}{n} \delta I + \&c.$$

quæ lex continuationis etsi non est constans, sed ab exponents potestatis z pendet, tamen eidem Seriei alia convenit lex progressionis constans, quam denominator evolutus præbet, na-

LIB. I. turæ Serierum recurrentium consentaneam. Illa vero lex non constans tantum locum habet si numerator fractionis fuerit unitas seu quantitas constans; si enim quoque aliquot potestates ipsius z contineret, tum illa lex multo magis fieret complicata, id quod post tradita calculi differentialis principia facilius patebit.

69. Quoniam hætenus posuimus primum denominatoris terminum constantem non esse $= 0$, ejusque loco unitatem collocavimus; nunc videamus cujusmodi Series oriantur, si in denominatore terminus constans evanescat. His casibus ergo Functio fracta hujusmodi formam habebit

$$\frac{a + bz + cz^2 + \mathcal{E}c.}{z(1 - az - bz^2 - \mathcal{E}c.)}, \text{ convertatur ergo, neglecto denominatoris Factore } z, \text{ reliqua fractio } \frac{a + bz + cz^2 + \mathcal{E}c.}{1 - az - bz^2 - \mathcal{E}c.} \text{ in Seriem recurrentem } A + Bz + Cz^2 + Dz^3 + \mathcal{E}c. \text{ atque manifestum est fore } \frac{a + bz + cz^2 + \mathcal{E}c.}{z(1 - az - bz^2 - \gamma z^3 - \mathcal{E}c.)} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \mathcal{E}c. \text{ Simili modo erit } \frac{a + bz + cz^2 + \mathcal{E}c.}{z^2(1 - az - bz^2 - \mathcal{E}c.)} = \frac{A}{z^2} + \frac{B}{z} + C + Dz + Ez^2 + \mathcal{E}c., \text{ atque generatim erit } \frac{a + bz + cz^2 + \mathcal{E}c.}{z^m(1 - az - bz^2 - \gamma z^3 - \mathcal{E}c.)} = \frac{A}{z^m} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + \frac{D}{z^{m-3}} + \mathcal{E}c. \text{ quicunque numerus fuerit exponens } m.$$

70. Quoniam per substitutionem loco z alia variabilis x in Functionem fractam introduci, hocque pacto Functio fracta quævis in innumerabiles formas diversas transmutari potest; hoc modo eadem Functio fracta infinitis modis per Series recurrentes explicari poterit. Sit scilicet proposita hæc fractio $y = \frac{1 + z}{1 - z - z^2}$ & per Seriem recurrentem $y = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \mathcal{E}c.$: ponatur $z = \frac{1}{x}$ erit $y =$

$$= \frac{xx + x}{xx - x - 1} = \frac{-x(1+x)}{1+x-xx}. \text{ Jam } \frac{1+x}{1+x-xx} = 1 + \text{CAP. IV.}$$

$0x + xx - x^3 + 2x^4 - 3x^5 + 5x^6 - \&c.$; unde erit $y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 +$

$\&c.$ Vel ponatur $z = \frac{1-x}{1+x}$, erit $y = \frac{-2-2x}{1-4x-xx}$; un-

de fit $y = -2 - 10x - 42xx - 178x^3 - 754x^4 - \&c.$ cujusmodi Series recurrentes pro y innumerabiles inveniri possunt.

71. Functiones irrationales ex hoc theoremate universali

in Series infinitas transformari solent, quod fit $(P+Q)^{\frac{m}{n}}$

$$= P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n \cdot 2n} P^{\frac{m-2n}{n}} Q^2 + \frac{m(m-n)(m-2n)}{n \cdot 2n \cdot 3n} P^{\frac{m-3n}{n}} Q^3 + \&c. : \text{ hi enim}$$

termini, nisi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m & n numeros definitos scribendo.

$$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2} P^{-\frac{1}{2}} Q - \frac{1 \cdot 1}{2 \cdot 4} P^{-\frac{3}{2}} Q^2 +$$

$$\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} P^{-\frac{5}{2}} Q^3 - \&c.$$

$$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1 \cdot 3}{2 \cdot 4} P^{-\frac{5}{2}} Q^2 -$$

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} P^{-\frac{7}{2}} Q^3 + \&c.$$

$$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3} P^{-\frac{2}{3}} Q - \frac{1 \cdot 2}{3 \cdot 6} P^{-\frac{5}{3}} Q^2 +$$

$$\frac{1 \cdot 2 \cdot 5}{3 \cdot 6 \cdot 19} P^{-\frac{8}{3}} Q^3 - \&c.$$

(P

$$\text{LIB. I. } (P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3} P^{-\frac{4}{3}} Q + \frac{1 \cdot 4}{3 \cdot 6} P^{-\frac{7}{3}} Q^2 - \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} P^{-\frac{10}{3}} Q^3 + \&c.$$

$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3} P^{-\frac{1}{3}} Q - \frac{2 \cdot 1}{3 \cdot 6} P^{-\frac{4}{3}} Q^2 + \frac{2 \cdot 1 \cdot 4}{3 \cdot 6 \cdot 9} P^{-\frac{7}{3}} Q^3 - \&c.$$

&c.

72. Hujusmodi ergo Serierum termini ita progrediuntur ut quilibet ex antecedente formari possit: sit enim Series, quæ ex

$$(P+Q)^{\frac{m}{n}} \text{ nascitur, terminus quilibet} = M P^{\frac{m-kn}{n}} Q^k$$

$$\text{erit sequens} = \frac{m-kn}{(k+1)n} M R^{\frac{m-(k+1)n}{n}} Q^{k+1}.$$

Notandum autem est in quovis termino sequente exponentem ipsius P unitate decrescere, contra vero exponentem ipsius Q unitate crescere. Quo autem hæc facilius ad quemvis casum accom-

modentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi potest $P^{\frac{m}{n}} (1 + \frac{Q}{P})^{\frac{m}{n}}$: evoluta enim formula $(1 + \frac{Q}{P})^{\frac{m}{n}}$ Serieque

resultante per $P^{\frac{m}{n}}$ multiplicata, prodibit ipsa Series ante data. Tum vero si m non solum numeros integros denotet, sed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis, si pro $\frac{Q}{P}$, quæ est Functio ipsius z , ponatur Z , habebitur

$$(1+Z)^{\frac{m}{n}} = 1 + \frac{m}{1} Z + \frac{m(m-1)}{1 \cdot 2} Z^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} Z^3 + \&c.$$

Ad sequentes progressionum leges autem observandas conveniet hanc formulæ generalis in Seriem conversionem notasse

$$\text{tasse } (1+Z)^{m-1} = 1 + \frac{(m-1)}{1} Z + \frac{(m-1)(m-2)}{1 \cdot 2} Z^2 + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} Z^3 + \&c. \quad \text{CAP. IV.}$$

73. Sit igitur primum $Z = az$, eritque $(1+az)^{m-1}$
 $= 1 + \frac{m-1}{1} az + \frac{(m-1)(m-2)}{1 \cdot 2} a^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} a^3 z^3 + \&c.$ Scribatur pro hac
 Serie ista forma generalis

$$1 + Az + Bz^2 + Cz^3 + \dots + Mz^{n-1} + Nz^n + \&c.$$

atque quilibet coëfficiens N ex præcedente M ita determina-
 bitur ut fit $N = \frac{m-n}{n} a M$. Sic, posito $n=1$, cum fit

$$M=1, \text{ erit } N=A = \frac{m-1}{1} a; \text{ tum facto } n=2, \text{ ob}$$

$$M=A = \frac{m-1}{1} a, \text{ erit } N=B = \frac{m-2}{2} a M = \frac{(m-1)(m-2)}{1 \cdot 2} a^2; \text{ similique modo porro } C = \frac{m-3}{3} a B = \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} a^3, \text{ uti Series ante inventa declarat.}$$

$$74. \text{ Sit } Z = az + \epsilon zz, \text{ eritque } (1+az+\epsilon zz)^{m-1} = 1 + \frac{(m-1)}{1} (az+\epsilon zz) + \frac{(m-1)(m-2)}{1 \cdot 2} (az+\epsilon zz)^2 + \&c.$$

Quod si ergo termini secundum potestates ipsius z disponantur

$$\text{erit } (1+az+\epsilon zz)^{m-1} = 1 + \frac{(m-1)}{1} az + \frac{(m-1)(m-2)}{1 \cdot 2} a^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} a^3 z^3 + \&c. \\ + \frac{(m-1)}{1} \epsilon z^2 + \frac{(m-1)(m-2)}{1 \cdot 2} 2a\epsilon z^3 + \&c.$$

LIB. I. Scribatur pro hac Serie ista forma generalis :

$1 + Az + Bz^2 + Cz^3 + \dots + Lz^{n-2} + Mz^{n-1} + Nz^n + \&c.$
 atque quilibet coëfficiens ex duobus antecedentibus ita definie-
 tur ut sit $N = \frac{m-n}{n} \alpha M + \frac{2m-n}{n} \epsilon L$, unde omnes
 termini ex primo, qui est 1, definiri poterunt. Erit nempe

$$A = \frac{m-1}{1} \alpha;$$

$$B = \frac{(m-2)}{2} \alpha A + \frac{(2m-2)}{2} \epsilon$$

$$C = \frac{(m-3)}{3} \alpha B + \frac{(2m-3)}{3} \epsilon A$$

$$D = \frac{(m-4)}{4} \alpha C + \frac{(2m-4)}{4} \epsilon B$$

&c.

75. Si fuerit $Z = \alpha z + \epsilon z^2 + \gamma z^3$, erit $(1 + \alpha z + \epsilon z^2 + \gamma z^3)^{m-1} = 1 + \frac{(m-1)}{1} (\alpha z + \epsilon z^2 + \gamma z^3) + \frac{(m-1)(m-2)}{1 \cdot 2} (\alpha z + \epsilon z^2 + \gamma z^3)^2 + \&c.$, quæ expres-
 sio, si omnes termini secundum potestates ipsius z ordinentur,
 abibit in hanc Seriem :

$$1 + \frac{(m-1)}{1} \alpha z + \frac{(m-1)(m-2)}{1 \cdot 2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} \alpha^3 z^3 + \frac{(m-1)}{1} \epsilon z^2 + \frac{(m-1)(m-2)}{1 \cdot 2} 2 \alpha \epsilon z^3 + \&c. + \frac{(m-1)}{1} \gamma z^3$$

ejus lex progressionis ut melius patefcat, ponatur ejus loco
 $1 + Az + Bz^2 + Cz^3 + \dots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n$,
 ejus Seriei quilibet coëfficiens ex tribus antecedentibus ita deter-
 minatur ut sit $N = \frac{(m-n)}{n} \alpha M + \frac{(2m-n)}{n} \epsilon L + \frac{(3m-n)}{n} \gamma K$.
 Cum

Cum igitur primus terminus sit $= 1$, & antecedentes nulli, CAP. IV.
erit

$$\begin{aligned} A &= \frac{m-1}{1} a \\ B &= \frac{(m-2)}{2} a A + \frac{(2m-2)}{2} \epsilon \\ C &= \frac{(m-3)}{3} a B + \frac{(2m-3)}{3} \epsilon A + \frac{(3m-3)}{3} \gamma \\ D &= \frac{(m-4)}{4} a C + \frac{(2m-4)}{4} \epsilon B + \frac{(3m-4)}{4} \gamma A \\ E &= \frac{(m-5)}{5} a D + \frac{(2m-5)}{5} \epsilon C + \frac{(3m-5)}{5} \gamma B \\ &\quad \&c. \end{aligned}$$

76. Generaliter ergo si ponatur $(1 + az + \epsilon z^2 + \gamma z^3 + \delta z^4 + \&c.)^{m-1} = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \&c.$, hujus Seriei singuli termini ita ex præcedentibus definiuntur, ut sit

$$\begin{aligned} A &= \frac{m-1}{1} a \\ B &= \frac{(m-2)}{2} a A + \frac{(2m-2)}{2} \epsilon \\ C &= \frac{(m-3)}{3} a B + \frac{(2m-3)}{3} \epsilon A + \frac{(3m-3)}{3} \gamma \\ D &= \frac{(m-4)}{4} a C + \frac{(2m-4)}{4} \epsilon B + \frac{(3m-4)}{4} \gamma A + \frac{(4m-4)}{4} \delta \\ E &= \frac{(m-5)}{5} a D + \frac{(2m-5)}{5} \epsilon C + \frac{(3m-5)}{5} \gamma B + \frac{(4m-5)}{5} \delta A + \\ &\quad \frac{(5m-5)}{5} \epsilon \\ &\quad \&c. \end{aligned}$$

quilibet scilicet terminus per tot præcedentes determinatur, quot habentur litteræ $a, \epsilon, \gamma, \delta, \&c.$ in Functione ipsius z cujus potestas in Seriem convertitur. Ceterum ratio hujus legis convenit cum ea, quam supra §. 68. ubi similem formam

$(1 - az - \epsilon z^2 - \gamma z^3 - \&c.)^{m-1}$ in Seriem infinitam

L I B. I. tam resolvimus; si enim loco m scribatur — m atque litteræ $a, b, c, d, &c.$ negative accipiantur, Series inventæ prorsus congruent. Interim hoc loco non licet rationem hujus progressionis legis a priori demonstrare, id quod per principia calculi differentialis demum commode fieri poterit; interea ergo sufficiet veritatem per applicationem ad omnis generis exempla comprobasse.

C A P U T V.

De Functionibus duarum pluriumve variabilium.

77. **Q**Uanquam plures hæcenus quantitates variabiles sumus contemplati, tamen eæ ita erant comparatæ, ut omnes unius essent Functiones, unaque determinata reliquæ simul determinarentur. Nunc autem ejusmodi considerabimus quantitates variabiles, quæ a se invicem non pendent, ita ut quamvis uni determinatus valor tribuatur, reliquæ tamen nihilominus maneant indeterminatæ ac variabiles. Ejusmodi ergo quantitates variabiles, cujusmodi sint x, y, z , ratione significationis convenient, cum quælibet omnes valores determinatos in se complectatur; at, si inter se comparentur maxime erunt diversæ, cum, licet pro una z valor quicumque determinatus substituatur, reliquæ tamen x & y æquale pateant, atque ante. Discrimen ergo inter quantitates variabiles a se pendentes, & non pendentes in hoc versatur, ut priori casu, si una determinetur, simul reliquæ determinantur; posteriori vero determinatio unius significationes reliquarum minime restringat.

78. *Functio ergo duarum pluriumve quantitarum variabilium, x, y, z , est expressio quomodocunque ex his quantitatibus composita.*

Ita erit $x^3 + xyz + az^2$ Functio quantitarum variabilium trium x, y, z . Hæc ergo Functio, si una determinetur variabilis,

riabilis, puta z , hoc est ejus loco constans numerus substitua- CAP. V.
tur, manebit adhuc quantitas variabilis, scilicet Functio ipsa-
rum x & y . Atque si, præter z , quoque y determinetur, tum
erit adhuc Functio ipsius x . Hujusmodi ergo plurium varia-
bilium Functio non ante valorem determinatum obtinebit,
quam singulæ quantitates variabiles fuerint determinatæ. Cum
igitur una quantitas variabilis infinitis modis determinari pos-
sit, Functio duarum variabilium, quia pro quavis determina-
tione unius infinitas determinationes suscipere potest, omnino
infinites infinitas determinationes admittet. Atque in Functio-
ne trium variabilium numerus determinationum erit adhuc in-
finites major; sicque porro crescet pro pluribus variabilibus.

79. *Hujusmodi Functiones plurium variabilium perinde atque
Functiones unius variabilis, commodissime dividuntur in algebrai-
cas ac transcendentes.*

Quarum illæ sunt, in quibus ratio compositionis in solis Al-
gebra operationibus est posita; hæ vero, in quarum forma-
tionem quoque operationes transcendentes ingrediuntur. In
his denuo species notari possent, prout operationes transcen-
dentes vel omnes quantitates variabiles implicant, vel aliquot,
vel tantum unicam. Sic ista expressio $zz + y \log. z$, quia
Logarithmus ipsius z inest, erit quidem Functio transcendens
ipsarum y & z , verum ideo minus transcendens est putanda,
quod si variabilis z determinetur, supersit Functio algebraica
ipsius y . Interim tamen non expedit hujusmodi subdivisionibus
tractationem amplificari.

80. *Functiones deinde algebraica subdividuntur in rationales
& irrationales; rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex Capite primo jam abunde
intelligitur. Functio scilicet rationalis omnino est libera ab om-
ni irrationalitate quantitates variabiles, quarum Functio dici-
tur, afficiente; hæcque erit integra si nullis fractionibus inqui-
netur, contra vero fracta. Sic Functionis integræ duarum va-
riabilium y & z hæc erit forma generalis: $a + by + yz + dy^2 +$
 $eyz + \xi z^2 + \eta z^3 + \theta y^2z + \iota yz^2 + \kappa z^3 + \&c.$ Quod fi

LIB. I. si ergo P & Q denotent hujusmodi Functiones integras, five duarum five plurium variabilium, erit $\frac{P}{Q}$ forma generalis Functionum fractarum. Functio denique irrationalis est vel explicita, vel implicita; illa per signa radicalia jam penitus est evoluta, hæc autem per æquationem irresolubilem exhibetur: sic V erit Functio implicita irrationalis ipsarum y & z , si fuerit $V^5 = (ayz + z^3)V^2 + (y^4 + z^4)V + y^5 + 2ayz^3 + z^5$.

81. *Multiformitas deinde in his Functionibus æque notari debet, atque in iis, quæ ex unica variabili constant.*

Sic Functiones rationales erunt uniformes, quia singulis quantitatibus variabilibus determinatis, unicum valorem determinatum exhibent. Denotent P, Q, R, S , &c. Functiones rationales seu uniformes variabilium x, y, z , eritque V Functio biformis earundem variabilium, si fuerit $V^2 - PV + Q = 0$; quicunque enim valores determinati quantitatibus x, y , & z tribuuntur, Functio V non unum sed duplicem perpetuo habebit valorem determinatum. Simili modo erit V Functio triformis si fuerit $V^3 - PV^2 + QV - R = 0$: atque Functio quadriformis si fuerit $V^4 - PV^3 + QV^2 - RV + S = 0$: hocque modo ratio Functionum multiformium ulteriorum erit comparata.

82. Quemadmodum si Functio unius variabilis z nihilo æqualis ponitur, quantitas variabilis z valorem consequitur determinatum vel simplicem vel multiplicem; ita si Functio duarum variabilium y & z nihilo æqualis ponitur, tum altera variabilis per alteram definitur, ejusque ideo Functio evadit, cum ante a se mutuo non penderent. Simili modo si Functio trium variabilium x, y, z , nihilo æqualis statuatur, tum una variabilis per duas reliquas definitur, earumque Functio existit. Idem evenit si Functio non nihilo sed quantitati constanti vel etiam alii Functioni æqualis ponatur; ex omni enim æquatione, quotcunque variabiles involvat, semper una variabilis per reliquas definitur earumque fit Functio; duæ autem æquationes

tiones diversæ inter easdem variables ortæ binas per reliquas CAP. V. definient, atque ita porro.

83. *Functionum autem duarum pluriumve variabilium divisio maxime notatu digna est in homogeneas & heterogeneas.*

Functio homogenea est per quam ubique idem regnat variabilium numerus dimensionum: Functio autem heterogenea est, in qua diversi occurrunt dimensionum numeri. Censetur vero unaquæque variabilis unam dimensionem constituere; quadratum uniuscujusque atque productum ex duabus, duas; productum ex tribus variabilibus, sive iisdem sive diversis, tres & ita porro; quantitates autem constantes ad dimensionum numerationem non admittuntur. Ita in his formulis ay ; cz , unica dimensio inesse dicitur; in his vero ay^2 ; cyz ; yz^2 duæ insunt dimensiones: in his ay^3 ; cy^2z ; yyz^2 ; dz^3 , tres; in his vero ay^4 ; cy^3z ; yy^2z^2 ; dyz^3 ; ez^4 ; quatuor, sicque porro.

84. Applicemus primum hanc distinctionem ad Functiones integras, atque duas tantum variables inesse ponamus, quoniam plurium par est ratio.

Functio igitur integra erit homogenea in cujus singulis terminis idem existit dimensionum numerus.

Subdividentur ergo hujusmodi Functiones commodissime secundum numerum dimensionum, quem variables in ipsis ubique constituunt. Sic erit $ay + cz$ forma generalis Functionum integranum unius dimensionis: hæc vero expressio $ay^2 + cyz + yz^2$ erit forma generalis Functionum duarum dimensionum, tum forma generalis Functionum trium dimensionum erit: $ay^3 + cy^2z + yyz^2 + dz^3$; quatuor dimensionum vero hæc: $ay^4 + cy^3z + yy^2z^2 + dyz^3 + ez^4$; & ita porro. Ad analogiam igitur erit quantitas constans sola a Functio nullius dimensionis.

85. *Functio porro fracta erit homogenea, si ejus Numerator ac Denominator fuerint Functiones homogeneæ.*

Sic hæc Fractio $\frac{ayy + bzz}{ay + cz}$ erit Functio homogenea ipsa-

rum

LIB. I. rum y & z ; numerus dimensionum autem habebitur, si a numero dimensionum Numeratoris subtrahatur numerus dimensionum Denominatoris: atque ob hanc rationem Fractio allata erit Functio unius dimensionis. Hæc vero Fractio

$\frac{y^5 + z^5}{yy + zz}$ erit Functio trium dimensionum. Quando ergo in

Numeratore ac Denominatore idem dimensionum numerus inest, tum Fractio erit Functio nullius dimensionis, uti evenit in hac Fractioe $\frac{y^3 + z^3}{yyz}$, vel etiam in his $\frac{y}{z}$; $\frac{azz}{yy}$; $\frac{6y^3}{z^3}$. Quod

si igitur in Denominatore plures sint dimensiones quam in Numeratore, numerus dimensionum Fractiois erit negativus:

sic $\frac{y}{zz}$ erit Functio — 1 dimensionis: $\frac{y + z}{y^4 + z^4}$ erit Functio — 3

dimensionum: $\frac{1}{y^5 + ayz^4}$ erit Functio — 5 dimensionum, quia in Numeratore nulla inest dimensio. Ceterum sponte intelligitur plures Functiones homogeneas, in quibus singulis idem regnat dimensionum numerus, sive additas sive subtractas præbere Functionem quoque homogeneam ejusdem dimensionum

numeri. Sic hæc expressio $ay + \frac{6zz}{y} + \frac{yy^4 - z^4}{yyz + yzz}$ erit Fun-

ctio unius dimensionis: hæc autem $a + \frac{6y}{z} + \frac{yzz}{yy} + \frac{yy + zz}{yy - zz}$ erit Functio nullius dimensionis.

86. Natura Functionum homogenearum quoque ad expressiones irrationales extenditur. Si enim fuerit P Functio quæcunque homogenea, puta n dimensionum, tum \sqrt{P} erit Functio $\frac{1}{2} n$ dimensionum; $\sqrt[3]{P}$ erit Functio $\frac{1}{3} n$ dimensionum,

& generatim $P^{\frac{\mu}{\nu}}$ erit Functio $\frac{\mu}{\nu} n$ dimensionum. Sic $\sqrt{(yy + zz)}$ erit Functio unius dimensionis; $\sqrt[3]{(y^3 + z^3)}$ erit Functio trium dimensionum: $(yz + zz)^{\frac{3}{4}}$ erit Functio $\frac{3}{2}$ dimensionum: at-

que

que $\frac{yy + zz}{\sqrt{(y^4 + z^4)}}$ erit Functio nullius dimensionis. His ergo
cum præcedentibus conjunctis intelligetur hæc expressio $\frac{1}{y}$
 $+ \frac{y\sqrt{(yy + zz)}}{z^2} - \frac{y}{\sqrt{(y^6 - z^6)}} + \frac{y\sqrt{z}}{zz\sqrt{y + \sqrt{(y^5 + z^5)}}}$ esse
Functio homogenea — 1 dimensionis.

87. Utrum Functio irrationalis implicita sit homogenea necne, ex his facile colligi potest. Sit V hujusmodi Functio implicita ac $V^3 + PV^2 + QV + R = 0$, existentibus P , Q & R Functionibus ipsarum y & z . Primum igitur patet V Functionem homogeneam esse non posse, nisi P , Q , & R sint Functiones homogeneæ. Præterea vero si ponamus V esse Functionem n dimensionum, erit V^2 Functio $2n$, & V^3 Functio $3n$ dimensionum; cum igitur ubique idem debeat esse numerus dimensionum, oportet, ut P sit Functio n dimensionum, Q Functio $2n$ dimensionum, & R Functio $3n$ dimensionum. Si ergo vicissim litteræ P , Q , R Functiones homogeneæ respective n , $2n$, $3n$ dimensionum, hinc concludetur fore V Functionem n dimensionum. Ita si fuerit $V^3 + (y^4 + z^4)V^2 + ay^5V - z^{10} = 0$ erit V Functio homogenea duarum dimensionum, ipsarum y & z .

88. Si fuerit V Functio homogenea n dimensionum ipsarum y & z , in eaque ponatur ubique $y = uz$, Functio V abit in productum ex potestate z^n in Functionem quandam variabilis u .

Per hanc enim substitutionem $y = uz$, in singulos terminos tantæ inducentur potestates ipsius z , quantæ ante inerant ipsius y . Cum igitur in singulis terminis dimensiones ipsarum y & z conjunctim æquassent numerum n , nunc sola variabilis z ubique habebit n dimensiones, ideoque ubique inerit ejus potestas z^n . Per hanc ergo potestatem Functio V fiet divisibilis & quotus erit Functio variabilem tantum u involvens. Hoc primum patebit in Functionibus integris; si enim sit $V = ay^3 + by^2z + cyz^2 + dz^3$, posito $y = uz$, fiet $V = z^3$
Euleri *Introduct. in Anal. infin. parv.* I $(au^3 +$

LIB. I.

($au^3 + \epsilon u^2 + \gamma u + \delta$). Deinde vero idem manifestum est in fractis: fit enim $V = \frac{ay + \epsilon z}{yy + zz}$, nempe Functio — 1 dimensionis, facto $y = uz$ fiet $V = z^{-1} \left(\frac{au + \epsilon}{uu + 1} \right)$. Neque etiam Functiones irrationales hinc excipiuntur, si enim sit $V = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{(y^3 + z^3)}}$, quæ est Functio — $\frac{3}{2}$ dimensionum; posito $y = uz$, prodibit $V = z^{-\frac{3}{2}} \left(\frac{u + \sqrt{(uu + 1)}}{\sqrt{(u^3 + 1)}} \right)$. Hoc itaque modo Functiones homogeneæ duarum tantum variabilium reducentur ad Functiones unius variabilis; neque enim potestas ipsius z , quia est Factor, Functionem illam ipsius u inquinat.

89. Functio ergo homogenea V duarum variabilium y & z nullius dimensionis, posito $y = uz$, transmutabitur in Functionem unice variabilis u puram.

Cum enim numerus dimensionum sit nullus, Potestas ipsius z , quæ Functionem ipsius u multiplicabit, erit $z^0 = 1$; hocque casu variabilis z prorsus ex computo egredietur. Ita si fuerit $V = \frac{y + z}{y - z}$, facto $y = uz$, orietur $V = \frac{u + 1}{u - 1}$: atque in irrationalibus si sit $V = \frac{y - \sqrt{(yy - zz)}}{z}$ posito $y = uz$ erit $V = u - \sqrt{(uu - 1)}$.

90. Functio integra homogenea duarum variabilium y & z , resolvi poterit in tot Factores simplices formæ $ay + \epsilon z$, quot habuerit dimensiones.

Cum enim Functio sit homogenea, posito $y = uz$, transibit in productum ex z^n in Functionem quandam ipsius u integram, quæ Functio propterea in Factores simplices formæ $au + \epsilon$ resolvi poterit. Multiplicentur singuli Factores hi per z , eritque uniuscujusque forma $auz + \epsilon z = ay + \epsilon z$ ob $uz = y$. Propter multiplicatorem autem z^n , tot hujusmodi Factores nascentur quot exponens n contineat unitates; Factores autem
hi

hi simplices erunt vel reales vel imaginarii, hoc est coëfficien- CAP. V.
tes α , & ζ erunt vel reales vel imaginarii.

Ex hoc itaque sequitur Functionem duarum dimensionum $ayy + byz + czz$ duos habere Factores simplices formæ $\alpha y + \zeta z$; Functio autem $ay^3 + by^2z + cyz^2 + dz^3$ habebit tres Factores simplices formæ $\alpha y + \zeta z$; sicque porro Functionum homogenearum integrarum, quæ plures habent dimensiones, natura erit comparata.

91. Quemadmodum ergo hæc expressio $\alpha y + \zeta z$ continet formam generalem Functionum integrarum unius dimensionis - ita $(\alpha y + \zeta z)(\gamma y + \delta z)$ erit forma generalis Functionum in-
tegrarum duarum dimensionum: atque in hac forma $(\alpha y + \zeta z)(\gamma y + \delta z)(\epsilon y + \xi z)$ continebuntur omnes Functiones integræ trium dimensionum, sicque omnes Functiones integræ homogeneæ per producta ex tot hujusmodi Factoribus $\alpha y + \zeta z$ exhiberi poterunt, quot Functiones illæ contineant dimensio-
nes. Isti autem Factores eodem modo per resolutionem æqua-
tionum reperiuntur, quo supra Factores simplices Functionum integrarum unius variabilis invenire docuimus. Ceterum hæc
proprietas Functionum homogenearum duarum variabilium non
extenditur ad Functiones homogeneas trium pluriumve varia-
bilium: forma enim generalis hujusmodi Functionum duarum
tantum dimensionum, quæ est $ayy + byz + cyx + dxy + exx + fzz$ generaliter non reduci potest ad hujusmodi productum
 $(\alpha y + \zeta z + \gamma x)(\delta y + \epsilon z + \xi x)$; multoque minus Fun-
ctiones plurium dimensionum ad hujusmodi producta revocari
possunt.

92. Ex his, quæ de Functionibus homogeneis sunt dicta, simul intelligitur, quid sit Functio heterogenea: in cujus scili-
cet terminis non ubique idem dimensionum numerus depre-
henditur. Possunt autem Functiones heterogeneæ subdividi
pro multiplicitate dimensionum, quæ in ipsis occurrunt. Sic
Functio bifida erit, in qua duplex dimensionum numerus
occurrit, eritque adeo aggregatum duarum Functionum homo-

LIB. I. generarum, quarum numeri dimensionum differunt; ita $y^5 + 2y^3z^2 + yy + zz$ erit Functio bifida, quia partim quinque, partim duas continet dimensiones. Functio autem trifida est, in qua tres diversi dimensionum numeri insunt, seu quæ in tres Functiones homogeneas distribui possunt, uti $y^6 + y^2z^2 + z^4 + y - z$.

Præterea autem dantur Functiones heterogeneæ fractæ vel irrationales tantopere permixtæ, quæ in Functiones homogeneas resolvi non possunt, cujusmodi sunt $\frac{y^3 + ayz}{by + zz}$,

$$\frac{a + \sqrt{(yy + zz)}}{yy - bz}.$$

93. Interdum Functio heterogenea ope substitutionis idoneæ, vel loco unius vel utriusque variabilis factæ, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quædam attulisse, quibus ejusmodi reductio locum habet. Si scilicet hæc proposita sit Functio $y^5 + zzy + y^3z + \frac{z^3}{y}$; post levem attentionem apparebit, eam ad homogeneitatem perducere, posito $z = xx$: prodibit enim $y^5 + x^4y + y^3xx + \frac{x^6}{y}$, Functio homogenea 5 dimensionum ipsarum x & y . Deinde hæc Functio $y + y^2x + y^3xx + y^5x^4 + \frac{a}{x}$ ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$, prodit enim Functio unius dimensionis $y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az$. Multo difficiliore autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur Functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero qui in Functione inest. Sic $xx + yy + zz + ay - aa$ est Functio secundi ordinis, quia duæ dimensiones occurrunt. Et $y^4 + yz^2 - ay^2z + abyz -$

$abyz - aayy + b^4$ pertinet ad Functiones quarti ordinis. Ad CAP. V. hanc divisionem potissimum in doctrina de lineis curvis respici solet; unde adhuc una Functionum integrarum divisio commemoranda venit.

95. Superest scilicet divisio Functionum integrarum in complexas atque incomplexas. Functio autem complexa est, quæ in Factores rationales resolvi potest, seu quæ est productum ex duabus Functionibus pluribusve rationalibus; cuiusmodi est $y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy$, quæ est productum ex his duabus Functionibus $(yy + zz - az + by)$ $(yy - zz + az - by)$. Ita vidimus omnem Functionem integram homogeneam, quæ tantum duas variables complectatur, esse Functionem complexam, quoniam tot Factores simplices formæ $ay + bz$ habet, quot continet dimensiones. Functio igitur integra erit incomplexa, si in Factores rationales resolvi omnino nequeat; uti $yy + zz - aa$, cuius nullos dari Factores rationales facile intelligitur. Ex inquisitione Divisorum patebit, utrum Functio proposita sit complexa an incomplexa.

CAPUT VI.

De Quantitatibus exponentialibus ac Logarithmis.

96. **Q**uanquam notio Functionum transcendentium in calculo integrali demum perpendetur, tamen antequam eo perveniamus, quasdam species magis obvias, atque ad plures investigationes aditum aperientes, evolvere conveniet. Primum ergo considerandæ sunt quantitates exponentiales, seu Potestates, quarum Exponens ipse est quantitas variabilis. Perspicuum enim est huiusmodi quantitates ad Functiones algebraicas referri non posse, cum in his Exponentes non nisi constantes locum habeant. Multiplices autem sunt quan-

LIB. I. titates exponentiales, prout vel solus Exponens est quantitas
 ——— variabilis, vel præterea etiam ipsa quantitas elevata; prioris
 generis est a^z , hujus vero y^z ; quin etiam ipse Exponens
 potest esse quantitas exponentialis uti in his formis a^{a^z} ; a^{y^z} ;
 y^{a^z} ; x^{y^z} . Hujusmodi autem quantitatum non plura constitue-
 mus genera, cum earum natura satis clarè intelligi queat, si pri-
 mam tantum speciem a^z evolverimus.

97. Sit igitur proposita hujusmodi quantitas exponentialis
 a^z , quæ est Potestas quantitatis constantis a , Exponentem ha-
 bens variabilem z . Cum igitur iste Exponens z , omnes nu-
 meros determinatos in se complectatur, primum patet si loco
 z omnes numeri integri affirmativi successive substituantur, lo-
 co a^z hos prodituros esse valores determinatos a^1 ; a^2 ; a^3 ; a^4 ;
 a^5 ; a^6 ; &c. Sin autem pro z ponantur successive numeri ne-
 gativi — 1, — 2, — 3, &c. prodibunt $\frac{1}{a}$; $\frac{1}{a^2}$; $\frac{1}{a^3}$; $\frac{1}{a^4}$; &c.
 ac, si fuerit $z = 0$, habebitur semper $a^0 = 1$. Quod si lo-
 co z numeri fracti ponantur, ut $\frac{1}{2}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{3}{4}$; &c.
 orientur isti valores \sqrt{a} ; $\sqrt[3]{a}$; $\sqrt[3]{aa}$; $\sqrt[4]{a}$; $\sqrt[4]{a^3}$; &c., qui in se
 spectati geminos pluresve induunt valores, cum radicum ex-
 tractio semper valores multiformes producat. Interim tamen
 hoc loco valores tantum primarii, reales scilicet atque affir-
 mativi admitti solent; quia quantitas a^z tanquam Functio uni-
 formis ipsius z spectatur. Sic $a^{\frac{5}{2}}$ medium quendam tenebit
 locum inter a^2 & a^3 , eritque ideo quantitas ejusdem generis;
 & quamvis valor $a^{\frac{5}{2}}$ sit æque $= -aa\sqrt{a}$, ac $= +aa\sqrt{a}$,
 tamen posterior tantum in censum venit. Eodem modo res
 se habet, si Exponens z valores irrationales accipiat, quibus
 casibus cum difficile sit numerum valorum involutorum conci-
 pere,

pere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor CAP. VI.
determinatus intra limites a^2 & a^3 comprehensus.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri constantis a pendent. Si enim fuerit $a=1$, semper erit $a^z=1$, quicunque valores Exponenti z tribuatur; si autem fuerit $a > 1$, tum valor ipsius a^z eo erunt majores, quo major numerus loco z substituatur, atque adeo, posito $z=\infty$, in infinitum excrefcunt; si fuerit $z=0$, fiet $a^z=1$, & si sit $z < 0$ valores a^z fient unitate minores, quoad posito $z=-\infty$ fiat $a^z=0$. Contrarium evenit si sit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipsius a^z decrescent, crescente z supra 0; crescent vero, si pro z numeri negativi substituuntur. Cum enim sit $a < 1$, erit $\frac{1}{a} > 1$; posito ergo $\frac{1}{a}=b$; erit $a^z=b^{-z}$, unde posterior casus ex priori dijudicari poterit.

99. Si sit $a=0$, ingens saltus in valoribus ipsius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus seu major nihilo, erit perpetuo $a^z=0$; si sit $z=0$ erit $a^0=1$; si autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z=-3$; erit $a^z=0^{-3}=\frac{1}{0^3}=\frac{1}{0}$, ideoque infinitum. Multo majores autem saltus occurrent, si quantitas constans a habeat valorem negativum, puta -2 ; tum enim ponendis loco z numeris integris valores ipsius a^z alternatim erunt affirmativi & negativi, ut ex hac Serie intelligitur

$$a^{-4}; a^{-3}; a^{-2}; a^{-1}; a^0; a^1; a^2; a^3; a^4; \&c.$$

+

LIB. I. $+\frac{1}{16}; -\frac{1}{8}; +\frac{1}{4}; -\frac{1}{2}; 1; -2; +4; -8; +16.$

Præterea vero si Exponenti z valores tribuantur fracti, Potestas $a^z = (-2)^z$ mox reales mox imaginarios induet valores, erit enim $a^{\frac{1}{2}} = \sqrt{-2}$, imaginarium; at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, si Exponenti z tribuantur valores irrationales, Potestas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a substituendorum commemoratis, statuamus a esse numerum affirmativum, & unitate quidem majorem, quia huc quoque illi casus, quibus a est numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z substituendo omnes numeros reales, qui intra limites $+\infty$ & $-\infty$ continentur, y adipiscetur omnes valores affirmativos intra limites $+\infty$ & 0 contentos. Si enim sit $z = \infty$ erit $y = \infty$; si $z = 0$ erit $y = 1$, & si $z = -\infty$ fiet $y = 0$. Vicissim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis respondens pro z ita ut sit $a^z = y$; sin autem ipsi y tribueretur valor negativus, Exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y Functio quædam ipsius z , & quemadmodum y a z pendeat, ex natura Potestatum facile intelligitur; hinc enim quicunque valor ipsi z tribuatur, valor ipsius y determinatur. Erit autem $yy = a^{2z}$; $y^3 = a^{3z}$; & generaliter erit $y^n = a^{nz}$; unde sequitur fore $\sqrt{y} = a^{\frac{1}{2}z}$; $\sqrt[3]{y} = a^{\frac{1}{3}z}$ & $\frac{1}{y} = a^{-z}$; $\frac{1}{yy} = a^{-2z}$; & $\frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$, & ita porro. Præterea, si fuerit $v = a^x$ erit $vy = a^{x+z}$ & $\frac{v}{y} = a^{x-z}$, quorum subsidiorum beneficio eo facilius valor ipsius y ex dato valore ipsius z inveniri potest.

E X E M-

E X E M P L U M.

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promptu erit valores ipsius y exhibere, si quidem pro z numeri integri ponantur. Erit enim $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$; $10^4 = 10000$; & $10^0 = 1$; item $10^{-1} = \frac{1}{10} = 0,1$; $10^{-2} = \frac{1}{100} = 0,01$; $10^{-3} = \frac{1}{1000} = 0,001$: fin autem pro z Fractiones ponantur, ope radicum extractionis valores ipsius y indicari possunt: sic erit $10^{\frac{1}{2}} = \sqrt{10} = 3,162277$, &c.

102. Quemadmodum autem, dato numero a , ex quovis valore ipsius z reperiri potest valor ipsius y , ita vicissim, dato valore quocunque affirmativo ipsius y , conveniens dabitur valor ipsius z , ut sit $a^z = y$; iste autem valor ipsius z , quatenus tanquam Functio ipsius y spectatur, vocari solet LOGARITHMUS ipsius y . Supponit ergo doctrina Logarithmorum numerum certum constantem loco a substituendum, qui propterea vocatur *basis* Logarithmorum; qua assumpta erit Logarithmus cujusque numeri y Exponens Potestatis a^z , ita ut ipsa Potestas a^z æqualis sit numero illi y ; indicari autem Logarithmus numeri y solet hoc modo ly . Quod si ergo fuerit $a^z = y$, erit $z = ly$: ex quo intelligitur, basin Logarithmorum, etiamsi ab arbitrio nostro pendeat, tamen esse debere numerum unitate majorem: hincque nonnisi numerorum affirmativorum Logarithmos realiter exhiberi posse.

103. Quicumque ergo numerus pro basi Logarithmica a accipiatur, erit semper $l1 = 0$; si enim in æquatione $a^z = y$, quæ convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$. Deinde numerorum unitate majorum Logarithmi erunt affirmativi, pendentes a valore basis a , sic erit $la = 1$; $laa = 2$; $la^3 = 3$;

Euleri *Introduc. in Anal. infin. parv.* K $la^4 = 4$,

LIB. I. $la^4 = 4$; &c., unde a posteriori intelligi potest, quantus numerus pro basi sit assumptus, scilicet ille numerus, cujus Logarithmus est $= 1$, erit basis Logarithmica. Numerorum autem unitate minorum, affirmativorum tamen, Logarithmi erunt negativi; erit enim $l \frac{1}{a} = -1$; $l \frac{1}{a a} = -2$; $l \frac{1}{a^3} = -3$, &c.; numerorum autem negativorum Logarithmi non erunt reales, sed imaginarii, uti jam notavimus.

104. Simili modo si fuerit $ly = z$; erit $lyy = 2z$; $ly^3 = 3z$; & generaliter $ly^n = nz$, seu $ly^n = nly$, ob $z = ly$. Logarithmus igitur cujusque Potestatis ipsius y æquatur Logarithmo ipsius y per Exponentem Potestatis multiplicato; sic erit $l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly$; $l \frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$; & ita porro; unde ex dato Logarithmo cujusque numeri inveniri possunt Logarithmi quarumcunque ipsius Potestatum. Sin autem jam inventi sint duo Logarithmi, nempe $ly = z$ & $lv = x$: cum sit $y = a^z$ & $v = a^x$ erit $luy = x + z = lv + ly$; hinc Logarithmus Producti duorum numerorum æquatur summæ Logarithmorum Factorum; simili vero modo erit $l \frac{y}{v} = z - x = ly - lv$; hincque Logarithmus Fractionis æquatur Logarithmo Numeratoris dempto Logarithmo Denominatoris, quæ regulæ inserviunt Logarithmis plurium numerorum inveniendis, ex cognitis jam aliquot Logarithmis.

105. Ex his autem patet aliorum numerorum non dari Logarithmos rationales, nisi Potestatum baseos a ; nisi enim numerus alius b fuerit Potestas basis a , ejus Logarithmus numero rationali exprimi non poterit. Neque vero etiam Logarithmus ipsius b erit numerus irrationalis; si enim foret $lb = \sqrt[n]{n}$, tum esset $a^{\sqrt[n]{n}} = b$; id quod fieri nequit, si quidem numeri a & b rationales statuantur; solent autem imprimis numerorum rationa-

tionalium & integrorum Logarithmi desiderari, quia ex his Logarithmi Fractionum ac numerorum surdorum inveniri possunt. Cum igitur Logarithmi numerorum, qui non sunt Potestates basis a , neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates transcendentes referuntur, hincque Logarithmi quantitatibus transcendentes annumerari solent.

106. Hanc ob rem Logarithmi numerorum vero tantum proxime per Fractiones decimales exprimi solent, qui eo minus à veritate discrepabunt, ad quo plures figuras fuerint exacti. Atque hoc modo per solam radicis quadratæ extractionem cuiusque numeri Logarithmus vero proxime determinari poterit.

Cum enim, posito $ly = z$ & $lv = x$, sit $lvv = \frac{x+z}{2}$; si numerus propositus b contineatur intra limites a^2 & a^3 , quorum Logarithmi sunt 2 & 3, quæratuur valor ipsius $a^{2\frac{1}{2}}$ seu $a^2 \sqrt{a}$, atque b vel intra limites a^2 & $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ & a^3 continebitur, utrumvis accadat, sumendo medio proportionali, denuo limites propiores prodibunt, hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat, & quibuscum numerus propositus b sine errore confundi possit. Quoniam vero horum singulorum limitum Logarithmi dantur, tandem Logarithmus numeri b reperietur.

EXEMPLUM.

Ponatur basis Logarithmica $a = 10$, quod in tabulis usu receptis fieri solet; & quæratuur vero tantum proxime Logarithmus numeri 5; quia hic continetur intra limites 1 & 10 quorum Logarithmi sunt 0 & 1; sequenti modo radicum extractio continua instituat, quoad ad limites à numero propositio 5 non amplius discrepantes perveniatur.

LIB. I.	$A = 1, 000000;$	$lA = 0, 0000000$	fit
—	$B = 10, 000000;$	$lB = 1, 0000000;$	$C = \sqrt{AB}$
	$C = 3, 162277;$	$lC = 0, 5000000;$	$D = \sqrt{BC}$
	$D = 5, 623413;$	$lD = 0, 7500000;$	$E = \sqrt{CD}$
	$E = 4, 216964;$	$lE = 0, 6250000;$	$F = \sqrt{DE}$
	$F = 4, 869674;$	$lF = 0, 6875000;$	$G = \sqrt{DF}$
	$G = 5, 232991;$	$lG = 0, 7187500;$	$H = \sqrt{FG}$
	$H = 5, 048065;$	$lH = 0, 7031250;$	$I = \sqrt{FH}$
	$I = 4, 958069;$	$lI = 0, 6953125;$	$K = \sqrt{HI}$
	$K = 5, 002865;$	$lK = 0, 6992187;$	$L = \sqrt{IK}$
	$L = 4, 980416;$	$lL = 0, 6972656;$	$M = \sqrt{KL}$
	$M = 4, 991627;$	$lM = 0, 6982421;$	$N = \sqrt{KM}$
	$N = 4, 997142;$	$lN = 0, 6987304;$	$O = \sqrt{KN}$
	$O = 5, 000052;$	$lO = 0, 6989745;$	$P = \sqrt{NO}$
	$P = 4, 998647;$	$lP = 0, 6988525;$	$Q = \sqrt{OP}$
	$Q = 4, 999350;$	$lQ = 0, 6989135;$	$R = \sqrt{OQ}$
	$R = 4, 999701;$	$lR = 0, 6989440;$	$S = \sqrt{OR}$
	$S = 4, 999876;$	$lS = 0, 6989592;$	$T = \sqrt{OS}$
	$T = 4, 999963;$	$lT = 0, 6989668;$	$V = \sqrt{OT}$
	$V = 5, 000008;$	$lV = 0, 6989707;$	$W = \sqrt{TV}$
	$W = 4, 999984;$	$lW = 0, 6989687;$	$X = \sqrt{WV}$
	$X = 4, 999997;$	$lX = 0, 6989697;$	$Y = \sqrt{VX}$
	$Y = 5, 000003;$	$lY = 0, 6989702;$	$Z = \sqrt{XY}$
	$Z = 5, 000000;$	$lZ = 0, 6989700;$	

Sic ergo mediis proportionalibus sumendis tandem perventum est ad $Z = 5, 000000$, ex quo Logarithmus numeri 5 quaesitus est $0, 698970$, posita basi Logarithmica $= 10$. Quare erit

proxime $10^{\frac{69897}{100000}} = 5$. Hoc autem modo computatus est canon Logarithmorum vulgaris à BRIGGIO & VLACQUIO, quamquam postea eximia inventa sunt compendia, quorum ope multo expeditius Logarithmi supputari possunt.

107. Dantur ergo tot diversa Logarithmorum systemata quot varii numeri pro basi α accipi possunt, atque ideo numerus systema-

tematum Logarithmicorum erit infinitus. Perpetuo autem in CAP. VI. duobus systematis Logarithmi ejusdem numeri eandem inter se servant rationem. Sit basis unius systematis $= a$, alterius $= b$, atque numeri n Logarithmus in priori systemate $= p$, in posteriori $= q$; erit $a^p = n$ & $b^q = n$; unde $a^p = b^q$; ideoque $a = b^{\frac{q}{p}}$. Oportet ergo ut Fractio $\frac{q}{p}$ constantem obtineat valorem, quicumque numerus pro n fuerit assumptus. Quod si ergo pro uno systemate Logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream Logarithmi pro quovis alio systemate reperiri possunt. Sic, cum dentur Logarithmi pro basi 10, hinc Logarithmi pro quavis alia basi, puta 2, inveniri possunt; quaratur enim Logarithmus numeri n pro basi 2, qui sit $= q$, cum ejusdem numeri n Logarithmus sit $= p$ pro basi 10. Quoniam pro basi 10 est $\lg 2 = 0,3010300$, & pro basi 2, est $\lg 2 = 1$, erit $0,3010300 : 1 = p : q$ ideoque $q = \frac{p}{0,3010300} = 3,3219277 \cdot p$, si ergo omnes Logarithmi communes multiplicentur per numerum 3,3219277, prodibit tabula Logarithmorum pro basi 2.

108. *Hinc sequitur duorum numerorum Logarithmos in quocunque systemate eandem tenere rationem*

Sint enim duo numeri M & N , quorum pro basi a Logarithmi sint m & n , erit $M = a^m$ & $N = a^n$: hinc fiet

$m^m = M^n = N^m$, ideoque $M = N^{\frac{m}{n}}$; in qua æquatione cum basis a non amplius insit, perspicuum est Fractionem $\frac{m}{n}$ habere valorem à basi a non pendentem. Sint enim pro alia basi b numerorum eorundem M & N Logarithmi μ & ν ,

pari modo colligetur fore $M = N^{\frac{\mu}{\nu}}$. Erit ergo $N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$, hincque $\frac{m}{n} = \frac{\mu}{\nu}$, seu $m : n = \mu : \nu$. Ita jam vidimus

LIB. I. in omni Logarithmorum systemate Logarithmos diversarum ejusdem numeri Potestatum ut y^m & y^n tenere rationem Exponentium $m:n$.

109. Ad canonem ergo Logarithmorum pro basi quacunque a condendum sufficit numerorum tantum primorum Logarithmos methodo ante tradita, vel alia commodiori, supputasse. Cum enim Logarithmi numerorum compositorum sint æquales summis Logarithmorum singulorum Factorum, Logarithmi numerorum compositorum per solam additionem reperientur. Sic, si habeantur Logarithmi numerorum 3 & 5, erit $l_{15} = l_3 + l_5$; $l_{45} = 2 l_3 + l_5$. Atque, cum supra pro basi $a = 10$, inventus sit $l_5 = 0,6989700$, præterea autem sit $l_{10} = 1$ erit $l_{\frac{10}{5}} = l_2 = l_{10} - l_5$, ideoque oriatur $l_2 = 1 - 0,6989700 = 0,3010300$; ex his autem numerorum primorum 2 & 5 Logarithmis inventis reperientur Logarithmi omnium numerorum ex his 2 & 5 compositorum; cujusmodi sunt isti 4, 8, 16, 32, 64, &c; 20, 40, 80, 25, 50; &c.

110. Tabularum autem Logarithmicarum amplissimus est usus in contrahendis calculis numericis, propterea quod ex ejusmodi tabulis non solum dati cujusque numeri Logarithmus, sed etiam cujusque Logarithmi propositi numerus conveniens reperiri potest. Sic, si c, d, e, f, g, h , denotent numeros quoscunque, citra multiplicationem reperiri poterit valor istius expressionis $\frac{cc d \sqrt{e}}{f \sqrt[3]{g h}}$, erit enim hujus expressionis Logarithmus $= 2 l_c + l_d + \frac{1}{2} l_e - l_f - \frac{1}{3} l_g - \frac{1}{3} l_h$, cui Logarithmo si quæratur numerus respondens, habebitur valor quæsitus. Inprimis autem inserviunt tabulæ Logarithmicæ dignitatibus atque radicibus intricatissimis inveniendis, quarum operationum loco in Logarithmis tantum multiplicatio & divisio adhibetur.

EXEM-

EXEMPLUM I.

Quærat^{ur} valor hujus Potestatis $2^{\frac{7}{12}}$: quoniam ejus Logarithmus est $= \frac{7}{12} l 2$, multiplicetur Logarithmus binarii ex tabulis qui est 0,3010300 per $\frac{7}{12}$ hoc est per $\frac{1}{2} + \frac{1}{12}$ erit, $l 2^{\frac{7}{12}} = 0,1756008$, cui Logarithmo respondet numerus 1,498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM II.

Si numerus incolarum cujuspiam provinciæ quotannis sui parte trigesima augeatur, initio autem in provincia habitaverint 100000 hominum, quæritur post 100 annos incolarum numerus. Sit brevitatis gratia initio incolarum numerus $= n$, ita ut sit $n = 100000$, anno elapso uno erit incolarum numerus $= (1 + \frac{1}{30})n = \frac{31}{30} n$: post duos annos $= (\frac{31}{30})^2 n$: post tres annos $= (\frac{31}{30})^3 n$, hincque post centum annos $= (\frac{31}{30})^{100} n = (\frac{31}{30})^{100} 100000$; cujus Logarithmus est $= 100 l \frac{31}{30} + l 100000$. At est $l \frac{31}{30} = l 31 - l 30 = 0,014240439$, unde $100 l \frac{31}{30} = 1,4240439$, ad quem si addatur $l 100000 = 5$, erit Logarithmus numeri incolarum quæsitⁱ $= 6,4240439$, cui respondet numerus $= 2654874$. Post centum ergo annos numerus incolarum fit plus quam vices sexies cum semisse major.

EXEM-

EXEMPLUM III.

Cum post diluvium à sex hominibus genus humanum sit propagatum, si ponamus ducentis annis post, numerum hominum jam ad 1000000 excrevisse, quæritur quanta sui parte numerus hominum quotannis augeri debuerit. Ponamus hoc tempore numerum hominum parte sua $\frac{1}{x}$ quotannis increvisse, atque post ducentos annos prodierit necesse est numerus hominum $= (\frac{1+x}{x})^{200} 6 = 1000000$, unde fit $\frac{1+x}{x} = (\frac{1000000}{6})^{\frac{1}{200}}$. Erit ergo $\frac{1+x}{x} = \frac{1}{200} \frac{1000000}{6} = \frac{1}{200} 5, 2218487 = 0, 0261092$, ideoque $\frac{1+x}{x} = \frac{1061963}{1000000}$, & $1000000 = 61963x$, unde fit $x = 16$ circiter. Ad tantam ergo hominum multiplicationem suffecisset, si quotannis decima sexta sui parte increverint; quæ multiplicatio ob longævam vitam non nimis magna censi potest. Quod si autem eadem ratione per intervallum 400 annorum numerus hominum crescere perrexisset, tum numerus hominum ad 1000000. $\frac{1000000}{6} = 166666666666$ ascendere debuisset, quibus sustentandis universus orbis terrarum nequaquam par fuisset.

EXEMPLUM IV.

Si singulis seculis numerus hominum duplicetur, quæritur incrementum annuum. Si quotannis hominum numerum parte sua $\frac{1}{x}$ crescere ponamus, & initio numerus hominum fuerit $= n$,

erit is post centum annos $= (\frac{1+x}{x})^{100} n$, qui cum esse debeat

beat $= 2n$, erit $\frac{1+x}{x} = 2^{\frac{1}{100}}$ & $l \frac{1+x}{x} = \frac{1}{100} l 2 =$

$0,0030103$; hinc $\frac{1+x}{x} = \frac{10069555}{10000000}$; ergo $x =$

$\frac{10000000}{69555} = 144$, circiter; sufficit ergo si numerus hominum

quotannis parte sua $\frac{1}{144}$ augeatur. Quam ob causam maxime ridiculæ sunt eorum incredulorum hominum objectiones, qui negant tam brevi temporis spatio ab uno homine universam terram incolis impleri potuisse.

III. Potissimum autem Logarithmorum usus requiritur ad ejusmodi æquationes resolvendas, in quibus quantitas incognita in Exponentem ingreditur. Sic, si ad hujusmodi perveniatur æquationem $a^x = b$, ex qua incognitæ x valorem erui oporteat, hoc non nisi per Logarithmos effici poterit. Cum enim fit $a^x = b$ erit $l a^x = x l a = l b =$ ideoque $x = \frac{l b}{l a}$, ubi quidem perinde est, quonam systemate Logarithmico utatur, cum in omni systemate Logarithmi numerorum a & b eandem inter se teneant rationem.

EXEMPLUM I.

Si numerus hominum quotannis centesima sui parte augeatur; quæritur post quot annos numerus hominum fiat decuplo major. Ponamus hoc evenire post x annos, & initio hominum numerum fuisse $= n$, erit is ergo elapsis x annis $=$

$(\frac{101}{100})^x n$, qui cum æqualis sit $10n$, fiet $(\frac{101}{100})^x = 10$;

ideoque $x l \frac{101}{100} = l 10$ & $x = \frac{l 10}{l 101 - l 100}$. Prodibit ita-

que $x = \frac{10000000}{43214} = 231$. Post annos ergo 231 fiet homi-

Euleri *Introduct. in Anal. infin. parv.*

L

num

LIB. I. num numerus, quorum incrementum annum tantum centesimam partem efficit, decuplo major; hinc post 462 annos fiet centies, & post 693 annos millies major.

EXEMPLUM. II.

Quidam debet 400000 florenos hac conditione ut quotannis usuram 5 de centenis solvere teneatur; exsolvit autem singulis annis 25000 florenos: quæritur post quot annos debitum penitus extinguatur. Scribamus a pro debita summa 400000 fl. & b pro summa 25000 fl. quotannis soluta; debebit ergo elapso uno anno $\frac{105}{100} a - b$; elapsis duobus annis $(\frac{105}{100})^2 a - \frac{105}{100} b - b$; elapsis tribus annis $(\frac{105}{100})^3 a - (\frac{105}{100})^2 b - \frac{105}{100} b - b$; hinc, posito brevitatis causa, n pro $\frac{105}{100}$, elapsis x annis adhuc debebit $n^x a - n^{x-1} b - n^{x-2} b - n^{x-3} b - \dots - b = n^x a - b(1 + n + n^2 + \dots + n^{x-1})$. Cum igitur sit ex natura progressionum geometricarum, $1 + n + n^2 + \dots + n^{x-1} = \frac{n^x - 1}{n - 1}$, post x annos debitor adhuc debebit $n^x a - \frac{n^x b - b}{n - 1}$ flor., quod debitum nihilo æquale positum dabit hanc æquationem $n^x a = \frac{n^x b - b}{n - 1}$, seu $(n - 1) n^x a = n^x b - b$, ideoque $(b - na + a) n^x = b$ & $n^x = \frac{b}{b - (n - 1)a}$, unde fit $x = \frac{\ln b - \ln(b - (n - 1)a)}{\ln n}$. Cum jam sit $a = 400000$, $b = 25000$, $n = \frac{105}{100}$, erit $(n - 1) a = 20000$ & $b - (n - 1) a = 5000$, atque annorum, quibus debitum penitus extinguatur, numerus $x =$

$$\frac{l 25000 - l 5000}{l \frac{105}{100}} = \frac{l 5}{l \frac{21}{20}} = \frac{6989700}{211893}; \text{ erit ergo } x \text{ aliquanto mi-}$$

nor quam 33; scilicet elapsis annis 33 non solum debitum ex-
tinguetur, sed creditor debitori reddere tenebitur $\frac{(n^{33} - 1)b}{n - 1}$

$$- n^{33} a = \frac{(\frac{21}{20})^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000 (\frac{21}{20})^{33} - 500000$$

flor. Quia vero est $l \frac{21}{20} = 0,0211892991$, erit $l (\frac{21}{20})^{33} =$
 $0,69924687$, & $l 100000 (\frac{21}{20})^{33} = 5,6992469$, cui respon-
det hic numerus 500318,8; unde creditor debitori post 33 an-
nos restituere debet $318 \frac{4}{5}$ florenos.

112. Logarithmi autem vulgares super basi $= 10$ extracti,
præter hunc usum, quem Logarithmi in genere præstant, in
Arithmetica decimali usu recepta singulari gaudent commodo,
atque ob hanc causam præ aliis systematibus insignem afferunt
utilitatem. Cum enim Logarithmi omnium numerorum, præ-
ter denarii Potestates, in Fractionibus decimalibus exhibeantur,
numerorum inter 1 & 10 contentorum Logarithmi intra limi-
tes 0 & 1, numerorum autem inter 10 & 100 contentorum
Logarithmi inter limites 1 & 2, & ita porro, continebuntur.
Constat ergo Logarithmus quisque ex numero integro & Fra-
ctione decimali; & ille numerus integer vocari solet CHA-
RACTERISTICA; Fractio decimalis autem MANTISSA.
Characteristica itaque unitate deficiet a numero notarum, qui-
bus numerus constat; ita Logarithmi numeri 78509 Characte-
ristica erit 4, quia is ex quinque notis seu figuris constat.
Hinc ex Logarithmo cujusvis numeri statim intelligitur, ex quot
figuris numerus sit compositus. Sic numerus Logarithmo
7,5804631 respondens ex 8 figuris constabit.

113. Si ergo duorum Logarithmorum Mantissæ conveniant,
Characteristica vero tantum discrepent, tum numeri his Loga-

LIB. I. rithmis respondentes rationem habebunt, ut Potestas denarii ad unitatem, ideoque ratione figurarum, quibus constant, convenient. Ita horum Logarithmorum 4, 9130187 & 6 9130187 numeri erunt 81850 & 8185000; Logarithmo autem 3 9130187 conveniet 8185, & Logarithmo huic 0, 9130187 convenit 8, 185. Sola ergo Mantissa indicabit figuras numerum componentes, quibus inventis, ex Characteristica patebit, quot figuræ a sinistra ad integra referri debeant, reliquæ ad dextram vero dabunt Fractiones decimales. Sic, si hic Logarithmus fuerit inventus 2, 7603429, Mantissa indicabit has figuras 5758945, Characteristica 2 autem numerum illi Logarithmo determinat, ut sit 575, 8945; si Characteristica esset 0, foret numerus 5, 758945; sin denuo unitate minuatur ut sit — 1, erit numerus respondens decies minor, nempe 0, 5758945; & Characteristicæ — 2 respondebit 0, 05758945 &c.: loco Characteristicarum autem hujusmodi negativarum — 1, — 2, — 3, &c. scribi solent 9, 8, 7, &c., atque subintelligitur hos Logarithmos denario minui debere. Hæc vero in manductionibus ad tabulas Logarithmorum fusius exponi solent.

EXEMPLUM.

Si hæc progressio 2, 4, 16, 256, &c., cujus quisque terminus est quadratum precedentis, continuetur usque ad terminum vigesimum quintum; quaritur magnitudo hujus termini ultimi. Termini hujus progressionis per Exponentes ita commodius exprimuntur: 2^1 , 2^2 , 2^4 , 2^8 , &c. ubi patet Exponentes progressionem geometricam constituere, atque termini vigesimi quinti exponentem fore $2^{24} = 16777216$, ita ut ipse terminus quæsitus sit $= 2^{16777216}$, hujus ergo Logarithmus erit $= 16777216. l_2$. Cum ergo sit $l_2 = 0, 301029995663981195$, erit numeri quæsitæ Logarithmus $= 5050445, 25973367$, ex cujus Characteristica patet numerum quæsitum more solito expressum constare ex 5050446 figuris. Mantissa autem 259733675932 in tabula

la Logarithmorum quæ sita dabit figuras initiales numeri quæ- CAP. VI.
 siti, quæ erunt 181858. Quanquam ergo iste numerus nullo
 modo exhiberi potest, tamen affirmari potest cum omnino ex
 5050446 figuris constare, atque figuras initiales sex esse 181858,
 quas dextrorsum adhuc 5050440 figuræ sequantur, quarum
 insuper nonnullæ ex majori Logarithmorum canone definiri pos-
 sent, undecim scilicet figuræ initiales erunt 18185852986.

C A P U T V I I.

*De quantitatibus exponentialium ac Logarithmorum
 per Series explicatione.*

114. **Q**uia est $a^0 = 1$, atque crescente Exponente ipsius
 a simul valor Potestatis augetur, si quidem a est
 numerus unitate major; sequitur si Exponens infinite parum
 cyphram excedat, Potestatem ipsam quoque infinite parum
 unitatem esse superaturam. Sit ω numerus infinite parvus, seu
 Fractio tam exigua, ut tantum non nihilo sit æqualis, erit
 $a^\omega = 1 + \psi$, existente ψ quoque numero infinite parvo. Ex
 præcedente enim capite constat nisi ψ esset numerus infinite
 parvus, neque ω talem esse posse. Erit ergo vel $\psi = \omega$, vel
 $\psi > \omega$, vel $\psi < \omega$, quæ ratio utique a quantitate litteræ a
 pendeat, quæ cum adhuc sit incognita, ponatur $\psi = k\omega$,
 ita ut sit $a^\omega = 1 + k\omega$; &, sumpta a pro basi Logarithmica,
 erit $\omega = l(1 + k\omega)$.

E X E M P L U M.

Quo clarius appareat, quemadmodum numerus k pendeat a
 basi a , ponamus esse $a = 10$; atque ex tabulis vulgaribus
 quæramus Logarithmum numeri quam minime unitatem supe-

LIB. I. rantis, puta $1 + \frac{1}{1000000}$, ita ut sit $k \omega = \frac{1}{1000000}$; erit

$$l\left(1 + \frac{1}{1000000}\right) = l\frac{1000001}{1000000} = 0,00000043429 = \omega. \text{ Hinc,}$$

$$\text{ob } k \omega = 0,000001000000, \text{ erit } \frac{1}{k} = \frac{43429}{100000} \text{ \& } k =$$

$$\frac{100000}{43429} = 2,30258 : \text{ unde patet } k \text{ esse numerum finitum pen-}$$

dentem a valore basis a . Si enim alius numerus pro basi a statuatur, tum Logarithmus ejusdem numeri $1 + k \omega$ ad priorem datam tenebit rationem, unde simul alius valor litteræ k prodiret.

115. Cum sit $a^\omega = 1 + k \omega$, erit $a^{i\omega} = (1 + k \omega)^i$, quicunque numerus loco i substituatur. Erit ergo $a^{i\omega} = 1 + \frac{i}{1} k \omega + \frac{i(i-1)}{1.2} k^2 \omega^2 + \frac{i(i-1)(i-2)}{1.2.3} k^3 \omega^3 + \&c.$

Quod si ergo statuatur $i = \frac{z}{\omega}$, & z denotet numerum quemcunque finitum, ob ω numerum infinite parvum, fiet i numerus infinite magnus, hincque $\omega = \frac{z}{i}$, ita ut sit ω Fractio denominatorem habens infinitum, adeoque infinite parva, qualis est assumpta. Substituatur ergo $\frac{z}{i}$ loco ω , eritque $a^{\frac{z}{i}} = \left(1 + \frac{kz}{i}\right)^i = 1 + \frac{1}{1} k z + \frac{1(i-1)}{1.2i} k^2 z^2 + \frac{1(i-1)(i-2)}{1.2i.3i} k^3 z^3 + \frac{1(i-1)(i-2)(i-3)}{1.2i.3i.4i} k^4 z^4 + \&c.$, quæ æquatio erit vera si pro i numerus infinite magnus substituatur. Tum vero est k numerus definitus ab a pendens, uti modo vidimus.

116. Cum autem i sit numerus infinite magnus, erit $\frac{i-1}{i} = 1$; patet enim quo major numerus loco i substituatur, eo propius valorem Fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum, hinc si
 i sit

i fit numerus omni assignabili major, Fractio quoque $\frac{i-1}{i}$ CAP. VII.
 ipsam unitatem adæquabit. Ob similem autem rationem erit
 $\frac{i-2}{i} = 1; \frac{i-3}{i} = 1; \& \text{ ita porro; hinc sequitur fore}$
 $\frac{i-1}{2i} = \frac{1}{2}; \frac{i-2}{3i} = \frac{1}{3}; \frac{i-3}{4i} = \frac{1}{4}; \& \text{ ita porro. His}$
 igitur valoribus substitutis, erit $a^z = 1 + \frac{kz}{1} + \frac{k^2 z^2}{1.2} + \frac{k^3 z^3}{1.2.3} +$
 $\frac{k^4 z^4}{1.2.3.4} + \&c. \text{ in infinitum.}$ Hæc autem æquatio simul re-
 lationem inter numeros a & k ostendit, posito enim $z = 1$,
 erit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} + \&c.$, atque
 ut a sit $= 10$, necesse est ut sit circiter $k = 2,30258$, uti
 ante invenimus.

117. Ponamus esse $b = a^n$, erit, sumto numero a pro basi
 Logarithmica, $lb = n$. Hinc, cum sit $b^z = a^{nz}$, erit per Se-
 riem infinitam $b^z = 1 + \frac{k n z}{1} + \frac{k^2 n^2 z^2}{1.2} + \frac{k^3 n^3 z^3}{1.2.3} + \frac{k^4 n^4 z^4}{1.2.3.4} +$
 $\&c.$, posito vero lb pro n , erit $b^z = 1 + \frac{k z}{1} lb + \frac{k^2 z^2}{1.2} (lb)^2 +$
 $\frac{k^3 z^3}{1.2.3} (lb)^3 + \frac{k^4 z^4}{1.2.3.4} (lb)^4 + \&c.$ Cognito ergo valore
 litteræ k ex dato valore basis a , quantitas exponentialis quæ-
 cunque b^z per Seriem infinitam exprimi poterit, cujus termini
 secundum Potestates ipsius z procedant. His expositis osten-
 damus quoque quomodo Logarithmi per Series infinitas ex-
 plicari possint.

118. Cum sit $a^\omega = 1 + k\omega$, existente ω Fractione infinite
 parva, atque ratio inter a & k definiatur per hanc æquatio-
 nem $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \&c.$, si a sumatur pro
 basi Logarithmica, erit $\omega = l(1 + k\omega)$ & $i\omega = l(1 + k\omega)^i$.
Mani-

LIB. I. Manifestum autem est, quo major numerus pro i sumatur, eo magis Potestatem $(1 + k\omega)^i$ unitatem esse superaturam; atque statuendo $i =$ numero infinito, valorem Potestatis $(1 + k\omega)^i$ ad quemvis numerum unitate majorem ascendere. Quod si ergo ponatur $(1 + k\omega)^i = 1 + x$, erit $l(1 + x) = i\omega$, unde, cum fit $i\omega$ numerus finitus, Logarithmus scilicet numeri $1 + x$, perspicuum est, i esse debere numerum infinite magnum, alioquin enim $i\omega$ valorem finitum habere non posset.

119. Cum autem positum sit $(1 + k\omega)^i = 1 + x$, erit $1 + k\omega = (1 + x)^{\frac{1}{i}}$ & $k\omega = (1 + x)^{\frac{1}{i}} - 1$, unde fit $i\omega = \frac{i}{k} ((1 + x)^{\frac{1}{i}} - 1)$. Quia vero est $i\omega = l(1 + x)$, erit $l(1 + x) = \frac{i}{k} ((1 + x)^{\frac{1}{i}} - 1)$, posito i numero infinite magno. Est autem $(1 + x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{2i}x^2 + \frac{1(i-1)(2i-1)}{i \cdot 2i \cdot 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i \cdot 2i \cdot 3i \cdot 4i}x^4 + \&c.$ Ob i autem numerum infinitum, erit $\frac{i-1}{2i} = \frac{1}{2}$, $\frac{2i-1}{3i} = \frac{2}{3}$, $\frac{3i-1}{4i} = \frac{3}{4}$, &c.; hinc erit $i(1 + x)^{\frac{1}{i}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$, & consequenter $l(1 + x) = \frac{1}{k} (\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.)$, posita basi Logarithmica $= a$ ac denotante k numerum huic basi convenientem, ut scilicet sit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \&c.$

120. Cum igitur habeamus Seriem Logarithmo numeri $1 + x$ æqualem, ejus ope ex data basi a definire poterimus valorem numeri

numeri k . Si enim ponamus $1 + x = a$, ob $la = 1$, erit CAP. VII.

$$1 = \frac{1}{k} \left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c. \right),$$

$$\text{hincque habebitur } k = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} -$$

$$\frac{(a-1)^4}{4} + \&c., \text{ cujus ideo Seriei infinitæ valor, si ponatur}$$

$$a = 10, \text{ circiter esse debet } = 2,30258; \text{ quanquam diffi-}$$

$$\text{culter intelligi potest esse } 2,30258 = \frac{9}{1} - \frac{9^2}{2} + \frac{9^3}{3} -$$

$$\frac{9^4}{4} + \&c., \text{ quoniam hujus Seriei termini continuo fiunt majo-}$$

res, neque adeo aliquot terminis sumendis summa vero propin-
qua haberi potest: cui incommodo mox remedium afferetur.

$$121. \text{ Quoniam igitur est } l(1+x) = \frac{1}{k} \left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \&c. \right), \text{ erit, posito } x \text{ negativo, } l(1-x) = -\frac{1}{k}$$

$$\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c. \right). \text{ Subtrahatur Series poste-}$$

$$\text{rior a priori, erit } l(1+x) - l(1-x) = l \frac{1+x}{1-x} = \frac{2}{k} \times$$

$$\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \&c. \right). \text{ Nunc ponatur } \frac{1+x}{1-x} = a,$$

$$\text{ut sit } x = \frac{a-1}{a+1}, \text{ ob } la = 1 \text{ erit } k = 2 \left(\frac{a-1}{a+1} + \right.$$

$$\left. \frac{(a-1)^3}{3(a+1)^3} + \frac{(a-1)^5}{5(a+1)^5} + \&c. \right), \text{ ex qua æquatione valor}$$

numeri k ex basi a inveniri poterit. Si ergo basis a ponatur

$$= 10 \text{ erit } k = 2 \left(\frac{9}{11} + \frac{9^3}{3 \cdot 11^3} + \frac{9^5}{5 \cdot 11^5} + \frac{9^7}{7 \cdot 11^7} + \&c. \right), \text{ cu-}$$

jus Seriei termini sensibilibiter decrescunt, ideoque mox valorem
pro k satis propinquum exhibent.

122. Quoniam ad systema Logarithmorum condendum ba-
sin a pro lubitu accipere licet, ea ita assumi poterit ut fiat
 $k=1$. Ponamus ergo esse $k=1$, eritque per Seriem supra

Euleri *Introduc. in Anal. infin. parv.*

M

(116)

LIB. I.

(116) inventam, $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$, qui termini, si in fractiones decimales convertantur atque actu addantur, præbunt hunc valorem pro $a = 2,71828182845904523536028$, cujus ultima adhuc nota veritati est consentanea. Quod si jam ex hac basi Logarithmi construuntur, ii vocari solent Logarithmi *naturales* seu *hyperbolici*, quoniam quadratura hyperbolæ per istiusmodi Logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc $2,718281828459 \&c.$ constanter litteram e , quæ ergo denotabit basin Logarithmorum naturalium seu hyperbolicorum, cui respondet valor litteræ $k = 1$; sive hæc littera e quoque exprimet summam hujus Seriei $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$ in infinitum.

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1 + \omega$ Logarithmus sit $= \omega$, denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k = 1$ innotescat, omnium numerorum Logarithmi hyperbolici exhiberi poterunt. Erit ergo, posita e pro numero supra invento, perpetuo $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$ ipsi vero Logarithmi hyperbolici ex his Seriebus invenientur, quibus est $l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$, & $l \frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \&c.$, quæ Series vehementer convergunt, si pro x statuatur fractio valde parva: ita ex Serie posteriori facili negotio inveniuntur Logarithmi numerorum unitate non multo majorum. Posito namque $x = \frac{1}{5}$, erit $l \frac{6}{4} = l \frac{3}{2} = \frac{2}{1.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} + \frac{2}{7.5^7} + \&c.$, & facto $x = \frac{1}{7}$, erit $l \frac{4}{3} = \frac{2}{1.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} + \frac{2}{7.7^7}$

$\frac{2}{7 \cdot 7^7} + \&c.$, facto $x = \frac{1}{9}$, erit $l \frac{5}{4} = \frac{2}{1 \cdot 9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} + \frac{2}{7 \cdot 9^7} + \&c.$ CAP.VII.

Ex Logarithmis vero harum fractionum reperientur Logarithmi numerorum integrorum, erit enim ex natura Logarithmorum $l \frac{3}{2} + l \frac{4}{3} = l_2$; tum $l \frac{3}{2} + l_2 = l_3$; & $2 l_2 = l_4$; porro $l \frac{5}{4} + l_4 = l_5$; $l_2 + l_3 = l_6$; $3 l_2 = l_8$; $2 l_3 = l_9$; & $l_2 + l_5 = l_{10}$.

E X E M P L U M.

Hinc Logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

l_1	=	0,	00000	00000	00000	00000	00000
l_2	=	0,	69314	71805	59945	30941	72321
l_3	=	1,	09861	22886	68109	69139	52452
l_4	=	1,	38629	43611	19890	61883	44642
l_5	=	1,	60943	79124	34100	37460	07593
l_6	=	1,	79175	94692	28055	00081	24773
l_7	=	1,	94591	01490	55313	30510	54639
l_8	=	2,	07944	15416	79835	92825	16964
l_9	=	2,	19722	45773	36219	38279	04905
l_{10}	=	2,	30258	50929	94045	68401	79914

Hi scilicet Logarithmi omnes ex superioribus tribus Seriebus sunt deducti, præter l_7 , quem hoc compendio sum affecutus.

Posui nimirum in Serie posteriori $x = \frac{1}{99}$ ficque obtinui $l \frac{100}{98} =$

$l \frac{50}{49} = 0, 0202027073175194484078230$, qui subtractus a $l_{50} = 2 l_5 + l_2 = 3 9120230054281460586187508$, relinquit l_{49} , cujus semissis dat l_7 .

LIB. I. 124. Ponatur Logarithmus hyperbolicus ipsius $1+x$ seu $l(1+x) = y$; erit $y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$ Sumto autem numero a pro basi Logarithmica, fit numeri ejusdem $1+x$ Logarithmus $= v$; erit, ut vidimus, $v = \frac{1}{k} (x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.) = \frac{y}{k}$; hincque $k = \frac{y}{v}$; ex quo commodissime valor ipsius k basi a respondens ita definitur ut fit æqualis cujuscvis numeri Logarithmo hyperbolico diviso per Logarithmum ejusdem numeri ex basi a formati. Posito ergo numero hoc $= a$, erit $v = 1$, hincque fit $k =$ Logarithmo hyperbolico basis a . In systemate ergo Logarithmorum communium, ubi est $a = 10$, erit $k =$ Logarithmo hyperbolico ipsius 10 , unde fit $k = 2,3025850929940456840179914$, quem valorem jam supra satis prope collegimus. Si ergo singuli Logarithmi hyperbolici per hunc numerum k dividantur, vel, quod eodem redit, multiplicentur per hanc fractionem decimalem $0,4342944819032518276511289$, prodibunt Logarithmi vulgares basi $a = 10$ convenientes.

125. Cum fit $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \&c.$, si ponatur $a^y = e^z$, erit, sumtis Logarithmis hyperbolicis, $yla = z$, quia est $le = 1$, quo valore loco z substituto, erit $a^y = 1 + \frac{yla}{1} + \frac{y^2(la)^2}{1.2} + \frac{y^3(la)^3}{1.2.3} + \&c.$, unde quælibet quantitas exponentialis ope Logarithmorum hyperbolicorum per Seriem infinitam explicari potest. Tum vero, denotante i numerum infinite magnum, tam quantitates exponentiales quam Logarithmi per potestates exponi possunt. Erit enim $e^z = (1 + \frac{z}{i})^i$, hincque $a^y = (1 + \frac{yla}{i})^i$, deinde pro Logarithmis hyperbolicis habetur $l(1+x) = i((1+x)^{\frac{1}{i}} - 1)$. De cetero

C A P U T V I I I.

De quantitatibus transcendentibus ex Circulo ortis.

126. **P**ost Logarithmos & quantitates exponentiales considerari debent Arcus circulares eorumque Sinus & Cosinus, quia non solum aliud quantitarum transcendentium genus constituunt, sed etiam ex ipsis Logarithmis & exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo Radium Circuli seu Sinum totum esse $= 1$, atque satis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non posse, per approximationes autem inventa est Semicircumferentia hujus Circuli esse $= 3, 1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132723066470938446 +$, pro quo numero, brevitatis ergo, scribam π , ita ut sit $\pi =$ Semicircumferentiae Circuli, cujus Radius $= 1$, seu π erit longitudo Arcus 180 graduum.

127. Denotante z Arcum hujus Circuli quemcunque, cujus Radium perpetuo assumo $= 1$; hujus Arcus z considerari potissimum solent Sinus & Cosinus. Sinum autem Arcus z in posterum hoc modo indicabo, *sin. A. z*, seu tantum *sin. z*. Cosinum vero hoc modo *cos. A. z*, seu tantum *cos. z*. Ita, cum π sit Arcus 180°, erit *sin. 0* $\pi = 0$; *cos. 0* $\pi = 1$; & *sin. 1* $\pi = 1$, *cos. 1* $\pi = 0$; *sin. 2* $\pi = 0$; *cos. 2* $\pi = -1$; *sin. 3* $\pi = -1$; *cos. 3* $\pi = 0$; *sin. 4* $\pi = 0$; & *cos. 4* $\pi = 1$.

Omnes ergo Sinus & Cosinus intra limites $+ 1$ & $- 1$ con-

LIB. I. tinentur. Erit autem porro $\cos. z = \sin. (\frac{1}{2} \pi - z)$, &

$$\sin. z = \cos. (\frac{1}{2} \pi - z), \text{ atque } (\sin. z)^2 + (\cos. z)^2 = 1.$$

Præter has denominationes notandæ sunt quoque hæ: $\tan. z$, quæ denotat Tangentem Arcus z ; $\cot. z$ Cotangentem Arcus z ; constatque esse $\tan. z = \frac{\sin. z}{\cos. z}$ & $\cot. z = \frac{\cos. z}{\sin. z} = \frac{1}{\tan. z}$; quæ omnia ex Trigonometria sunt nota.

128. Hinc vero etiam constat si habeantur duo Arcus y & z , fore $\sin. (y + z) = \sin. y. \cos. z + \cos. y. \sin. z$, & $\cos. (y + z) = \cos. y. \cos. z - \sin. y. \sin. z$, itemque $\sin. (y - z) = \sin. y. \cos. z - \cos. y. \sin. z$ & $\cos. (y - z) = \cos. y. \cos. z + \sin. y. \sin. z$.

Hinc loco y substituendo Arcus $\frac{1}{2} \pi$; π ; $\frac{3}{2} \pi$, &c., erit

$\sin. (\frac{1}{2} \pi + z) = + \cos. z$	$\sin. (\frac{1}{2} \pi - z) = + \cos. z$
$\cos. (\frac{1}{2} \pi + z) = - \sin. z$	$\cos. (\frac{1}{2} \pi - z) = + \sin. z$
$\sin. (\pi + z) = - \sin. z$	$\sin. (\pi - z) = + \sin. z$
$\cos. (\pi + z) = - \cos. z$	$\cos. (\pi - z) = - \cos. z$
$\sin. (\frac{3}{2} \pi + z) = - \cos. z$	$\sin. (\frac{3}{2} \pi - z) = - \cos. z$
$\cos. (\frac{3}{2} \pi + z) = + \sin. z$	$\cos. (\frac{3}{2} \pi - z) = - \sin. z$
$\sin. (2\pi + z) = + \sin. z$	$\sin. (2\pi - z) = - \sin. z$
$\cos. (2\pi + z) = + \cos. z$	$\cos. (2\pi - z) = + \cos. z$

Si ergo n denotet numerum integrum quemcunque, erit

CAP.
VIII.

$\sin.(\frac{4n+1}{2}\pi+z) = + \cos.z$	$\sin.(\frac{4n+1}{2}\pi-z) = + \cos.z$
$\cos.(\frac{4n+1}{2}\pi+z) = - \sin.z$	$\cos.(\frac{4n+1}{2}\pi-z) = - \sin.z$
$\sin.(\frac{4n+2}{2}\pi+z) = - \sin.z$	$\sin.(\frac{4n+2}{2}\pi-z) = + \sin.z$
$\cos.(\frac{4n+2}{2}\pi+z) = - \cos.z$	$\cos.(\frac{4n+2}{2}\pi-z) = + \cos.z$
$\sin.(\frac{4n+3}{2}\pi+z) = - \cos.z$	$\sin.(\frac{4n+3}{2}\pi-z) = - \cos.z$
$\cos.(\frac{4n+3}{2}\pi+z) = + \sin.z$	$\cos.(\frac{4n+3}{2}\pi-z) = - \sin.z$
$\sin.(\frac{4n+4}{2}\pi+z) = + \sin.z$	$\sin.(\frac{4n+4}{2}\pi-z) = - \sin.z$
$\cos.(\frac{4n+4}{2}\pi+z) = + \cos.z$	$\cos.(\frac{4n+4}{2}\pi-z) = + \cos.z$

Quæ formulæ veræ sunt five n sit numerus affirmativus five negativus integer.

129. Sit $\sin. z = p$ & $\cos. z = q$ erit $pp + qq = 1$; & $\sin. y = m$; $\cos. y = n$; ut sit quoque $mm + nn = 1$; Arcuum ex his compositorum Sinus & Cosinus ita se habebunt.

$\sin. z = p$	$\cos. z = q$
$\sin. (y+z) = mq + np$	$\cos. (y+z) = nq - mp$
$\sin. (2y+z) = 2mnq + (nn - mm)p$	$\cos. (2y+z) = (nn - mm)q - 2mnp$
$\sin. (3y+z) = (3mn^2 - m^3)q + (n^3 - 3m^2n)p$	$\cos. (3y+z) = (n^3 - 3m^2n)q - (3mn^2 - m^3)p$
&c.	&c.

Arcus isti z , $y+z$, $2y+z$, $3y+z$, &c., in arithmetica progressionē progrediuntur; eorum vero tam Sinus quam Cosinus progressionem recurrentem constituunt, qualis ex denominatore $1 - 2nx + (mm + nn)xx$ oritur; est enim

$\sin.$

LIB. I. $\sin. (2y + z) = 2n \sin. (y + z) - (mm + nn) \sin. z$ five
 $\sin. (2y + z) = 2 \cos. y. \sin. (y + z) - (\sin. z)$; atque simili modo
 $\cos. (2y + z) = 2 \cos. y. \cos. (y + z) - \cos. z$. Eodem modo erit porro
 $\sin. (3y + z) = 2 \cos. y. \sin. (2y + z) - \sin. (y + z)$, &
 $\cos. (3y + z) = 2 \cos. y. \cos. (2y + z) - \cos. (y + z)$, itemque
 $\sin. (4y + z) = 2 \cos. y. \sin. (3y + z) - \sin. (2y + z)$, &
 $\cos. (4y + z) = 2 \cos. y. \cos. (3y + z) - \cos. (2y + z)$ &c.

Cujus legis beneficio Arcuum in progressionem arithmetica progredientium tam Sinus quam Cofinus quousque libuerit expedite formari possunt.

130. Cum sit $\sin. (y + z) = \sin. y. \cos. z + \cos. y. \sin. z$ atque $\sin. (y - z) = \sin. y. \cos. z - \cos. y. \sin. z$, erit his expressionibus vel addendis vel subtrahendis :

$$\sin. y. \cos. z = \frac{\sin. (y + z) + \sin. (y - z)}{2}$$

$$\cos. y. \sin. z = \frac{\sin. (y + z) - \sin. (y - z)}{2}$$

Quia porro est $\cos. (y + z) = \cos. y. \cos. z - \sin. y. \sin. z$, atque $\cos. (y - z) = \cos. y. \cos. z + \sin. y. \sin. z$, erit pari modo

$$\cos. y. \cos. z = \frac{\cos. (y - z) + \cos. (y + z)}{2}$$

$$\sin. y. \sin. z = \frac{\cos. (y - z) - \cos. (y + z)}{2}.$$

Sit $y = z = \frac{1}{2} v$, erit ex his postremis formulis :

$$\left(\cos. \frac{1}{2} v \right)^2 = \frac{1 + \cos. v}{2}, \text{ \& } \cos. \frac{1}{2} v = \sqrt{\frac{1 + \cos. v}{2}}$$

$$\left(\sin. \frac{1}{2} v \right)^2 = \frac{1 - \cos. v}{2}, \text{ \& } \sin. \frac{1}{2} v = \sqrt{\frac{1 - \cos. v}{2}}$$

unde, ex dato Cofinu cujusque anguli reperiuntur ejus semissis Sinus & Cofinus.

131. Ponatur Arcus $y + z = a$, & $y - z = b$; erit $y = \frac{a + b}{2}$ & $z = \frac{a - b}{2}$, quibus in superioribus formulis substitutis

tutis

tutis, habebuntur hæ æquationes, tanquam totidem Theore-
mata. CAP.
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$$\sin. a + \sin. b = 2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2}$$

$$\sin. a - \sin. b = 2 \cos. \frac{a+b}{2} \sin. \frac{a-b}{2}$$

$$\cos. a + \cos. b = 2 \cos. \frac{a+b}{2} \cos. \frac{a-b}{2}$$

$$\cos. b - \cos. a = 2 \sin. \frac{a+b}{2} \sin. \frac{a-b}{2}$$

ex his porro nascuntur, ope divisionis, hæc Theoremata

$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \frac{\tan. \frac{a+b}{2} \cot. \frac{a-b}{2}}{\tan. \frac{a-b}{2}} = \frac{\tan. \frac{a+b}{2}}{\tan. \frac{a-b}{2}}$$

$$\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \tan. \frac{a+b}{2}$$

$$\frac{\sin. a + \sin. a}{\cos. b - \cos. a} = \cot. \frac{a-b}{2}$$

$$\frac{\sin. a - \sin. b}{\cos. a + \cos. b} = \tan. \frac{a-b}{2}$$

$$\frac{\sin. a - \sin. a}{\cos. b - \cos. a} = \cot. \frac{a+b}{2}$$

$$\frac{\cos. a + \cos. b}{\cos. b - \cos. a} = \cot. \frac{a+b}{2} \cot. \frac{a-b}{2}$$

Ex his denique deducuntur ista Theoremata

$$\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \frac{\cos. b - \cos. a}{\sin. a - \sin. b};$$

$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} \times \frac{\cos. a + \cos. b}{\cos. b - \cos. a} = \left(\cot. \frac{a-b}{2} \right)^2$$

$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} \times \frac{\cos. b - \cos. a}{\cos. a + \cos. b} = \left(\tan. \frac{a+b}{2} \right)^2$$

132. Cum sit $(\sin. z)^2 + (\cos. z)^2 = 1$ erit, Factoribus
sumendis, $(\cos. z + \sqrt{-1} \sin. z)(\cos. z - \sqrt{-1} \sin. z) = 1$;
qui Factores, etsi imaginarii, tamen ingentem præstant usum in
Arcubus combinandis & multiplicandis. Quærat enim pro-
ductum horum Factorum $(\cos. z + \sqrt{-1} \sin. z)(\cos. y + \sqrt{-1} \sin. y)$ ac reperietur
 $\cos. y \cos. z - \sin. y \sin. z + (\cos. y \sin. z + \sin. y \cos. z)$
Euleri *Introduct. in Anal. infin. parv.* N \sqrt

LIB. I. $\sqrt{-1}$. Cum autem sit $\cos. y. \cos. z - \sin. y. \sin. z = \cos. (y+z)$
 & $\cos. y. \sin. z + \sin. y. \cos. z = \sin. (y+z)$ erit hoc productum
 $(\cos. y. + \sqrt{-1} \sin. y.) (\cos. z + \sqrt{-1} \sin. z) = \cos. (y+z) +$
 $\sqrt{-1} \sin. (y+z)$

& simili modo

$$(\cos. y - \sqrt{-1} \sin. y) (\cos. z - \sqrt{-1} \sin. z) = \cos. (y+z) - \sqrt{-1} \sin. (y+z)$$

item

$$(\cos. x \pm \sqrt{-1} \sin. x) (\cos. y \pm \sqrt{-1} \sin. y) (\cos. z \pm \sqrt{-1} \sin. z) = \cos. (x+y+z) \pm \sqrt{-1} \sin. (x+y+z).$$

133. Hinc itaque sequitur fore $(\cos. z \pm \sqrt{-1} \sin. z)^2 = \cos. 2z \pm \sqrt{-1} \sin. 2z$, & $(\cos. z \pm \sqrt{-1} \sin. z)^3 = \cos. 3z \pm \sqrt{-1} \sin. 3z$.

ideoque generaliter erit $(\cos. z \pm \sqrt{-1} \sin. z)^n = \cos. nz \pm \sqrt{-1} \sin. nz$:

Unde, ob signorum ambiguitatem, erit

$$\cos. nz = \frac{(\cos. z + \sqrt{-1} \sin. z)^n + (\cos. z - \sqrt{-1} \sin. z)^n}{2} \text{ \& }$$

$$\sin. nz = \frac{(\cos. z + \sqrt{-1} \sin. z)^n - (\cos. z - \sqrt{-1} \sin. z)^n}{2 \sqrt{-1}}$$

Evolutis ergo binomiis hisce erit per Series:

$$\cos. nz = (\cos. z)^n - \frac{n(n-1)}{1 \cdot 2} (\cos. z)^{n-2} (\sin. z)^2 +$$

$$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} (\cos. z)^{n-4} (\sin. z)^4 -$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\cos. z)^{n-6} (\sin. z)^6 + \&c., \text{ \& }$$

$$\sin. nz = \frac{n}{1} (\cos. z)^{n-1} \sin. z - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (\cos. z)^{n-3} (\sin. z)^3 +$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\cos. z)^{n-5} (\sin. z)^5 - \&c.$$

134. Sit Arcus z infinite parvus, erit $\sin. z = z$ & $\cos. z$ CAP. VIII.
 $= 1$: fit autem n numerus infinite magnus, ut fit Arcus $n z$
 finitæ magnitudinis, puta, $n z = v$; ob $\sin. z = z = \frac{v}{n}$ erit

$$\cos. v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c., \&$$

$$\sin. v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c. \text{ Da-}$$

to ergo Arcu v , ope harum Serierum ejus Sinus & Cofinus
 inveniri poterunt; quarum formularum usus quo magis pateat,
 ponamus Arcum v esse ad quadrantem, seu 90° , ut m ad n , seu
 esse $v = \frac{m}{n} \cdot \frac{\pi}{2}$; Quia nunc valor ipsius π constat, si is ubi-
 que substituitur, prodibit

$$\sin. A. \frac{m}{n} 90^\circ =$$

$$+ \frac{m}{n} \cdot 1, 5707963267948966192313216916$$

$$- \frac{m^3}{n^3} \cdot 0, 6459640975062462536557565636$$

$$+ \frac{m^5}{n^5} \cdot 0, 0796926262461670451205055488$$

$$- \frac{m^7}{n^7} \cdot 0, 0046817541353186881006854632$$

$$+ \frac{m^9}{n^9} \cdot 0, 0001604411847873598218726605$$

$$- \frac{m^{11}}{n^{11}} \cdot 0, 0000035988432352120853404580$$

$$+ \frac{m^{13}}{n^{13}} \cdot 0, 0000000569217292196792681171$$

$$- \frac{m^{15}}{n^{15}} \cdot 0, 0000000006688035109811467224$$

$$+ \frac{m^{17}}{n^{17}} \cdot 0, 0000000000060669357311061950$$

100 DE QUANTITATIBUS TRANSCENDENT.

LIB. I.

$$\begin{aligned}
 & - \frac{m^{19}}{n^{19}}. 0, 00000000000000000000437706546731370 \\
 & + \frac{m^{21}}{n^{21}}. 0, 000000000000000000002571422892856 \\
 & - \frac{m^{23}}{n^{23}}. 0, 0000000000000000000012538995403 \\
 & + \frac{m^{25}}{n^{25}}. 0, 00000000000000000000051564550 \\
 & - \frac{m^{27}}{n^{27}}. 0, 0000000000000000000000181239 \\
 & + \frac{m^{29}}{n^{29}}. 0, 000000000000000000000000549
 \end{aligned}$$

atque *cos.* A. $\frac{m}{n} 90^\circ =$

$$\begin{aligned}
 & + 1, 00000000000000000000000000000000 \\
 & - \frac{m^2}{n^2}. 1, 2337005501361698273543113745 \\
 & + \frac{m^4}{n^4}. 0, 2536695079010480136365633659 \\
 & - \frac{m^6}{n^6}. 0, 0208634807633529608730516364 \\
 & + \frac{m^8}{n^8}. 0, 0009192602748394265802417158 \\
 & - \frac{m^{10}}{n^{10}}. 0, 0000252020423730606054810526 \\
 & + \frac{m^{12}}{n^{12}}. 0, 0000004710874778818171503665 \\
 & - \frac{m^{14}}{n^{14}}. 0, 0000000063866030837918522408 \\
 & + \frac{m^{16}}{n^{16}}. 0, 0000000000656596311497947230 \\
 & - \frac{m^{18}}{n^{18}}. 0, 0000000000005294400200734620 \\
 & + \frac{m^{20}}{n^{20}}. 0, 0000000000000034377391790981 \\
 & - \frac{m^{22}}{n^{22}}. 0, 0000000000000000183599165212
 \end{aligned}$$

+

$$\begin{aligned}
& + \frac{m^{24}}{n^{24}} \cdot 0, 000000000000000000000000820675327 \\
& - \frac{m^{26}}{n^{26}} \cdot 0, 0000000000000000000000003115285 \\
& + \frac{m^{28}}{n^{28}} \cdot 0, 000000000000000000000000010165 \\
& - \frac{m^{30}}{n^{30}} \cdot 0, 00000000000000000000000000026
\end{aligned}$$

Cum igitur sufficiat Sinus & Cofinus angulorum ad 45° nosse, fractio $\frac{m}{n}$ semper minor erit quam $\frac{1}{2}$, hincque etiam ob Potestates fractionis $\frac{m}{n}$, Series exhibitæ maxime convergent, ita ut plerumque aliquot tantum termini sufficiant, præcipue, si Sinus & Cofinus non ad tot figuras desiderentur.

135. Inventis Sinibus & Cofinibus inveniri quidem possunt Tangentes & Cotangentes, per analogias consuetas, at quia in hujusmodi ingentibus numeris multiplicatio & divisio vehementer est molesta, peculiari modo eas exprimere convenit. Erit ergo

$$\begin{aligned}
\tang. v &= \frac{\sin. v}{\cos. v} = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c. \\
&= 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c. \\
&\& \cot. v = \frac{\cos. v}{\sin. v} = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c. \\
&= v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c.
\end{aligned}$$

LIB. I. si jam fit Arcus $v = \frac{m}{n} 90^\circ$ erit eodem modo quo ante

$\text{tang. A. } \frac{m}{n} 90^\circ =$	$\text{cot. A. } \frac{m}{n} 90^\circ =$
$+\frac{2mn}{nn-mmm} \cdot 0, 6366197723675$	$+\frac{n}{m} \cdot 0, 6366197723675$
$+\frac{m}{n} \cdot 0, 2975567820597$	$-\frac{4mm}{4nn-mmm} \cdot 0, 3183098861837$
$+\frac{m^3}{n^3} \cdot 0, 0186886502773$	$-\frac{m}{n} \cdot 0, 2052888894145$
$+\frac{m^5}{n^5} \cdot 0, 0018424752034$	$-\frac{m^3}{n^3} \cdot 0, 0065510747882$
$+\frac{m^7}{n^7} \cdot 0, 0001975800714$	$-\frac{m^5}{n^5} \cdot 0, 0003450292554$
$+\frac{m^9}{n^9} \cdot 0, 0000216977245$	$-\frac{m^7}{n^7} \cdot 0, 0000202791060$
$+\frac{m^{11}}{n^{11}} \cdot 0, 0000024011370$	$-\frac{m^9}{n^9} \cdot 0, 0000012366527$
$+\frac{m^{13}}{n^{13}} \cdot 0, 0000002664132$	$-\frac{m^{11}}{n^{11}} \cdot 0, 0000000764959$
$+\frac{m^{15}}{n^{15}} \cdot 0, 0000000295864$	$-\frac{m^{13}}{n^{13}} \cdot 0, 0000000047597$
$+\frac{m^{17}}{n^{17}} \cdot 0, 0000000032867$	$-\frac{m^{15}}{n^{15}} \cdot 0, 0000000002969$
$+\frac{m^{19}}{n^{19}} \cdot 0, 0000000003651$	$-\frac{m^{17}}{n^{17}} \cdot 0, 0000000000185$
$+\frac{m^{21}}{n^{21}} \cdot 0, 0000000000405$	$-\frac{m^{19}}{n^{19}} \cdot 0, 0000000000011$
$+\frac{m^{23}}{n^{23}} \cdot 0, 0000000000045$	
$+\frac{m^{25}}{n^{25}} \cdot 0, 0000000000005$	

quarum Serierum ratio infra fusius exponetur.

136. Ex superioribus quidem constat, si cogniti fuerint omnium angulorum semirecto minorum Sinus & Cofinus, inde simul omnium angulorum majorum Sinus & Cofinus haberi. Verum si tantum angulorum 30° minorum habeantur Sinus

Sinus & Cofinus, ex iis, per solam additionem & subtractionem, omnium angulorum majorum Sinus & Cofinus inveniri CAP. VIII.

possunt. Cum enim fit $\sin. 30^\circ = \frac{1}{2}$, erit, posito $y = 30^\circ$ ex (130) $\cos. z = \sin. (30 + z) + \sin. (30 - z)$; & $\sin. z = \cos. (30 - z) - \cos. (30 + z)$, ideoque ex Sinibus & Cofinibus angulorum z & $30 - z$, reperiuntur $\sin. (30 + z) = \cos. z - \sin. (30 - z)$ & $\cos. (30 + z) = \cos. (30 - z) - \sin. z$, unde Sinus & Cofinus angulorum a 30° ad 60° , hincque omnes majores definiuntur.

137. In Tangentibus & Cotangentibus simile subsidium usu venit. Cum enim fit $\tan. (a + b) = \frac{\tan. a + \tan. b}{1 - \tan. a. \tan. b}$, erit

$$\tan. 2a = \frac{2 \tan. a}{1 - \tan. a. \tan. a}, \text{ \& } \cot. 2a = \frac{\cot. a - \tan. a}{2}$$

unde ex Tangentibus & Cotangentibus Arcuum 30° minorum inveniuntur Cotangentes usque ad 60° .

Sit jam $a = 30 - b$ erit $2a = 60 - 2b$ & $\cot. 2a = \tan. (30 + 2b)$; erit ergo $\tan. (30 + 2b) = \frac{\cot. (30 - b) - \tan. (30 - b)}{2}$, unde etiam Tangentes Arcuum 30° majorum obtinentur.

Secantes autem & Cosecantes ex Tangentibus per solam subtractionem inveniuntur; est enim $\csc. z = \cot. \frac{1}{2} z - \cot. z$, & hinc $\sec. z = \cot. (45^\circ - \frac{1}{2} z) - \tan. z$. Ex his ergo luculenter perspicitur, quomodo canones Sinuum construi poterint.

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & fit n numerus infinite magnus i , ut iz obtineat valorem finitum v . Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde $\sin. z = \frac{v}{i}$ & $\cos. z = 1$; his substitutis fit $\cos. v =$

(1 +

LIB. I.

$$\frac{(1 + \frac{v\sqrt{-1}}{i})^i + (1 - \frac{v\sqrt{-1}}{i})^i}{2}; \text{ atque } \sin. v =$$

$$\frac{(1 + \frac{v\sqrt{-1}}{i})^i - (1 - \frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}. \text{ In Capite autem}$$

præcedente vidimus esse $(1 + \frac{z}{i})^i = e^z$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $\cos. v =$

$$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2} \text{ \& } \sin. v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = \cos. v + \sqrt{-1} \sin. v$ & $e^{-v\sqrt{-1}} = \cos. v - \sqrt{-1} \sin. v$.

139. Sit jam in iisdem formulis §. 130. n numerus infinite parvus, seu $n = \frac{1}{i}$, existente i numero infinite magno, erit $\cos. nz = \cos. \frac{z}{i} = 1$ & $\sin. nz = \sin. \frac{z}{i} = \frac{z}{i}$; Arcus enim evanescentis $\frac{z}{i}$ Sinus est ipsi æqualis, Cosinus vero $= 1$. His positis habebitur

$$1 = \frac{(\cos. z + \sqrt{-1} \sin. z)^{\frac{1}{i}} + (\cos. z - \sqrt{-1} \sin. z)^{\frac{1}{i}}}{2} \text{ \& }$$

$$\frac{z}{i} = \frac{(\cos. z + \sqrt{-1} \sin. z)^{\frac{1}{i}} - (\cos. z - \sqrt{-1} \sin. z)^{\frac{1}{i}}}{2\sqrt{-1}}. \text{ Su-}$$

mendis autem Logarithmis hyperbolicis supra (125) ostendimus esse $\ell(1+x) = i(1+x)^{\frac{1}{i}} - i$, seu $y^{\frac{1}{i}} = 1 + \frac{1}{i}\ell y$,
posito

posito y loco $1+x$. Nunc igitur, posito loco y , partim $\cos. z + \sqrt{1 - \sin. z}$ partim $\cos. z - \sqrt{1 - \sin. z}$, prodibit $1 =$ CAP. VIII.

$$1 + \frac{1}{i} l(\cos. z + \sqrt{1 - \sin. z}) + 1 + \frac{1}{i} l(\cos. z - \sqrt{1 - \sin. z})$$

$= 1$, ob Logarithmos evanescentes, ita ut hinc nil sequatur. Altera vero æquatio pro Sinu suppeditat:

$$\frac{z}{i} = \frac{\frac{1}{i} l(\cos. z + \sqrt{1 - \sin. z}) - \frac{1}{i} l(\cos. z - \sqrt{1 - \sin. z})}{2\sqrt{1 - \sin. z}}$$

ideoque $z = \frac{1}{2\sqrt{1 - \sin. z}} l \frac{\cos. z + \sqrt{1 - \sin. z}}{\cos. z - \sqrt{1 - \sin. z}}$, unde patet quemadmodum Logarithmi imaginarii ad Arcus circulares revo-

140. Cum sit $\frac{\sin. z}{\cos. z} = \tan. z$, Arcus z per suam Tangentem ita exprimetur ut sit $z = \frac{1}{2\sqrt{1 - \sin. z}} l \frac{1 + \sqrt{1 - \sin. z}}{1 - \sqrt{1 - \sin. z}}$. Supra vero (§. 123) vidimus esse $l \frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \&c.$. Posito ergo $x = \sqrt{1 - \sin. z}$, fiet $z = \frac{\tan. z}{1} - \frac{(\tan. z)^3}{3} + \frac{(\tan. z)^5}{5} - \frac{(\tan. z)^7}{7} + \&c.$. Si ergo ponamus $\tan. z = t$, ut sit z Arcus, cujus Tangens est t , quem ita indicabimus $A. \tan. t$, ideoque erit $z = A. \tan. t$. Cognita ergo Tangente t erit Arcus respondens $z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \&c.$ Cum igitur, si Tangens t æquetur Radio 1, fiat Arcus $z =$ Arcui 45° seu $z = \frac{\pi}{4}$, erit $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$, quæ est Series a LEIBNITZIO primum producta, ad valorem Peripheriæ Circuli exprimendum.

141. Quo autem ex hujusmodi Serie longitudo Arcus Circuli
Euleri *Introducet. in Anal. infin. parv.* O culi

LIB. I. culi expedite definiri possit, perspicuum est pro Tangente & fractionem satis parvam substitui debere. Sic ope hujus Seriei facile reperietur longitudo Arcus z , cujus Tangens & æquetur $\frac{1}{10}$, foret enim iste Arcus $z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \&c.$, cujus Seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali Arcu cognito nihil pro longitudine totius Circumferentiæ concludere licebit, cum ratio, quam Arcus, cujus Tangens est $= \frac{1}{10}$, ad totam Peripheriam tenet, non sit assignabilis. Hanc ob rem ad Peripheriam indagandam, ejusmodi Arcus quæri debet, qui sit simul pars aliquota Peripheriæ, & cujus Tangens satis exigua commode exprimi queat. Ad hoc ergo institutum sumi solet Arcus 30° , cujus Tangens est $= \frac{1}{\sqrt{3}}$, quia minorum Arcuum cum Peripheria commensurabilium Tangentes nimis fiunt irrationales. Quare, ob Arcum $30^\circ = \frac{\pi}{6}$, erit $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2\sqrt{3}} - \&c.$, & $\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} + \&c.$, cujus Seriei ope valor ipsius π ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo major est, quod primum singuli termini sint irrationales, tum vero quisque tantum, circiter, triplo sit minor quam præcedens. Huic itaque incommodo ita occurri poterit: sumatur Arcus 45° seu $\frac{\pi}{4}$ cujus valor, etsi

per Seriem vix convergentem $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$, exprimitur, tamen is retineatur, atque in duos Arcus a & b dispertiatur ut sit $a + b = \frac{\pi}{4} = 45^\circ$. Cum igitur sit tang.

$$(a+b) = 1 = \frac{\text{tang. } a + \text{tang. } b}{1 - \text{tang. } a \cdot \text{tang. } b} \text{ erit } 1 - \text{tang. } a \cdot \text{tang. } b = \text{tang. } a :$$

$\text{tang. } a + \text{tang. } b \text{ \& } \text{tang. } b = \frac{1 - \text{tang. } a}{1 + \text{tang. } a}$. Sit nunc $\text{tang. } a = \frac{1}{2}$, erit $\text{tang. } b = \frac{1}{3}$, hinc uterque Arcus a & b per Seriem rationalem multo magis, quam superior, convergentem exprimetur, eorumque summa dabit valorem Arcus $\frac{\pi}{4}$; hinc itaque erit

$$\pi = 4 \cdot \left\{ \begin{array}{l} \frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \frac{1}{9 \cdot 2^9} - \&c. \\ \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} - \&c. \end{array} \right\}$$

hoc ergo modo multo expeditius longitudo semicircumferentiae π inveniri potuisset, quam quidem factum est ope Seriei ante commemoratae.

CAPUT IX.

De investigatione Factorum trinomialium.

143. **Q**uemadmodum Factores simplices cujusque Functionis integræ inveniri oporteat, supra quidem ostendimus hoc fieri per resolutionem æquationum. Si enim proposita sit Functio quæcunque integra $a + \epsilon z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, hujusque quærantur Factores simplices formæ $p - qz$, manifestum est, si $p - qz$ fuerit Factor Functionis $a + \epsilon z + \gamma z^2 + \&c.$, tum, posito $z = \frac{p}{q}$, quo casu Factor $p - qz$ fit $= 0$, etiam ipsam Functionem propositam evanescere debere. Hinc $p - qz$ erit Factor vel divisor Functionis $a + \epsilon z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, sequitur fore hanc

O 2

expres-

LIB. I. expressionem $\alpha + \frac{\zeta p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\varepsilon p^4}{q^4} + \&c. = 0$. Unde vicissim, si omnes radices $\frac{p}{q}$ hujus æquationis eruantur, singulæ dabunt totidem Factores simplices Functionis integræ propositæ $\alpha + \zeta z + \gamma z^2 + \delta z^3 + \&c.$, nempe $p - qz$. Patet autem simul numerum Factorum hujusmodi simplicium ex maxima Potestate ipsius z definiri.

144. Hoc autem modo plerumque difficulter Factores imaginarii eruantur, quamobrem hoc Capite methodum peculiarem tradam, cujus ope sæpenumero Factores simplices imaginarii inveniri queant. Quoniam vero Factores simplices imaginarii ita sunt comparati, ut binorum productum fiat reale, hos ipsos Factores imaginarios reperiemus, si Factores investigemus duplices, seu hujus formæ $p - qz + rzz$, reales quidem, sed quorum Factores simplices sint imaginarii. Quod si enim Functionis $\alpha + \zeta z + \gamma z^2 + \delta z^3 + \&c.$, constent omnes Factores reales duplices hujus formæ trinomialis $p - qz + rzz$, simul omnes Factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ Factores simplices habebit imaginarios, si fuerit $4pr > qq$ seu $\frac{q}{2\sqrt{pr}} < 1$. Cum igitur Sinus & Cosinus Angulorum sint unitate minores, formula $p - qz + rzz$ Factores simplices habebit imaginarios si fuerit $\frac{q}{2\sqrt{pr}} = \text{Sinui vel Cosinui cujuspian Anguli}$. Sit ergo

$\frac{q}{2\sqrt{pr}} = \text{cos. } A\phi$, seu $q = 2\sqrt{pr} \cdot \text{cos. } \phi$, atque trinomium $p - qz + rzz$ continebit Factores simplices imaginarios. Ne autem irrationalitas molestiam facessat, assumo hanc formam $p\dot{p} - 2pqz \cdot \text{cos. } \phi + qqzz$, cujus Factores simplices imaginarii erunt hi, $qz - p(\text{cos. } \phi + \sqrt{-1} \cdot \text{sin. } \phi)$ & $qz - p(\text{cos. } \phi - \sqrt{-1} \cdot \text{sin. } \phi)$. Ubi quidem patet si fuerit $\text{cos. } \phi = \pm 1$, tum ambos Factores, ob $\text{sin. } \phi = 0$, fieri æquales & reales.

146. Proposita ergo Functione integra $\alpha + \zeta z + \gamma z^2 + \delta z^3 + \&c.$,

&c. , ejus Factores simplices imaginarii eruentur , si determi- CAP. IX.
nentur litteræ p & q cum Angulo ϕ , ut hoc trinomium
 $pp - 2pqz \cdot \cos. \phi + q q z z$ fiat Factor Functionis. Tum enim
simul inerunt isti Factores simplices imaginarii $qz - p(\cos. \phi + \sqrt{-1} \cdot \sin. \phi)$ & $qz - p(\cos. \phi - \sqrt{-1} \cdot \sin. \phi)$. Quam
ob rem Functio proposita evanescet , si ponatur tam $z = \frac{p}{q} \times$
 $(\cos. \phi + \sqrt{-1} \cdot \sin. \phi)$ quam $z = \frac{p}{q} (\cos. \phi - \sqrt{-1} \cdot \sin. \phi)$.
Hinc , facta substitutione utraque , duplex nascetur æquatio , ex
quibus tam fractio $\frac{p}{q}$ quam Arcus ϕ definiri poterunt.

147. Hæ autem substitutiones loco z faciendæ , etiamsi pri-
mo intuitu difficiles videantur , tamen per ea , quæ in Capite
præcedente sunt tradita , satis expedite absolventur. Cum enim
fuerit ostensum esse $(\cos. \phi \pm \sqrt{-1} \cdot \sin. \phi)^n = \cos. n\phi \pm \sqrt{-1} \times$
 $\sin. n\phi$, sequentes formulæ loco singularum ipsius z Potesta-
tum habebuntur substituendæ.

pro priori Factore	pro altero Factore
$z = \frac{p}{q} (\cos. \phi + \sqrt{-1} \cdot \sin. \phi)$	$z = \frac{p}{q} (\cos. \phi - \sqrt{-1} \cdot \sin. \phi)$
$z^2 = \frac{p^2}{q^2} (\cos. 2\phi + \sqrt{-1} \cdot \sin. 2\phi)$	$z^2 = \frac{p^2}{q^2} (\cos. 2\phi - \sqrt{-1} \cdot \sin. 2\phi)$
$z^3 = \frac{p^3}{q^3} (\cos. 3\phi + \sqrt{-1} \cdot \sin. 3\phi)$	$z^3 = \frac{p^3}{q^3} (\cos. 3\phi - \sqrt{-1} \cdot \sin. 3\phi)$
$z^4 = \frac{p^4}{q^4} (\cos. 4\phi + \sqrt{-1} \cdot \sin. 4\phi)$	$z^4 = \frac{p^4}{q^4} (\cos. 4\phi - \sqrt{-1} \cdot \sin. 4\phi)$
&c.	&c.

Ponatur brevitatis gratia $\frac{p}{q} = r$, factaque substitutione sequen-
tes duæ nascentur æquationes.

$$\begin{aligned} 0 &= \left\{ \alpha + 6r \cdot \cos. \phi + \gamma r^2 \cdot \cos. 2\phi + \delta r^3 \cdot \cos. 3\phi + \&c. \right\} \\ &\quad \left\{ + 6r \sqrt{-1} \cdot \sin. \phi + \gamma r^2 \sqrt{-1} \cdot \sin. 2\phi + \delta r^3 \sqrt{-1} \cdot \sin. 3\phi + \&c. \right\} \\ 0 &= \left\{ \alpha + 6r \cdot \cos. \phi + \gamma r^2 \cdot \cos. 2\phi + \delta r^3 \cdot \cos. 3\phi + \&c. \right\} \\ &\quad \left\{ - 6r \sqrt{-1} \cdot \sin. \phi - \gamma r^2 \sqrt{-1} \cdot \sin. 2\phi - \delta r^3 \sqrt{-1} \cdot \sin. 3\phi - \&c. \right\} \end{aligned}$$

() 3

148. Quod

148. Quod si hæ duæ æquationes invicem addantur & subtrahantur, & posteriori casu per $2\sqrt{-1}$ dividantur, prodibunt hæ duæ æquationes reales:

$$\begin{aligned} 0 &= a + 6r \cos \phi + \gamma r^2 \cos 2\phi + \delta r^3 \cos 3\phi + \&c. \\ 0 &= 6r \sin \phi + \gamma r^2 \sin 2\phi + \delta r^3 \sin 3\phi + \&c. \end{aligned}$$

quæ statim ex forma Functionis propositæ

$$a + 6z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$$

formari possunt, ponendo primum pro unaquaque ipsius z potestate $z^n = r^n \cos n\phi$, deinceps $z^n = r^n \sin n\phi$. Sic enim ob $\sin 0\phi = 0$ & $\cos 0\phi = 1$, pro z^0 seu 1 in termino constanti priori casu ponitur 1, posteriori autem 0. Si ergo ex his duabus æquationibus definiantur incognitæ r & ϕ , ob $r = \frac{p}{q}$, habebitur Factor Functionis trinomialis $pp - 2pqz \cos \phi + qqz^2$, duos Factores simplices imaginarios involvens.

149. Si æquatio prior multiplicetur per $\sin m\phi$; posterior per $\cos m\phi$, atque producta vel addantur vel subtrahantur, prodibunt istæ duæ æquationes:

$$\begin{aligned} 0 &= a \sin m\phi + 6r \sin (m+1)\phi + \gamma r^2 \sin (m+2)\phi + \delta r^3 \sin (m+3)\phi + \&c. \\ 0 &= a \cos m\phi + 6r \cos (m+1)\phi + \gamma r^2 \cos (m+2)\phi + \delta r^3 \cos (m+3)\phi + \&c. \end{aligned}$$

Sin autem æquatio prior multiplicetur per $\cos m\phi$ & posterior per $\sin m\phi$, per additionem ac subtractionem sequentes emergent æquationes.

$$\begin{aligned} 0 &= a \cos m\phi + 6r \cos (m-1)\phi + \gamma r^2 \cos (m-2)\phi + \delta r^3 \cos (m-3)\phi + \&c. \\ 0 &= a \sin m\phi + 6r \sin (m+1)\phi + \gamma r^2 \sin (m+2)\phi + \delta r^3 \sin (m+3)\phi + \&c. \end{aligned}$$

Hujus-

Hujusmodi ergo duæ æquationes quæcunque conjunctæ deter-
minabunt incognitas r & ϕ ; quod cum plerumque pluribus
modis fieri possit, simul plures Factores trinomiales obtinen-
tur, sique adeo omnes, quos Functio proposita in se com-
plectitur. CAP. IX.

150. Quo usus harum regularum clarius appareat, quarum-
dam Functionum sæpius occurrentium Factores trinomiales hic
indagabimus, ut eos, quoties occasio postulaverit, hinc depro-
mere liceat. Sit itaque proposita hæc Functio $a^n + z^n$, cujus
Factores trinomiales formæ $pp - 2pqz \cdot \cos. \phi + qq$ determinari
oporteat; posito ergo $r = \frac{p}{q}$, habebuntur hæ duæ æquationes:

$0 = a^n + r^n \cdot \cos. n\phi$ & $0 = r^n \cdot \sin. n\phi$, quarum poste-
rior dat $\sin. n\phi = 0$; unde erit $n\phi$ Arcus vel hujus formæ
 $(2k+1)\pi$ vel $2k\pi$, denotante k numerum integrum. Casus
hos ideo distinguo, quod eorum Cosinus sint differentes; priori
enim casu erit $\cos. (2k+1)\pi = -1$ posteriori casu autem
 $\cos. 2k\pi = +1$. Patet autem priorem formam $n\phi = (2k+1)\pi$
sumi debere, quippe quæ dat $\cos. n\phi = -1$, unde fit $0 = a^n$
 $- r^n$, hincque porro $r = a = \frac{p}{q}$. Erit ergo $p = a$, $q = 1$,

& $\phi = \frac{(2k+1)\pi}{n}$, unde Functionis $a^n + z^n$ Factor erit

$aa - 2az \cdot \cos. \frac{(2k+1)\pi}{n} + zz$. Cum igitur pro k nume-

rum quemque integrum ponere liceat, prodeunt hoc modo plu-
res Factores, neque tamen infiniti, quoniam si $2k+1$, ultra n
augetur, Factores priores recurrunt, quod ex exemplis clarius
patebit, cum sit $\cos. (2\pi \pm \phi) = \cos. \phi$. Deinde si n est
numerus impar, posito $2k+1 = n$, erit Factor quadratus
 $aa + 2az + zz$ neque vero hinc sequitur quadratum $(a+z)^2$

esse Factorem Functionis $a^n + z^n$, quoniam (in §. 148) uni-
ca æquatio resultat, qua tantum patet $a+z$ esse Divisorem
formulæ

LIB. I. formulæ $a^n + z^n$; quæ regula semper est tenenda quoties $\cos. \Phi$ fit vel $+1$ vel -1 .

EXEMPLUM.

Evolvamus aliquot casus, quo isti Factores clarius ob oculos ponantur, atque hos casus in duas classes distribuamus, prout n fuerit numerus vel par vel impar.

<p>Si $n = 1$ Formulæ $a + z$ Factor est $a + z$</p>	<p>Si $n = 2$ Formulæ $a^2 + z^2$ Factor est $a^2 + z^2$</p>
<p>Si $n = 3$ Formulæ $a^3 + z^3$ Factores sunt $aa - 2az.\cos.\frac{1}{3}\pi + zz$ $a + z$</p>	<p>Si $n = 4$ Formulæ $a^4 + z^4$ Factores sunt $aa - 2az.\cos.\frac{1}{4}\pi + zz$ $aa - 2az.\cos.\frac{3}{4}\pi + zz$</p>
<p>Si $n = 5$ Formulæ $a^5 + z^5$ Factores sunt $aa - 2az.\cos.\frac{1}{5}\pi + zz$ $aa - 2az.\cos.\frac{3}{5}\pi + zz$ $a + z$</p>	<p>Si $n = 6$ Formulæ $a^6 + z^6$ Factores sunt $aa - 2az.\cos.\frac{1}{6}\pi + zz$ $aa - 2az.\cos.\frac{3}{6}\pi + zz$ $aa - 2az.\cos.\frac{5}{6}\pi + zz$</p>

Ex quibus exemplis patet omnes Factores obtineri, si loco $z k + 1$ omnes numeri impares non majores, quam Exponens

n

n , substituantur, iis vero casibus quibus Factor quadratus prodit, CAP. IX. tantum ejus radicem Factoribus annumerari debere.

151. Si proposita sit hæc Functio $a^n - z^n$, ejus Factor trinomialis erit $pp - 2pqz \cdot \cos. \phi + qqzz$, si posito $r = \frac{p}{q}$, fuerit $0 = a^n - r^n \cdot \cos. n\phi$ & $0 = r^n \cdot \sin. n\phi$. Erit ergo iterum $\sin. n\phi = 0$, ideoque $n\phi = (2k+1)\pi$ vel $n\phi = 2k\pi$. Hoc autem casu valor posterior sumi debet, ut sit $\cos. n\phi = +1$, qui dat $0 = a^n - r^n$ & $r = \frac{p}{q} = a$. Habebitur itaque $p = a$; $q = 1$; & $\phi = \frac{2k\pi}{n}$; unde Factor trinomialis formulæ propositæ erit $aa - 2az \cdot \cos. \frac{2k}{n}\pi + zz$; quæ forma, si loco $2k$ omnes numeri pares non majores quam n ponantur, simul dabit omnes Factores; ubi de Factoribus quadratis idem est tenendum quod ante monuimus. Ac primo quidem, posito $k = 0$, prodit Factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, si n fuerit numerus par & ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ esse divisorem formulæ $a^n - z^n$.

EXEMPLUM.

Casus Exponentis n ut ante tractati ita se habebunt, prout n fuerit numerus vel impar vel par.

LIB. I.

<p>Si $n = 1$ Formulæ $a - z$ ipsa erit Factor $a - z$</p> <hr/> <p>Si $n = 3$ Formulæ $a^3 - z^3$ Factores erunt $a - z$ $aa - 2az.\cosf.\frac{2}{3}\pi + zz$</p> <hr/> <p>Si $n = 5$ Formulæ $a^5 - z^5$ Factores erunt $a - z$ $aa - 2az.\cosf.\frac{2}{5}\pi + zz$ $aa - 2az.\cosf.\frac{4}{5}\pi + zz$</p>	<p>Si $n = 2$ Formulæ $a^2 - z^2$ Factores erunt $a - z$ $a + z$</p> <hr/> <p>Si $n = 4$ Formulæ $a^4 - z^4$ Factores erunt $a - z$ $aa - 2az.\cosf.\frac{2}{4}\pi + zz$ $a + z$</p> <hr/> <p>Si $n = 6$ Formulæ $a^6 - z^6$ Factores erunt $a - z$ $aa - 2az.\cosf.\frac{2}{6}\pi + zz$ $aa - 2az.\cosf.\frac{4}{6}\pi + zz$ $a + z$</p>
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152. His igitur confirmatur id, quod supra jam innuimus, omnem Functionem integram, si non in Factores simplices reales, tamen in Factores duplices reales resolvi posse. Vidimus enim hanc Functionem indefinitæ dimensionis $a^n + z^n$ semper in Factores duplices reales, præter simplices reales, resolvi posse. Progrediamur ergo ad Functiones magis compositas, uti: $a + \epsilon z^n + \gamma z^{2n}$, cujus quidem, si duos habeat Factores formæ $\eta + \theta z^n$, resolutio ex præcedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formæ $a + \epsilon z^n + \gamma z^{2n}$, eo casu, quo non habet duos Factores reales formæ $\eta + \theta z^n$, reso-

resolutionem in Factores reales, vel simplices vel duplices, doceamus. CAP. IX.

153. Consideremus ergo hanc Functionem : $a^{2n} - 2a^n z^n \times \cos. g + z^{2n}$, quæ in duos Factores formæ $\eta + \theta z^n$ reales resolvi nequit. Quod si ergo ponamus hujus Functionis Factorem duplicem realem esse $pp - 2pqz. \cos. \phi + qqzz$, posito $r = \frac{p}{q}$, duæ sequentes æquationes erunt resolvendæ :
 $0 = a^{2n} - 2a^n r^n. \cos. g. \cos. n\phi + r^{2n}. \cos. 2n\phi$ & $0 = -2a^n r^n. \cos. g. \sin. n\phi + r^{2n}. \sin. 2n\phi$. Vel, loco prioris æquationis sumatur ex §. 149, (ponendo $m = 2n$), hæc $0 = a^{2n}. \sin. 2n\phi - 2a^n r^n. \cos. g. \sin. n\phi$, quæ cum posteriori collata dat $r = a$; tum vero erit $\sin. 2n\phi = 2 \cos. g. \sin. n\phi$. At est $\sin. 2n\phi = 2 \sin. n\phi. \cos. n\phi$. unde fit $\cos. n\phi = \cos. g$. At est semper $\cos. (2k\pi \pm g) = \cos. g$, ex quo habetur $n\phi = 2k\pi \pm g$ & $\phi = \frac{2k\pi \pm g}{n}$. Hinc ergo Factor generalis duplex formæ propositæ erit $= aa - 2az. \cos. \frac{2k\pi \pm g}{n} + zz$; atque omnes Factores prodibunt, si pro $2k$ omnes numeri pares non majores quam n successive substituantur, uti ex applicatione ad casus videre licebit.

EXEMPLUM.

Consideremus ergo casus quibus n est 1, 2, 3, 4, &c., ut ratio Factorum appareat. Erit ergo

Formulæ	
$a^2 - 2az. \cos. g + zz$	
Unicus Factor	
$aa - 2az. \cos. g + zz$	
Formulæ	
$a^4 - 2a^2z^2. \cos. g + z^4$	
P 2	Facto-

DE INVESTIGATIONE

Factores duo

$$aa - 2az \cdot \cos. \frac{g}{2} + z^2$$

$$aa - 2az \cos. \left(\frac{2\pi \pm g}{2} \right) + zz \text{ seu } aa + 2az \cos. \frac{g}{2} + zz$$

Formulæ

$$a^6 - 2a^3 z^3 \cdot \cos. g + z^6$$

Factores tres

$$aa - 2az \cdot \cos. \frac{g}{3} + z^2$$

$$aa - 2az \cos. \frac{2\pi - g}{3} + z^2$$

$$aa - 2az \cos. \frac{2\pi + g}{3} + z^2$$

Formulæ

$$a^8 - 2a^4 z^4 \cos. g + z^8$$

Factores quatuor

$$aa - 2az \cdot \cos. \frac{g}{4} + zz$$

$$aa - 2az \cos. \frac{2\pi - g}{4} + zz$$

$$aa - 2az \cos. \frac{2\pi + g}{4} + zz$$

$$aa - 2az \cos. \frac{4\pi \pm g}{4} + zz \text{ seu } aa + 2az \cos. \frac{g}{4} + zz$$

Formulæ

$$a^{10} - 2a^5 z^5 \cdot \cos. g + z^{10}$$

Factores quinque

$$aa - 2az \cdot \cos. \frac{g}{5} + zz$$

$$aa - 2az \cos. \frac{2\pi - g}{5} + zz$$

$$aa - 2az \cos. \frac{2\pi + g}{5} + zz$$

$$aa - 2az \cos. \frac{4\pi - g}{5} + zz$$

$$aa - 2az \cos. \frac{4\pi + g}{5} + zz$$

Con-

Confirmatur ergo etiam his exemplis omnem Functionem integram in Factores reales, five simplices five duplices, resolvi posse. CAP. IX.

154. Hinc ulterius progredi licebit ad Functionem hanc $\alpha + \epsilon z^n + \gamma z^{2n} + \delta z^{3n}$, quæ certo habebit unum Factorem realem formæ $\eta + \theta z^n$, cujus igitur Factores reales, vel simplices vel duplices, exhiberi possunt; alter vero multiplicator formæ $\iota + \kappa z^n + \lambda z^{2n}$, utcumque fuerit comparatus, per §. præced. pari modo in Factores resolvi poterit. Deinde hæc Functio $\alpha + \epsilon z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n}$, cum perpetuo habeat duos Factores reales formæ hujus $\eta + \theta z^n + \iota z^{2n}$, similiter in Factores, vel simplices vel duplices, reales resolvitur.

Quin etiam progredi licet ad formam $\alpha + \epsilon z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \xi z^{5n}$ quæ, cum certo habeat unum Factorem formæ $\eta + \theta z^n$, alter Factor erit formæ præcedentis; unde etiam hæc Functio resolutionem in Factores reales, vel simplices vel duplices, admittet. Quare si ullum dubium mansisset circa hujusmodi resolutionem omnium Functionum integrarum, hoc nunc fere penitus tollitur.

155. Traduci vero etiam potest hæc in Factores resolutio ad Series infinitas; scilicet, quia vidimus supra esse $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. = e^x$; at vero esse $e^x = (1 + \frac{x}{i})$, denotante i numerum infinitum, perspicuum est Seriem $1 + \frac{1}{x} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ habere Factores infinitos simplices inter se æquales nempe $1 + \frac{x}{i}$. At si ab eadem Serie primus terminus dematur, erit $\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$

LIB. I.

$\&c. = e^{\frac{x}{i}} - 1 = \left(1 + \frac{x}{i}\right)^i - 1$, cujus formæ cum §. 151 comparatæ, quo fit $a = 1 + \frac{x}{i}$; $n = i$ & $z = 1$, Factor quicunque erit $= \left(1 + \frac{x}{i}\right)^2 - 2\left(1 + \frac{x}{i}\right)\cos\frac{2k}{i}\pi + 1$, unde, substituendo pro $2k$ omnes numeros pares, simul omnes Factores prodibunt. Posito autem $2k = 0$ prodit Factor quadratus $\frac{x x}{i i}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ sumi debet, erit ergo x Factor expressionis $e^{\frac{x}{i}} - 1$ quod quidem sponte patet. Ad reliquos Factores inveniendos notari oportet esse, ob Arcum $\frac{2k}{i}\pi$ infinite parvum, $\cos\frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$ (134), terminis sequentibus, ob i numerum infinitum, in nihilum abeuntibus. Hinc erit Factor quilibet $= \frac{x x}{i i} + \frac{4kk}{i i}\pi\pi + \frac{4kk\pi\pi}{i^3}x$, atque adeo forma $e^{\frac{x}{i}} - 1$ erit divisibilis per $1 + \frac{x}{i} + \frac{x x}{4kk\pi\pi}$. Quare expressio $e^{\frac{x}{i}} - 1 = x \left(1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.\right)$, præter Factorem x , habebit hos infinitos $\left(1 + \frac{x}{i} + \frac{x x}{4\pi\pi}\right) \left(1 + \frac{x}{i} + \frac{x x}{16\pi\pi}\right) \left(1 + \frac{x}{i} + \frac{x x}{36\pi\pi}\right) \left(1 + \frac{x}{i} + \frac{x x}{64\pi\pi}\right) \&c.$

156. Cum autem hi Factores contineant partem infinite parvam $\frac{x}{i}$, quæ, cum in singulis insit, atque per multiplicationem omnium, quorum numerus est $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non potest. Ad hoc ergo incommodum vitandum consideremus hanc expressionem $e^{\frac{x}{i}} - e^{-\frac{x}{i}} =$

$\left(1 + \frac{x}{i}\right)^i - \left(1 - \frac{x}{i}\right)^i = 2\left(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.\right)$

est

est enim $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \&c.$; quæ cum

§. 151. comparata dat $n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$

unde hujus expressionis Factor erit $= aa - 2az \cos. \frac{2k}{n} \pi +$

$zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}) \cos. \frac{2k}{i} \pi = \frac{4xx}{ii} + \frac{4kk}{ii} \pi \pi -$

$\frac{4kk\pi\pi xx}{i^4}$, ob $\cos. \frac{2k}{i} \pi = 1 - \frac{2kk\pi\pi}{ii}$. Functio ergo $e^x -$

e^{-x} divisibilis erit per $1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii}$, ubi autem termi-

nus $\frac{xx}{ii}$ tuto omittitur, quia etsi per i multiplicetur, tamen

manet infinite parvus. Præterea vero ut ante, si $k = 0$, erit

primus Factor $= x$. Quocirca, his Factoribus in ordinem re-

ductis, erit $\frac{e^x - e^{-x}}{2} = x(1 + \frac{xx}{\pi\pi})(1 + \frac{xx}{4\pi\pi})(1 + \frac{xx}{9\pi\pi})$

$(1 + \frac{xx}{16\pi\pi})(1 + \frac{xx}{25\pi\pi}) \&c. = x(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} +$

$\frac{x^6}{1.2.3.4.5.6} + \&c.)$. Singulis scilicet Factoribus per multiplica-

tionem constantis ejusmodi formam dedi, ut per actualement multiplicationem primus terminus x resultet.

157. Eodem modo cum fit $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} +$

$\frac{x^4}{1.2.3.4} + \&c. = \frac{(1 + \frac{x}{i})(1 - \frac{x}{i})}{2}$, hujus expressio-

nis cum superiori $a^n + z^n$ comparatio dabit $a = 1 + \frac{x}{i}$;

$z = 1 - \frac{x}{i}$ & $n = i$: erit ergo Factor quicunque $= aa - 2az \times$

$\cos. \frac{2k+1}{n} \pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}) \cos. \frac{2k+1}{i} \pi$. Est

autem

LIB. I. autem $\cos. \frac{2k+1}{i} \pi = 1 - \frac{(2k+1)^2 \pi \pi}{2 i i}$, unde forma Factoris erit $= \frac{4xx}{i i} + \frac{(2k+1)^2 \pi^2}{i i}$, evanescente termino cujus denominator est i^4 . Quoniam ergo omnis Factor expressionis $1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c.$ hujusmodi formam habere debet $1 + axx$, quo Factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2 \pi^2}{i i}$: hinc Factor formæ propositæ erit

$1 + \frac{4xx}{(2k+1)^2 \pi \pi}$, ex eoque omnes Factores infiniti inveniuntur, si loco $2k+1$ successive omnes numeri impares substituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c. =$$

$$(1 + \frac{4xx}{\pi \pi})(1 + \frac{4xx}{9\pi \pi})(1 + \frac{4xx}{25\pi \pi})(1 + \frac{4xx}{49\pi \pi}) \&c.$$

158. Si x fiat quantitas imaginaria, formulæ hæ exponentiales in Sinum & Cosinum cujuspiam Arcus realis abeunt.

Sit enim $x = z \sqrt{-1}$; erit $\frac{e^{z \sqrt{-1}} - e^{-z \sqrt{-1}}}{2 \sqrt{-1}} =$

$$\sin. z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7} + \&c.,$$

quæ adeo expressio hos habet Factores numero infinitos

$$z(1 - \frac{zz}{\pi \pi})(1 - \frac{zz}{4\pi \pi})(1 - \frac{zz}{9\pi \pi})(1 - \frac{zz}{16\pi \pi})(1 - \frac{zz}{25\pi \pi})$$

$$\&c., \text{ seu erit } \sin. z = z(1 - \frac{z}{\pi})(1 + \frac{z}{\pi})(1 - \frac{z}{2\pi})$$

$$(1 + \frac{z}{2\pi})(1 - \frac{z}{3\pi})(1 + \frac{z}{3\pi}) \&c. . \text{ Quoties ergo Arcus } z \text{ ita est comparatus, ut quispiam Factor evanescat, quod}$$

fit si $z = 0$, $z = \pm \pi$; $z = \pm 2\pi$, & generaliter si $z = \pm k\pi$, denotante k numerum quemcunque integrum, simul Si-

nus

nus ejus Arcus debet esse $= 0$, quod quidem ita patet, ut CAP. IX. hinc istos Factores a posteriori eruere licuisset.

Simili modo, cum sit $e^{\frac{z\sqrt{-1}}{2}} + e^{-\frac{z\sqrt{-1}}{2}} = \cos. z$,
erit quoque $\cos. z = (1 - \frac{4z^2}{\pi^2})(1 - \frac{4z^2}{9\pi^2})(1 - \frac{4z^2}{25\pi^2})$
 $(1 - \frac{4z^2}{49\pi^2})$ &c., seu, his Factoribus in binos resolvendis,
erit quoque $\cos. z = (1 - \frac{2z}{\pi})(1 + \frac{2z}{\pi})(1 - \frac{2z}{3\pi})(1 + \frac{2z}{3\pi})$
 $(1 - \frac{2z}{5\pi})(1 + \frac{2z}{5\pi})$ &c., ex qua pari modo patet, si fuerit
 $z = \pm \frac{(2k+1)\pi}{2}$, fore $\cos. z = 0$, id quod etiam ex natura
Circuli liquet.

159. Ex §. 152. etiam inveniri possunt Factores hujus ex-
pressionis $e^x - 2 \cos. g + e^{-x} = 2(1 - \cos. g + \frac{x^2}{1.2} +$
 $\frac{x^4}{1.2.3.4} + \text{&c.})$. Transit enim hæc expressio in hanc
 $(1 + \frac{x}{i}) - 2 \cos. g + (1 - \frac{x}{i})$, quæ cum illa forma
comparata dat $2n = i$; $a = 1 + \frac{x}{i}$, & $z = 1 - \frac{x}{i}$, unde
Factor quicunque hujus formulæ erit $= aa - 2az. \cos.$
 $\frac{2k\pi \pm g}{n} + 2z = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}). \cos. \frac{2(2k\pi \pm g)}{i}$;
at est $\cos. \frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii}$, unde Factor
erit $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$, seu hujus formæ $1 + \frac{xx}{(2k\pi \pm g)^2}$.
Si ergo expressio per $2(1 - \cos. g)$ dividatur, ut in Serie infini-
ta terminus constans sit $= 1$, erit, sumendis omnibus Factori-
bus, $\frac{e^x - 2 \cos. g + e^{-x}}{2(1 - \cos. g)} = (1 + \frac{xx}{gg})(1 + \frac{xx}{(2\pi - g)^2})$

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Q (1 +

LIB. I. $(1 + \frac{x^2}{(2\pi + g)^2})(1 + \frac{x^2}{(4\pi - g)^2})(1 + \frac{x^2}{(4\pi + g)^2})$
 $(1 + \frac{x^2}{(6\pi - g)^2})(1 + \frac{x^2}{(6\pi + g)^2}) \&c..$ Atque, si loco
 x ponatur $z \sqrt{-1}$, erit $\frac{\cos 2g - \cos g}{1 - \cos g} = (1 - \frac{z}{g})(1 + \frac{z}{g})$
 $(1 - \frac{z}{2\pi - g})(1 + \frac{z}{2\pi - g})(1 - \frac{z}{2\pi + g})(1 + \frac{z}{2\pi + g})$
 $(1 - \frac{z}{4\pi - g})(1 + \frac{z}{4\pi - g}) \&c., = 1 - \frac{z^2}{1.2(1 - \cos g)} +$
 $\frac{z^4}{1.2.3.4(1 - \cos g)} - \frac{z^6}{1.2...6(1 - \cos g)} + \&c..$ Hu-
 jus adeo Series in infinitum continuata Factores omnes cog-
 noscuntur.

160. Commode etiam hujusmodi Functionis $e^{b+x} \pm e^{c-x}$
 Factores inveniri omnesque assignari possunt. Transmutatur enim
 in hanc formam $(1 + \frac{b+x}{i})^i \pm (1 + \frac{c-x}{i})^i$, quæ comparata
 cum forma $a^i \pm z^i$, Factorem habebit $aa - 2az \cos \frac{m\pi}{i} + zz$,
 denotante m numerum imparem si valeat signum superius, con-
 tra vero numerum parem. Cum autem, ob i numerum infinite
 magnum, fit $\cos \frac{m\pi}{i} = 1 - \frac{mm\pi\pi}{2ii}$, erit Factor ille generalis
 $= (a - z)^2 + \frac{mm\pi\pi}{ii} az$. At hoc casu erit $a = 1 + \frac{b+x}{i}$
 $\& z = 1 + \frac{c-x}{i}$, unde fit $(a - z)^2 = \frac{(b - c + 2x)^2}{ii}$
 $\& az = 1 + \frac{b+c}{i} + \frac{bc + (c-b)x - xx}{ii}$, ideoque Fa-
 ctor erit per ii multiplicatus $= (b - c)^2 + 4(b - c)x +$
 $4xx + mm\pi\pi$, neglectis terminis per i vel ii divis, quoniam
 jam omnis generis termini adsunt, præ quibus hi evanescerent.
 Termino ergo constante ad unitatem per divisionem reducto
 erit Factor $= 1 + \frac{4(b-c)x + 4xx}{mm\pi\pi + (b-c)^2}$

161. Nunc,

161. Nunc, quoniam in omnibus Factoribus terminus con-
 stans est $= 1$, ipsa Functio $e^{b+x} \pm e^{c-x}$ per ejusmodi
 constantem dividi debet, ut terminus constans fiat $= 1$, seu
 ut ejus valor, posito $x = 0$, fiat $= 1$; talis Divisor erit
 $e^b \pm e^c$, & hanc ob rem expressio hæc $\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$ per

Factores numero infinitos exponi poterit. Erit ergo, si va-
 leat signum superius atque m denotet numerum imparem,
 $\frac{e^{b+x} + e^{c-x}}{e^b + e^c} = (1 + \frac{4(b-c)x + 4xx}{\pi\pi + (b-c)^2})(1 + \frac{4(b-c)x + 4xx}{9\pi\pi + (b-c)^2})$

$(1 + \frac{4(b-c)x + 4xx}{25\pi\pi + (b-c)^2})$ &c., sin autem signum inferius va-
 leat, atque ideo m denotet numerum parem, casuque $m = 0$

radix Factoris quadrati ponatur, erit $\frac{e^{b+x} - e^{c-x}}{e^b - e^c} =$

$(1 + \frac{2x}{b-c})(1 + \frac{4(b-c)x + 4xx}{4\pi\pi + (b-c)^2})(1 + \frac{4(b-c)x + 4xx}{16\pi\pi + (b-c)^2})$
 $(1 + \frac{4(b-c)x + 4xx}{36\pi\pi + (b-c)^2})$ &c.

162. Ponatur $b = 0$, quod sine detrimento universalitatis
 fieri potest, eritque $\frac{e^x + e^c e^{-x}}{1 + e^c} = (1 - \frac{4cx + 4xx}{\pi\pi + cc})$

$(1 - \frac{4cx + 4xx}{9\pi\pi + cc})(1 - \frac{4cx + 4xx}{25\pi\pi + cc})$ &c.; $\frac{e^x - e^c e^{-x}}{1 - e^c}$

$= (1 - \frac{2x}{c})(1 - \frac{4cx + 4xx}{4\pi\pi + cc})(1 - \frac{4cx + 4xx}{16\pi\pi + cc})$

$(1 - \frac{4cx + 4xx}{36\pi\pi + cc})$ &c.. Jam ponatur c negativum, atque

habebuntur hæ duæ æquationes: $\frac{e^x + e^{-c} e^{-x}}{1 + e^{-c}} =$
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$$\frac{e^x - e^{-x}}{1 - e^{-c}} = \left(1 + \frac{4cx + 4xx}{\pi\pi + cc}\right) \left(1 + \frac{4cx + 4xx}{9\pi\pi + cc}\right) \left(1 + \frac{4cx + 4xx}{25\pi\pi + cc}\right) \&c.,$$

$$\left(1 + \frac{4cx + 4xx}{16\pi\pi + cc}\right) \left(1 + \frac{4cx + 4xx}{36\pi\pi + cc}\right) \&c..$$

Multiplicetur forma prima per tertiam, ac prodibit $\frac{e^{2x} + e^{-2x} + e^c + e^{-c}}{2 + e^c + e^{-c}}$; ponatur ve-

ro y loco $2x$, eritque $\frac{e^y + e^{-y} + e^c + e^{-c}}{2 + e^c + e^{-c}} = \left(1 - \frac{2cy + yy}{\pi\pi + cc}\right)$

$$\left(1 + \frac{2cy + yy}{\pi\pi + cc}\right) \left(1 - \frac{2cy + yy}{9\pi\pi + cc}\right) \left(1 + \frac{2cy + yy}{9\pi\pi + cc}\right) \left(1 - \frac{2cy + yy}{25\pi\pi + cc}\right)$$

$$\left(1 + \frac{2cy + yy}{25\pi\pi + cc}\right) \&c..$$

Multiplicetur prima forma per quartam, erit productum $= \frac{e^{2x} - e^{-2x} + e^c - e^{-c}}{e^c - e^{-c}}$; po-

natur y pro $2x$, eritque $\frac{e^y - e^{-y} + e^c - e^{-c}}{e^c - e^{-c}} =$

$$\left(1 + \frac{y}{c}\right) \left(1 - \frac{2cy + yy}{\pi\pi + cc}\right) \left(1 + \frac{2cy + yy}{4\pi\pi + cc}\right) \left(1 - \frac{2cy + yy}{9\pi\pi + cc}\right)$$

$$\left(1 + \frac{2cy + yy}{16\pi\pi + cc}\right) \left(1 - \frac{2cy + yy}{25\pi\pi + cc}\right) \&c..$$

Si secunda forma per quartam multiplicetur, prodibit eadem æquatio nisi quod c capiendum sit negativum, erit nempe

$$\frac{e^c - e^{-c} - e^y + e^{-y}}{e^c - e^{-c}} = \left(1 - \frac{y}{c}\right) \left(1 + \frac{2cy + yy}{\pi\pi + cc}\right)$$

$$\left(1 - \frac{2cy + yy}{4\pi\pi + cc}\right) \left(1 + \frac{2cy + yy}{9\pi\pi + cc}\right) \left(1 - \frac{2cy + yy}{16\pi\pi + cc}\right)$$

$$\left(1 + \frac{2cy + yy}{25\pi\pi + cc}\right) \left(1 - \frac{2cy + yy}{36\pi\pi + cc}\right) \&c..$$

Multiplicetur de-
nique

nique forma secunda per quartam eritque $\frac{e^y + e^{-y} - e^c - e^{-c}}{2 - e^c - e^{-c}}$ CAP. IX.

$$= (1 - \frac{yy}{cc}) (1 - \frac{2cy + yy}{4\pi\pi + cc}) (1 + \frac{2cy + yy}{4\pi\pi + cc}) (1 - \frac{2cy + yy}{16\pi\pi + cc})$$

$$(1 + \frac{2cy + yy}{16\pi\pi + cc}) (1 - \frac{2cy + yy}{36\pi\pi + cc}) (1 + \frac{2cy + yy}{36\pi\pi + cc}) \&c.$$

163. Hæ quatuor combinationes nunc commode ad Circulum transferri possunt, ponendo $c = g\sqrt{-1}$ & $y = v\sqrt{-1}$: erit enim $e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} = 2 \cos. v$; $e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} = 2\sqrt{-1} \sin. v$. & $e^{g\sqrt{-1}} + e^{-g\sqrt{-1}} = 2 \cos. g$; $e^{g\sqrt{-1}} - e^{-g\sqrt{-1}} = 2\sqrt{-1} \sin. g$. Hinc prima combinatio dabit $\frac{\cos. v + \cos. g}{1 + \cos. g} = 1 - \frac{vv}{1.2(1 + \cos. g)} +$

$$\frac{v^4}{1.2.3.4(1 + \cos. g)} - \frac{v^6}{1.2...6(1 + \cos. g)} + \&c. = (1 + \frac{2gv - vv}{\pi\pi - gg})$$

$$(1 - \frac{2gv - vv}{\pi\pi - gg}) (1 + \frac{2gv - vv}{9\pi\pi - gg}) (1 - \frac{2gv - vv}{9\pi\pi - gg})$$

$$(1 + \frac{2gv - vv}{25\pi\pi - gg}) (1 - \frac{2gv - vv}{25\pi\pi - gg}) \&c. = (1 + \frac{v}{\varpi - g})$$

$$(1 - \frac{v}{\varpi + g}) (1 - \frac{v}{\varpi - g}) (1 + \frac{v}{\varpi + g}) (1 + \frac{v}{3\varpi - g})$$

$$(1 - \frac{v}{3\varpi + g}) (1 - \frac{v}{3\varpi - g}) (1 + \frac{v}{3\varpi + g}) \&c. =$$

$$(1 - \frac{vv}{(\varpi - g)^2}) (1 - \frac{vv}{(\varpi + g)^2}) (1 - \frac{vv}{(3\varpi - g)^2})$$

$$(1 - \frac{vv}{(3\varpi + g)^2}) (1 - \frac{vv}{(5\varpi - g)^2}) \&c..$$

Quarta vero combinatio dat $\frac{\cos. v - \cos. g}{1 - \cos. g} = 1 - \frac{vv}{1.2(1 - \cos. g)} +$

$$\frac{v^4}{1.2.3.4(1 - \cos. g)} - \frac{v^6}{1.2...6(1 - \cos. g)} + \&c. = (1 - \frac{vv}{gg})$$

$$(1 + \frac{2gv - vv}{4\varpi\varpi - gg}) (1 - \frac{2gv - vv}{4\varpi\varpi - gg}) (1 + \frac{2gv - vv}{16\varpi\varpi - gg})$$

$$(1 - \frac{2gv - vv}{16\varpi\varpi - gg}) \&c.$$

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$$\begin{aligned} & \left(1 - \frac{2gv - vv}{16\varpi\varpi - gg}\right) \&c. = \left(1 - \frac{v}{g}\right) \left(1 + \frac{v}{g}\right) \left(1 + \frac{v}{2\varpi - g}\right) \\ & \left(1 - \frac{v}{2\varpi + g}\right) \left(1 - \frac{v}{2\varpi - g}\right) \left(1 + \frac{v}{2\varpi + g}\right) \left(1 + \frac{v}{4\varpi - g}\right) \\ & \left(1 - \frac{v}{4\varpi + g}\right) \&c. = \left(1 - \frac{vv}{gg}\right) \left(1 - \frac{vv}{(2\varpi - g)^2}\right) \\ & \left(1 - \frac{vv}{(2\varpi + g)^2}\right) \left(1 - \frac{vv}{(4\varpi - g)^2}\right) \left(1 - \frac{vv}{(4\varpi + g)^2}\right) \&c.. \end{aligned}$$

Secunda combinatio dat $\frac{\sin. g + \sin. v}{\sin. g} = 1 + \frac{v}{\sin. g} - \frac{v^2}{1.2.3 \sin. g} +$

$$\begin{aligned} & \frac{v^3}{1.2.3 \sin. g} - \&c. = \left(1 + \frac{v}{g}\right) \left(1 + \frac{2gv - vv}{\varpi\varpi - gg}\right) \\ & \left(1 - \frac{2gv - vv}{4\varpi\varpi - gg}\right) \left(1 + \frac{2gv - vv}{9\varpi\varpi - gg}\right) \left(1 - \frac{2gv - vv}{16\varpi\varpi - gg}\right) \&c. \\ & = \left(1 + \frac{v}{g}\right) \left(1 + \frac{v}{\varpi - g}\right) \left(1 - \frac{v}{\varpi + g}\right) \left(1 - \frac{v}{2\varpi - g}\right) \\ & \left(1 + \frac{v}{2\varpi + g}\right) \left(1 + \frac{v}{3\varpi - g}\right) \left(1 - \frac{v}{3\varpi + g}\right) \left(1 - \frac{v}{4\varpi - g}\right) \&c.. \end{aligned}$$

Ac sumto v negativo prodit tertia combinatio.

164. Ipsæ vero etiam expressiones in §. 162. primum inventæ ad Arcus circulares traduci possunt hoc modo: cum sit

$$\begin{aligned} \frac{e^x + e^c e^{-x}}{1 + e^c} &= \frac{(1 + e^{-c})(e^x + e^c e^{-x})}{2 + e^c + e^{-c}} = \\ \frac{e^x + e^{-x} + e^{c-x} + e^{-c+x}}{2 + e^c + e^{-c}}, &\text{ si ponamus } c = g \sqrt{-1} \& \\ x = z \sqrt{-1}, &\text{ hæc expressio abit in hanc } \frac{\cos. z + \cos. (g - z)}{1 + \cos. g} = \end{aligned}$$

$$\begin{aligned} \cos. z + \frac{\sin. g \sin. z}{1 + \cos. g}. &\text{ Erit ergo (ob } \frac{\sin. g}{1 + \cos. g} = \tan. \frac{1}{2} g) \\ \cos. z + \tan. \frac{1}{2} g \sin. z &= 1 + \frac{z}{1} \tan. \frac{1}{2} g - \frac{z^2}{1.2} - \\ \frac{z^3}{1.2.3} \tan. \frac{1}{2} g + \frac{z^4}{1.2.3.4} + \frac{z^5}{1.2.3.4.5} \tan. \frac{1}{2} g - \&c. \\ &= \left(1 + \frac{4g^2 - 4z^2}{\varpi\varpi - gg}\right) \left(1 + \frac{4g^2 - 4z^2}{9\varpi\varpi - gg}\right) \left(1 + \frac{4g^2 - 4z^2}{25\varpi\varpi - gg}\right) \&c. \\ &= (1 + \end{aligned}$$

$$= \left(1 + \frac{2z}{w-g}\right) \left(1 - \frac{2z}{w+g}\right) \left(1 + \frac{2z}{3w-g}\right) \left(1 - \frac{2z}{3w+g}\right) \dots$$

$$\left(1 + \frac{2z}{5w-g}\right) \left(1 - \frac{2z}{5w+g}\right) \&c.. \text{ Simili modo altera ex-}$$

pressio, si Numerator & Denominator per $1 - e^{-c}$ multi-

plicetur, abit in $\frac{e^x + e^{-x} - e^c - e^{-c} - e^{-c+x}}{2 - e^c - e^{-c}}$; quæ,

facto $e = r\sqrt{-1}$ & $x = z\sqrt{-1}$, dat $\frac{\cos. z - \cos. (g-z)}{1 - \cos. g} =$

$\cos. z - \frac{\sin. g \sin. z}{1 - \cos. g} = \cos. z - \frac{\sin. z}{\tan. \frac{1}{2} g}$. Erit ergo $\cos. z =$

$\cot. \frac{1}{2} g \sin. z = 1 - \frac{z}{1} \cot. \frac{1}{2} g - \frac{z^2}{1.2} + \frac{z^3}{1.2.3} \cot. \frac{1}{2} g +$

$\frac{z^4}{1.2.3.4} - \frac{z^5}{1.2.3.5} \cot. \frac{1}{2} g + \&c. = \left(1 - \frac{2z}{g}\right) \left(1 + \frac{4gz - 4z^2}{4ww - gg}\right)$

$\left(1 + \frac{4gz - 4z^2}{16ww - gg}\right) \left(1 + \frac{4gz - 4z^2}{36\pi\pi - gg}\right) \&c. = \left(1 - \frac{2z}{g}\right)$

$\left(1 + \frac{2z}{2\pi - g}\right) \left(1 - \frac{2z}{2\pi + g}\right) \left(1 + \frac{2z}{4\pi - g}\right) \left(1 - \frac{2z}{4\pi + g}\right) \&c.$

Quod si ergo ponatur $v = 2z$ seu $z = \frac{1}{2} v$; habebitur

$\frac{\cos. \frac{1}{2} (g-v)}{\cos. \frac{1}{2} g} = \cos. \frac{1}{2} v + \tan. \frac{1}{2} g \sin. \frac{1}{2} v =$

$\left(1 + \frac{v}{w-g}\right) \left(1 - \frac{v}{w+g}\right) \left(1 + \frac{v}{3w-g}\right) \left(1 - \frac{v}{3w+g}\right) \&c.;$

$\frac{\cos. \frac{1}{2} (g+v)}{\cos. \frac{1}{2} g} = \cos. \frac{1}{2} v - \tan. \frac{1}{2} g \sin. \frac{1}{2} v =$

$\left(1 - \frac{v}{w-g}\right) \left(1 + \frac{v}{w+g}\right) \left(1 - \frac{v}{3w-g}\right) \left(1 + \frac{v}{3w+g}\right) \&c.;$

$\frac{\sin. \frac{1}{2} (g-v)}{\sin. \frac{1}{2} g} = \cos. \frac{1}{2} v - \cot. \frac{1}{2} g \sin. \frac{1}{2} v =$

$\left(1 - \frac{v}{g}\right) \left(1 + \frac{v}{2w-g}\right) \left(1 - \frac{v}{2w+g}\right) \left(1 + \frac{v}{4w-g}\right) \&c.$

$\frac{\sin. \frac{1}{2} (g+v)}{\sin. \frac{1}{2} g}$

$$\text{LIB. I. } \frac{\sin. \frac{1}{2} (g + v)}{\sin. \frac{1}{2} g} = \cos. \frac{1}{2} v + \cot. \frac{1}{2} g \cdot \sin. \frac{1}{2} v =$$

$$\left(1 + \frac{v}{g}\right) \left(1 - \frac{v}{2g}\right) \left(1 + \frac{v}{2g}\right) \left(1 - \frac{v}{4g}\right) \&c.$$

Quorum Factorum lex progressionis satis est simplex & uniformis; atque ex his expressionibus per multiplicationem oriuntur eæ ipsæ, quæ §. præcedente sunt inventæ.

C A P U T X.

*De usu Factorum inventorum in definiendis
summis Serierum infinitarum.*

165. **S**I fuerit $1 + Az + Bz^2 + Cz^3 + Dz^4 + \&c. =$
 $(1 + \alpha z)(1 + \epsilon z)(1 + \gamma z)(1 + \delta z) \&c.$, hi
 Factores, siue sint numero finiti siue infiniti, si in se actu mul-
 tiplicentur, illam expressionem $1 + A + Bz^2 + Cz^3 + Dz^4 +$
 $\&c.$, producere debent. Æquabitur ergo coëfficiens A summæ
 omnium quantitatuum $\alpha + \epsilon + \gamma + \delta + \&c.$. Coëfficiens
 vero B æqualis erit summæ productorum ex binis, eritque
 $B = \alpha\epsilon + \alpha\gamma + \alpha\delta + \epsilon\gamma + \epsilon\delta + \gamma\delta + \&c.$. Tum vero
 coëfficiens C æquabitur summæ productorum ex ternis, nem-
 pe erit $C = \alpha\epsilon\gamma + \alpha\epsilon\delta + \epsilon\gamma\delta + \alpha\gamma\delta + \&c.$. Atque
 ita porro erit $D =$ summæ productorum ex quaternis, $E =$
 summæ productorum ex quinis, $\&c.$, id quod ex Algebra
 communi constat.

166. Quia summa quantitatuum $\alpha + \epsilon + \gamma + \delta + \&c.$,
 datur una cum summa productorum ex binis, hinc summa
 Quadratorum $\alpha^2 + \epsilon^2 + \gamma^2 + \delta^2 + \&c.$, inveniri poterit,
 quippe quæ æqualis est Quadrato summæ demtis duplicibus
 productis ex binis. Simili modo summa Cuborum, Biquadra-
 torum & altiorum Potestatum definiri potest: si enim ponamus

$$P =$$

$$\begin{aligned}
P &= a + b + \gamma + \delta + \varepsilon + \&c. \\
Q &= a^2 + b^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c. \\
R &= a^3 + b^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \&c. \\
S &= a^4 + b^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \&c. \\
T &= a^5 + b^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \&c. \\
V &= a^6 + b^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \&c. \\
&\&c.
\end{aligned}$$

Valores P, Q, R, S, T, V &c. sequenti modo ex cognitis $A, B, C, D, \&c.$, determinabuntur.

$$\begin{aligned}
P &= A \\
Q &= AP - 2B \\
R &= AQ - BP + 3C \\
S &= AR - BQ + CP - 4D \\
T &= AS - BR + CQ - DP + 5E \\
V &= AT - BS + CR - DQ + EP - 6F \\
&\&c.
\end{aligned}$$

quarum formularum veritas examine instituto facile agnoscitur: interim tamen in calculo differentiali summo cum rigore demonstrabitur.

167. Cum igitur supra (§. 156.) invenerimus esse:

$$\begin{aligned}
\frac{e^x - e^{-x}}{2} &= x \left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2....7} + \&c. \right) = \\
&= x \left(1 + \frac{xx}{\pi\pi} \right) \left(1 + \frac{xx}{4\pi\pi} \right) \left(1 + \frac{xx}{9\pi\pi} \right) \left(1 + \frac{xx}{16\pi\pi} \right) \\
&\left(1 + \frac{xx}{25\pi\pi} \right) \&c., \text{ erit } 1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3....7} + \\
&\&c. = \left(1 + \frac{xx}{\pi\pi} \right) \left(1 + \frac{xx}{4\pi\pi} \right) \left(1 + \frac{xx}{9\pi\pi} \right) \left(1 + \frac{xx}{16\pi\pi} \right) \&c. \\
\text{Ponatur } xx &= \pi\pi z, \text{ eritque } 1 + \frac{\pi\pi}{1.2.3} z + \frac{\pi^4}{1.2.3.4.5} z^3 + \\
&\frac{\pi^6}{1.2.3....7} z^5 + \&c. = \left(1 + z \right) \left(1 + \frac{1}{4}z \right) \left(1 + \frac{1}{9}z \right) \left(1 + \frac{1}{16}z \right)
\end{aligned}$$

Euleri *Introduct. in Anal. infin. parv.*

R (1 +

LIB. I. $(1 + \frac{1}{25} \pi) \&c.$. Facta ergo applicatione superioris regulæ ad hunc casum, erit $A = \frac{\pi\pi}{6}$; $B = \frac{\pi^4}{120}$; $C = \frac{\pi^6}{5040}$; $D = \frac{\pi^8}{362880}$ &c.. Quod si ergo ponatur

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \&c.$$

$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \&c.$$

$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \&c.$$

$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \&c.$$

$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \&c.$$

&c.

atque harum litterarum valores ex A , B , C , D , &c. determinantur, prodibit.

$$P = \frac{\pi\pi}{6}$$

$$Q = \frac{\pi^4}{90}$$

$$R = \frac{\pi^6}{945}$$

$$S = \frac{\pi^8}{9450}$$

$$T = \frac{\pi^{10}}{93555}$$

&c.

168. Patet ergo omnium Serierum infinitarum in hac forma generali $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$, contentarum, quoties n fuerit numerus par, ope Peripheriæ Circuli π exhiberi posse; habebit enim semper summa Seriei ad π^n rationem rationalem.

lem. Quo autem valor harum summarum clarius perspiciatur, plures hujusmodi Serierum summas commodiori modo expressas hic adjiciam. CAP. X.

$$\begin{aligned}
 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. &= \frac{2^0 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 1} \pi^2 \\
 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c. &= \frac{2^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{1}{3} \pi^4 \\
 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c. &= \frac{2^4}{1 \cdot 2 \cdot 3 \dots 7} \cdot \frac{1}{3} \pi^6 \\
 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \&c. &= \frac{2^6}{1 \cdot 2 \cdot 3 \dots 9} \cdot \frac{3}{5} \pi^8 \\
 1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \&c. &= \frac{2^8}{1 \cdot 2 \cdot 3 \dots 11} \cdot \frac{5}{3} \pi^{10} \\
 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \&c. &= \frac{2^{10}}{1 \cdot 2 \cdot 3 \dots 13} \cdot \frac{691}{105} \pi^{12} \\
 1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} + \frac{1}{5^{14}} + \&c. &= \frac{2^{12}}{1 \cdot 2 \cdot 3 \dots 15} \cdot \frac{35}{1} \pi^{14} \\
 1 + \frac{1}{2^{16}} + \frac{1}{3^{16}} + \frac{1}{4^{16}} + \frac{1}{5^{16}} + \&c. &= \frac{2^{14}}{1 \cdot 2 \cdot 3 \dots 17} \cdot \frac{3617}{15} \pi^{16} \\
 1 + \frac{1}{2^{18}} + \frac{1}{3^{18}} + \frac{1}{4^{18}} + \frac{1}{5^{18}} + \&c. &= \frac{2^{16}}{1 \cdot 2 \cdot 3 \dots 19} \cdot \frac{43867}{21} \pi^{18} \\
 1 + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \frac{1}{4^{20}} + \frac{1}{5^{20}} + \&c. &= \frac{2^{18}}{1 \cdot 2 \cdot 3 \dots 21} \cdot \frac{1222277}{55} \pi^{20} \\
 1 + \frac{1}{2^{22}} + \frac{1}{3^{22}} + \frac{1}{4^{22}} + \frac{1}{5^{22}} + \&c. &= \frac{2^{20}}{1 \cdot 2 \cdot 3 \dots 23} \cdot \frac{854513}{3} \pi^{22} \\
 1 + \frac{1}{2^{24}} + \frac{1}{3^{24}} + \frac{1}{4^{24}} + \frac{1}{5^{24}} + \&c. &= \frac{2^{22}}{1 \cdot 2 \cdot 3 \dots 25} \cdot \frac{1181820455}{273} \pi^{24} \\
 1 + \frac{1}{2^{26}} + \frac{1}{3^{26}} + \frac{1}{4^{26}} + \frac{1}{5^{26}} + \&c. &= \frac{2^{24}}{1 \cdot 2 \cdot 3 \dots 27} \cdot \frac{76977927}{1} \pi^{26}
 \end{aligned}$$

Hucusque istos Potestatum ipsius π Exponentes artificio alibi exponendo continuare licuit, quod ideo hic adjunxi, quod
R 2 Seriei

LIB. I Seriei fractionum primo intuitu perquam irregularis $1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, \frac{35}{1}, \&c.$ in plurimis occasionibus eximius est usus.

169. Tractemus eodem modo æquationem §. 157. inven-

$$\text{tam, ubi erat } \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c., = (1 + \frac{4xx}{\pi\pi})(1 + \frac{4xx}{9\pi\pi})(1 + \frac{4xx}{25\pi\pi})(1 + \frac{4xx}{49\pi\pi}) \&c..$$

$$\text{Posito ergo } xx = \frac{\pi\pi z}{4} \text{ erit } 1 + \frac{\pi\pi}{1.2.4} z + \frac{\pi^4}{1.2.3.4.4^2} z^2 + \frac{\pi^6}{1.2....6.4^3} z^3 + \&c., = (1 + z)(1 + \frac{1}{9} z)(1 + \frac{1}{25} z)(1 + \frac{1}{49} z) \&c.. \text{ Unde, facta applicatione, erit } A = \frac{\pi\pi}{1.2.4};$$

$$B = \frac{\pi^4}{1.2.3.4.4^2}; C = \frac{\pi^6}{1.2.3....6.4^3}; \&c.. \text{ Quod si ergo ponamus}$$

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.$$

$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \&c.$$

$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \&c.$$

$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \&c.$$

reperientur sequentes pro $P, Q, R, S, \&c.$, valores :

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3}; \quad Q = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}$$

$$R = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7}; \quad S = \frac{272}{1.2.3....7} \cdot \frac{\pi^8}{2^9}$$

$T =$

$$T = \frac{7936}{1.2.3.....9} \cdot \frac{\pi^{10}}{2^{11}}; V = \frac{353792}{1.2.3.....11} \cdot \frac{\pi^{12}}{2^{13}}$$

$$W = \frac{22368256}{1.2.3.....13} \cdot \frac{\pi^{14}}{2^{15}}.$$

CAP. X.

170. Eadem summæ Potestatum numerorum imparium inveniri possunt ex summis præcedentibus, in quibus omnes numeri occurrunt; si enim fuerit $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \&c.$, erit ubique, per $\frac{1}{2^n}$ multiplicando, $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \&c.$, quæ Series numeros tantum pares continens, si a priori subtrahatur, relinquet numeros impares, eritque ideo $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \&c.$ Quod si autem Series $\frac{M}{2^n}$ bis sumpta subtrahatur ab M signa prodibunt alternantia, eritque $M - \frac{2}{2^n} M = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c.$ Per tradita ergo præcepta summari poterunt hæ Series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \&c.$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \&c.$$

Si quidem n sit numerus par, atque summa erit $= A a^n$ existente A numero rationali.

171. Præterea vero expressiones §. 164 exhibitæ simili modo

R 3

do

do Series notatu dignas suppeditabunt. Cum enim fit $\cos. \frac{1}{2} v + \tan. \frac{1}{2} g. \sin. \frac{1}{2} v = (1 + \frac{v}{\varpi - g}) (1 - \frac{v}{\varpi + g}) (1 + \frac{v}{3\varpi - g}) \&c.$, si ponamus $v = \frac{x}{n} \varpi$ & $g = \frac{m}{n} \pi$ erit $(1 + \frac{x}{n - m}) (1 - \frac{x}{n + m}) (1 + \frac{x}{3n - m}) (1 - \frac{x}{3n + m}) (1 + \frac{x}{5n - m}) (1 - \frac{x}{5n + m}) \&c. = \cos. \frac{x\varpi}{2n} + \tan. \frac{m\varpi}{2n} \sin. \frac{x\varpi}{2n} = 1 + \frac{\varpi x}{2n} \tan. \frac{m\varpi}{2n} - \frac{\varpi\varpi xx}{2.4nn} - \frac{\varpi^3 x^3}{2.4.6n^3} \tan. \frac{m\varpi}{2n} + \frac{\varpi^4 x^4}{2.4.6.8n^4} + \&c.$ Hæc expressio infinita cum §. 165 collata dabit hos valores $A = \frac{\varpi}{2n} \tan. \frac{m\varpi}{2n}$; $B = -\frac{\varpi\varpi}{2.4nn}$; $C = -\frac{\varpi^3}{2.4.6n^3} \tan. \frac{m\varpi}{2n}$; $D = \frac{\varpi^4}{2.4.6.8n^4}$; $E = \frac{\varpi^5}{2.4.6.8.10n^5} \tan. \frac{m\varpi}{2n} \&c.$ Tum vero erit $\alpha = \frac{1}{n - m}$; $\beta = -\frac{1}{n + m}$; $\gamma = \frac{1}{3n - m}$; $\delta = -\frac{1}{3n + m}$; $\epsilon = \frac{1}{5n - m}$; $\xi = -\frac{1}{5n + m} \&c.$

172. Hinc ergo ad normam §. 166 sequentes Series exorientur.

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \frac{1}{5n+m} + \&c.$$

$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \&c.$$

$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \&c.$$

$$S =$$

$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \text{CAP. X.} \\ \frac{1}{(5n-m)^4} + \&c.$$

$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \\ \frac{1}{(5n-m)^5} - \&c.$$

$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \\ \frac{1}{(5n-m)^6} + \&c.$$

&c.

Posito autem $\text{tang. } \frac{m\pi}{2n} = k$ erit, uti ostendimus,

$$P = A = \frac{k\pi}{2n} = \frac{k\pi}{2n}$$

$$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2 \cdot 4nn}$$

$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2 \cdot 4 \cdot 6n^3}$$

$$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2 \cdot 4 \cdot 6 \cdot 8n^4}$$

$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10n^5}$$

173. Pari modo ultima forma §. 164; $\text{cos. } \frac{1}{2}v + \text{cot. } \frac{1}{2}g \times$
 $\text{sin. } \frac{1}{2}v = (1 + \frac{v}{g})(1 - \frac{v}{2\pi - g})(1 + \frac{v}{2\pi + g})(1 - \frac{v}{4\pi - g})$
 $(1 + \frac{v}{4\pi + g}) \&c.$ Si ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$, &
 $\text{tang. } \frac{m\pi}{2n} = k$, ut sit $\text{cot. } \frac{1}{2}g = \frac{1}{k}$, dabit $\text{cos. } \frac{\pi x}{2n} + \frac{1}{k} \times$
 $\text{sin. } \frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi^2 x^2}{2 \cdot 4nn} - \frac{\pi^3 x^3}{2 \cdot 4 \cdot 6n^3k} + \frac{\pi^4 x^4}{2 \cdot 4 \cdot 6 \cdot 8n^4} +$
 $\frac{\pi^5 x^5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10n^5k} - \&c. = (1 + \frac{x}{m})(1 - \frac{x}{2n-m})(1 + \frac{x}{2n+m})$
 $(1 -$

LIB. I.

$(1 - \frac{x}{4n-m})(1 + \frac{x}{4n+m})$ &c.. Comparatione ergo cum forma generali (§. 165) instituta erit $A = \frac{\pi}{2nk}$; $B = \frac{\pi\pi}{2 \cdot 4n^2}$; $C = \frac{\pi^3}{2 \cdot 4 \cdot 6n^3 k}$; $D = \frac{\pi^4}{2 \cdot 4 \cdot 6 \cdot 8n^4}$; $E = \frac{\pi^5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10n^5 k}$; &c.; ex Factoribus vero habebitur $\alpha = \frac{1}{m}$; $\beta = \frac{1}{2n-m}$; $\gamma = \frac{1}{2n+m}$; $\delta = \frac{1}{4n-m}$; $\epsilon = \frac{1}{4n+m}$ &c.

174. Hinc ergo ad normam §. 166. sequentes Series formabuntur, earumque summæ assignabuntur

$$P = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{\&c.}$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{\&c.}$$

$$R = \frac{1}{m^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} - \frac{1}{(4n-m)^3} + \frac{1}{(4n+m)^3} - \text{\&c.}$$

$$S = \frac{1}{m^4} + \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \frac{1}{(4n-m)^4} + \frac{1}{(4n+m)^4} + \text{\&c.}$$

$$T = \frac{1}{m^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} - \frac{1}{(4n-m)^5} + \frac{1}{(4n+m)^5} - \text{\&c.}$$

&c.

Hæ

Hæ autem summæ P, Q, R, S, &c. ita se habebunt CAP. X.

$$\begin{aligned}
 P &= A = \frac{\pi}{2nk} &= \frac{1\pi}{2nk} \\
 Q &= \frac{(kk+1)\pi\pi}{4nnkk} &= \frac{(2+2kk)\pi^2}{2\cdot 4n^2k^2} \\
 R &= \frac{(kk+1)\pi^3}{8n^3k^3} &= \frac{(6+6kk)\pi^3}{2\cdot 4\cdot 6n^3k^3} \\
 S &= \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} &= \frac{(24+32kk+3k^4)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4k^4} \\
 T &= \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} &= \frac{(120+200kk+80k^4)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k^5} \\
 V &= \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} &= \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12n^6k^6}
 \end{aligned}$$

&c.

175. Series istæ generales merentur ut casus quosdam particulares inde derivemus, qui prodibunt si rationem m ad n in numeris determinemus. Sit igitur primum $m = 1$ & $n = 2$, fiet $k = \text{tang. } \frac{\pi}{4} = \text{tang. } 45^\circ = 1$, atque ambæ Serierum classes inter se congruent. Erit ergo

$$\begin{aligned}
 \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c. \\
 \frac{\pi\pi}{8} &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c. \\
 \frac{\pi^3}{32} &= 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c. \\
 \frac{\pi^4}{96} &= 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c. \\
 \frac{5\pi^5}{1536} &= 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c. \\
 \frac{\pi^6}{960} &= 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.
 \end{aligned}$$

&c.

Harum Serierum primam jam supra (§. 140) elicuimus, reliquarum illæ, quæ pares habent Dignitates, modo ante (§. 169)

Euleri *Introducēt. in Anal. infin. parv.*

S

sunt

LIB. I. sunt erutæ; ceteræ, in quibus Exponentes sunt numeri impar-
 res, hic primum occurrunt. Constat ergo omnium quoque ista-
 rum Serierum:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \&c.$$

summas per valorem ipsius π assignari posse.

176. Sit nunc $m = 1$, $n = 3$; erit $k = \text{tang. } \frac{\pi}{6} =$
 $\text{tang. } 30^\circ = \frac{1}{\sqrt{3}}$; atque Series §. 172 abibunt in has

$$\begin{aligned} \frac{\pi}{6\sqrt{3}} &= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c. \\ \frac{\pi\pi}{27} &= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \&c. \\ \frac{\pi^3}{162\sqrt{3}} &= \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \&c. \\ &\&c., \text{ five} \end{aligned}$$

$$\begin{aligned} \frac{\pi}{3\sqrt{3}} &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c. \\ \frac{4\pi\pi}{27} &= 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c. \\ \frac{4\pi^3}{81\sqrt{3}} &= 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \&c. \\ &\&c. \end{aligned}$$

in his Seriebus defunt omnes numeri per ternarium divisibi-
 les: hinc pares dimensiones ex jam inventis deducuntur hoc
 modo. Cum sit

$$\begin{aligned} \frac{\pi\pi}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c., \text{ erit} \\ \frac{\pi\pi}{6.9} &= \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \&c. = \frac{\pi\pi}{54}, \end{aligned}$$

quæ posterior Series continens omnes numeros per ternarium
 divisi-

IN DEFINIEND. SUMMIS SERIER. INFINIT. 139
 divisibiles, si subtrahatur a priore, remanebunt omnes numeri CAP. X.
 non divisibiles per 3: sicque erit $\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} +$
 $\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.$, uti invenimus.

177. Eadem hypothesi $m = 1$, $n = 3$, & $k = \frac{1}{\sqrt{3}}$, ad
 §. 174. accommodata has præbebit summationes

$$\begin{aligned}\frac{\pi}{2\sqrt{3}} &= 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \&c. \\ \frac{\pi\pi}{9} &= 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c. \\ \frac{\pi^3}{18\sqrt{3}} &= 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \&c. \\ &\&c.\end{aligned}$$

in quarum denominatoribus numeri tantum impares occurrunt
 exceptis iis, qui per ternarium sunt divisibiles. Ceterum pares
 dimensiones ex jam cognitis deduci possunt, cum enim sit

$$\begin{aligned}\frac{\pi\pi}{8} &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c., \text{ erit} \\ \frac{\pi\pi}{8 \cdot 9} &= \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \&c. = \frac{\pi\pi}{72}\end{aligned}$$

quæ Series, omnes numeros impares per 3 divisibiles continens,
 si subtrahatur a superiore, relinquet Seriem quadratorum nu-
 merorum imparium per 3 non divisibilium, eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c.$$

178. Si Series in §. §. 172. & 174 inventæ vel addantur vel
 subtrahantur, obtinebuntur aliæ Series notatu dignæ. Erit sci-

$$\text{licet } \frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m}$$

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$$\text{LIB. I. } \frac{1}{2n+m} + \&c. = \frac{(kk+1)\pi}{2nk} : \text{at est } k = \text{tang. } \frac{m\pi}{2n} =$$

$$\frac{\sin. \frac{m\pi}{2n}}{\cos. \frac{m\pi}{2n}}, \& 1+kk = \frac{1}{(\cos. \frac{m\pi}{2n})^2}, \text{ unde } \frac{2k}{1+kk} = 2 \sin. \frac{m\pi}{2n} \times$$

$$\cos. \frac{m\pi}{2n} = \sin. \frac{m\pi}{n}, \text{ quo valore substituto habebimus}$$

$$\frac{\pi}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} +$$

$$\frac{1}{3n-m} - \frac{1}{3n+m} - \&c.. \text{ Simili modo per subtractionem}$$

$$\text{erit } \frac{\pi}{2nk} - \frac{k\pi}{2n} = \frac{(1-kk)\pi}{2nk} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} -$$

$$\frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \&c., \text{ at est}$$

$$\frac{2k}{1-kk} = \text{tang. } 2. \frac{m\pi}{2n} = \text{tang. } \frac{m\pi}{n} = \frac{\sin. \frac{m\pi}{n}}{\cos. \frac{m\pi}{n}}, \text{ hinc erit}$$

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} -$$

$$\frac{1}{3n-m} + \&c.. \text{ Series Quadratorum \& altiorum Potestatum}$$

hinc ortæ facilius per differentiationem hinc deducuntur infra.

179. Quoniam casus, quibus $m=1$ & $n=2$ vel 3 , jam evolvimus, ponamus $m=1$ & $n=4$; erit $\sin. \frac{m\pi}{n} =$

$$\sin. \frac{\pi}{4} = \frac{1}{\sqrt{2}} \& \cos. \frac{\pi}{4} = \frac{1}{\sqrt{2}}. \text{ Hinc itaque habebitur}$$

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} +$$

&c.

&

$$\& \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{CAP. X.}$$

$$\&c.. \text{ Sit } m = 1 \& n = 8, \text{ erit } \frac{m\varpi}{n} = \frac{\pi}{8} \& \sin. \frac{\varpi}{8} =$$

$$\sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)} \& \cos. \frac{\pi}{8} = \sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)} \& \frac{\cos. \frac{\pi}{8}}{\sin. \frac{\pi}{8}} =$$

$1 + \sqrt{2}$. Hinc itaque erit

$$\frac{\varpi}{4\sqrt{(2-\sqrt{2})}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \&c.$$

$$\frac{\varpi}{8(\sqrt{2}-1)} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \&c.$$

$$\text{Sit nunc } m = 3 \& n = 8, \text{ erit } \frac{m\varpi}{n} = \frac{3\varpi}{8} \& \sin. \frac{3\varpi}{8} =$$

$$\sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)}, \& \cos. \frac{3\pi}{8} = \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)}, \text{ unde } \frac{\cos. \frac{3\varpi}{8}}{\sin. \frac{3\varpi}{8}} =$$

$\frac{1}{\sqrt{2}+1}$; ac prodibunt hæ Series

$$\frac{\varpi}{4\sqrt{(2+\sqrt{2})}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \&c.$$

$$\frac{\varpi}{8(\sqrt{2}+1)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \&c.$$

180. Ex his Seriebus per combinationem nascuntur:

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} -$$

$$\frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} +$$

$$\frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}+\sqrt{2}-1)}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} +$$

$$\frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

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$$\begin{aligned}
 \text{LIB. I. } \frac{\pi(\sqrt{4+2\sqrt{2}}-\sqrt{2+1})}{8} &= 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \\
 &\quad \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c. \\
 \frac{\pi(\sqrt{2+1}+\sqrt{4-2\sqrt{2}})}{8} &= 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \\
 &\quad \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c. \\
 \frac{\pi(\sqrt{2+1}-\sqrt{4-2\sqrt{2}})}{8} &= 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \\
 &\quad \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \&c.
 \end{aligned}$$

Simili modo, ponendo $n = 16$ & m vel 1 vel 3 vel 5 vel 7, ulterius progredi licet, hocque modo summæ reperiuntur Serierum 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, &c., in quibus signorum + & - vicissitudines alias leges sequantur.

181. Si in Seriebus §. 178. inventis bini termini in unam summam colligantur, erit

$$\begin{aligned}
 \frac{\pi}{n \sin. \frac{m\pi}{n}} &= \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} - \\
 &\quad \frac{2m}{16nn-mm} + \&c.
 \end{aligned}$$

ideoque

$$\begin{aligned}
 \frac{1}{nn-mm} - \frac{1}{4nn-mm} + \frac{1}{9nn-mm} - \&c. &= \\
 \frac{\pi}{2mn \sin. \frac{m\pi}{n}} - \frac{1}{2m-m} &
 \end{aligned}$$

Altera vero Series dabit

$$\begin{aligned}
 \frac{\pi}{n \tan. \frac{m\pi}{n}} &= \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \\
 &\quad \&c.
 \end{aligned}$$

hincque

$$\frac{1}{nn-mm} + \frac{1}{4nn-mm} + \frac{1}{9nn-mm} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn \operatorname{tang.} \frac{m}{n} \pi}.$$

Ex his autem conjunctis nascitur hæc

$$\frac{1}{nn-mm} + \frac{1}{9nn-mm} + \frac{1}{25nn-mm} + \&c. = \frac{\pi \operatorname{tang.} \frac{m}{2n} \pi}{4mn}.$$

Si in hac Serie sit $n=1$ & m numerus par quicumque $=2k$, ob $\operatorname{tang.} k\pi = 0$, erit semper, nisi sit $k=0$,

$$\frac{1}{1-4kk} + \frac{1}{9-4kk} + \frac{1}{25-4kk} + \frac{1}{49-4kk} + \&c. = 0,$$

sin autem in illa Serie fiat $n=2$ & m fuerit numerus quicumque impar $=2k+1$, ob $\frac{1}{\operatorname{tang.} \frac{m}{n} \pi} = 0$, erit $\frac{1}{4-(2k+1)^2} +$

$$\frac{1}{16-(2k+1)^2} + \frac{1}{36-(2k+1)^2} + \&c. = \frac{1}{2(2k+1)^2}.$$

182. Multiplicentur Series inventæ per nn sitque $\frac{m}{n}=p$, habebuntur istæ formæ

$$\frac{1}{1-pp} - \frac{1}{4-pp} + \frac{1}{9-pp} - \frac{1}{16-pp} + \&c. = \frac{\pi}{2p \sin. p\pi} - \frac{1}{2pp}.$$

$$\frac{1}{1-pp} + \frac{1}{4-pp} + \frac{1}{9-pp} + \frac{1}{16-pp} + \&c. = \frac{1}{2pp} - \frac{\pi}{2p \sin. p\pi}.$$

Sit $pp=a$, atque nascentur hæc Series

$$\frac{1}{1-a} - \frac{1}{4-a} + \frac{1}{9-a} - \frac{1}{16-a} + \&c. = \frac{\pi \sqrt{a}}{2a \sin. \pi \sqrt{a}} - \frac{1}{2a}$$

$$\frac{1}{1-a} + \frac{1}{4-a} + \frac{1}{9-a} + \frac{1}{16-a} + \&c. = \frac{1}{2a} - \frac{\pi \sqrt{a}}{2a \operatorname{tang.} \pi \sqrt{a}}$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, summa harum Serierum per Circulum exhiberi poterit.

183. Per reductionem autem exponentialium imaginariorum ad Sinus & Cofinus Arcuum circularium supra traditam poterimus quoque summas harum Serierum assignare si a sit numerus negativus. Cum enim sit $e^{x\sqrt{-1}} = \cos. x + \sqrt{-1} \times \sin. x$ & $e^{-x\sqrt{-1}} = \cos. x - \sqrt{-1} \cdot \sin. x$, erit vicissim, posito $y\sqrt{-1}$ loco x ; $\cos. y\sqrt{-1} = \frac{e^{-y} + e^y}{2}$ & $\sin. y\sqrt{-1} = \frac{e^{-y} - e^y}{2\sqrt{-1}}$. Quod si ergo $a = -b$ & $y = \omega\sqrt{b}$, erit $\cos. \omega\sqrt{-b} = \frac{e^{-\omega\sqrt{b}} + e^{\omega\sqrt{b}}}{2}$ & $\sin. \omega\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{2\sqrt{-1}}$; ideoque $\tan. \pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}})\sqrt{-1}}$. Hinc erit $\frac{\pi\sqrt{-b}}{\sin. \omega\sqrt{-b}} = \frac{-2\omega\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$; & $\frac{\omega\sqrt{-b}}{\tan. \pi\sqrt{-b}} = \frac{(e^{-\omega\sqrt{b}} + e^{\omega\sqrt{b}})\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$. His ergo notatis, erit $\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \&c. = \frac{1}{2b} - \frac{(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}})b}{\pi\sqrt{b}}$ $\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \&c. = \frac{(e^{\pi\sqrt{b}} + e^{-\pi\sqrt{b}})\omega\sqrt{b}}{2b(e^{\pi\sqrt{b}} - e^{-\omega\sqrt{b}})} - \frac{1}{2b}$. Eadem hæ Series deduci possunt ex §. 162. adhibendo eandem methodum, qua
in

in hoc capite sum usus. Quoniam vero hoc pacto reductio CAP. X. Sinuum & Cosinum Arcuum imaginariorum ad quantitates exponentiales reales, non mediocriter illustratur, hanc explanationem alteri præferendam duxi.

CAPUT XI.

De aliis Arcuum atque Sinuum expressionibus infinitis.

184. **Q**Uoniam supra (158.), denotante z Arcum Circuli quemcunque, vidimus esse $\sin. z = z \left(1 - \frac{z z}{\pi \pi} \right) \left(1 - \frac{z z}{4 \pi \pi} \right) \left(1 - \frac{z z}{9 \pi \pi} \right) \left(1 - \frac{z z}{16 \pi \pi} \right) \&c.$, & $\cos. z = \left(1 - \frac{4 z z}{\pi \pi} \right) \left(1 - \frac{4 z z}{9 \pi \pi} \right) \left(1 - \frac{4 z z}{25 \pi \pi} \right) \left(1 - \frac{4 z z}{49 \pi \pi} \right) \&c.$, ponamus esse Arcum $z = \frac{m \pi}{n}$, erit $\sin. \frac{m \pi}{n} = \frac{m \pi}{n} \left(1 - \frac{m m}{n n} \right) \left(1 - \frac{m m}{4 n n} \right) \left(1 - \frac{m m}{9 n n} \right) \left(1 - \frac{m m}{16 n n} \right) \&c.$, & $\cos. \frac{m}{n} \pi = \left(1 - \frac{4 m m}{n n} \right) \left(1 - \frac{4 m m}{9 n n} \right) \left(1 - \frac{4 m m}{25 n n} \right) \left(1 - \frac{4 m m}{49 n n} \right) \&c.$. Vel ponatur z loco n , ut prodeant hæ expressiones

$$\sin. \frac{m \pi}{2 n} = \frac{m \pi}{2 n} \cdot \left(\frac{4 n n - m m}{4 n n} \right) \cdot \left(\frac{16 n n - m m}{16 n n} \right) \cdot \left(\frac{36 n n - m m}{36 n n} \right) \&c.$$

$$\cos. \frac{m \pi}{2 n} = \left(\frac{n n - m m}{n n} \right) \cdot \left(\frac{9 n n - m m}{9 n n} \right) \cdot \left(\frac{25 n n - m m}{25 n n} \right) \cdot \left(\frac{49 n n - m m}{49 n n} \right) \&c.,$$

quæ, in Factores simplices resolutæ, dant

$$\sin. \frac{m \pi}{2 n} = \frac{m \pi}{2 n} \cdot \left(\frac{2 n - m}{2 n} \right) \cdot \left(\frac{2 n + m}{2 n} \right) \cdot \left(\frac{4 n - m}{4 n} \right) \cdot \left(\frac{4 n + m}{4 n} \right) \cdot \left(\frac{6 n - m}{6 n} \right) \&c.$$

LIB. I. $\cos. \frac{m\pi}{2n} = \left(\frac{n-m}{n}\right) \left(\frac{n+m}{n}\right) \left(\frac{3n-m}{3n}\right) \left(\frac{3n+m}{3n}\right) \left(\frac{5n-m}{5n}\right) \left(\frac{5n+m}{5n}\right) \&c.$

Ponatur $n-m$ loco m , quia est $\sin. \frac{(n-m)\pi}{2n} = \cos. \frac{m\pi}{2n}$ &
 $\cos. \frac{(n-m)\pi}{2n} = \sin. \frac{m\pi}{2n}$, provenient hæ expressiones.

$$\cos. \frac{m\pi}{2n} = \left(\frac{(n-m)\pi}{2n}\right) \left(\frac{n+m}{2n}\right) \left(\frac{3n-m}{2n}\right) \left(\frac{3n+m}{4n}\right) \left(\frac{5n-m}{4n}\right) \left(\frac{5n+m}{6n}\right) \&c.$$

$$\sin. \frac{m\pi}{2n} = \frac{m}{n} \left(\frac{2n-m}{n}\right) \left(\frac{2n+m}{3n}\right) \left(\frac{4n-m}{3n}\right) \left(\frac{4n+m}{5n}\right) \left(\frac{6n-m}{5n}\right) \&c.$$

185. Cum igitur pro Sinu & Cosinu Anguli $\frac{m\pi}{2n}$ binæ habeantur expressiones, si eæ inter se comparentur dividendo, erit $1 = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot \&c.$,

ideoque $\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13} \&c.$, quæ est

expressio pro Peripheria Circuli, quam WALLISIUS invenit in *Arithmetica infinitorum*. Similes autem huic innumeras expressiones exhibere licet ope primæ expressionis pro Sinu; ex ea enim deducitur fore:

$$\frac{\pi}{2} = \frac{n}{m} \cdot \sin. \frac{m\pi}{2n} \left(\frac{2n}{2n-m}\right) \left(\frac{2n}{2n+m}\right) \left(\frac{4n}{4n-m}\right) \left(\frac{4n}{4n+m}\right) \left(\frac{6n}{6n-m}\right) \&c.,$$

quæ, posito $\frac{m}{n} = 1$, præbet illam ipsam WALLISII formulam.

Sit

Sit ergo $\frac{m}{n} = \frac{1}{2}$, ob $\sin. \frac{1}{4} \pi = \frac{1}{\sqrt{2}}$, erit

$$\frac{\pi}{2} = \frac{\sqrt{2}}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{16}{15} \cdot \frac{16}{17} \cdot \&c.$$

Sit $\frac{m}{n} = \frac{1}{3}$, ob $\sin. \frac{1}{6} \pi = \frac{1}{2}$, erit

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \&c.$$

Quod si Expressio *Wallisiana* dividatur per illam ubi $\frac{m}{n} = \frac{1}{2}$,
erit $\sqrt{2} = \frac{2 \cdot 2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \cdot 18 \cdot 18}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19} \cdot \&c.$

186. Quoniam Tangens cujusque Anguli æquatur Sinui per Cofinum diviso, Tangens quoque per hujusmodi Factores infinitos exprimi poterit. Quod si autem prima Sinus expressio dividatur per alteram Cofinus expressionem, erit

$$\text{tang. } \frac{m \pi}{2n} = \frac{m}{n-m} \left(\frac{2n-m}{n+m} \right) \left(\frac{2n+m}{3n-m} \right) \left(\frac{4n-m}{3n+m} \right) \left(\frac{4n+m}{5n-m} \right) \&c.,$$

$$\text{cot. } \frac{m \pi}{2n} = \frac{n-m}{m} \left(\frac{n+m}{2n-m} \right) \left(\frac{3n-m}{2n+m} \right) \left(\frac{3n+m}{4n-m} \right) \left(\frac{5n-m}{4n+m} \right) \&c..$$

Simili modo autem Secantes & Cofecantes exprimentur

$$\text{sec. } \frac{m \pi}{2n} = \left(\frac{n}{n-m} \right) \left(\frac{n}{n+m} \right) \left(\frac{3n}{3n-m} \right) \left(\frac{3n}{3n+m} \right) \left(\frac{5n}{5n-m} \right) \left(\frac{5n}{5n+m} \right) \&c.$$

$$\text{cofec. } \frac{m \pi}{2n} = \frac{n}{m} \left(\frac{n}{2n-m} \right) \left(\frac{3n}{2n+m} \right) \left(\frac{3n}{4n-m} \right) \left(\frac{5n}{4n+m} \right) \left(\frac{5n}{6n-m} \right) \&c.$$

Sin autem alteræ Sinuum & Cofinum formulæ combinentur, erit

T 2

tang.

LIB. I.

$$\begin{aligned}
 \text{tang. } \frac{m\pi}{2n} &= \frac{\pi}{2} \cdot \frac{m}{n-m} \cdot \frac{1(2n-m)}{2(n+m)} \cdot \frac{3(2n+m)}{2(3n-m)} \cdot \frac{3(4n-m)}{4(3n+m)} \cdot \&c. \\
 \text{cot. } \frac{m\pi}{2n} &= \frac{\pi}{2} \cdot \frac{n-m}{m} \cdot \frac{1(n+m)}{2(2n-m)} \cdot \frac{3(3n-m)}{2(2n+m)} \cdot \frac{3(3n+m)}{4(4n-m)} \cdot \&c. \\
 \text{sec. } \frac{m\pi}{2n} &= \frac{2}{\pi} \cdot \frac{n}{n-m} \cdot \frac{2n}{n+m} \cdot \frac{2n}{3n-m} \cdot \frac{4n}{3n+m} \cdot \frac{4n}{5n-m} \cdot \&c. \\
 \text{cosec. } \frac{m\pi}{2n} &= \frac{2}{\pi} \cdot \frac{n}{m} \cdot \frac{2n}{2n-m} \cdot \frac{2n}{2n+m} \cdot \frac{4n}{4n-m} \cdot \frac{4n}{4n+m} \cdot \&c.
 \end{aligned}$$

187. Si loco m scribatur k , similique modo Anguli $\frac{k\pi}{2n}$ Sinus & Cofinus definiantur, ac per has expressiones illæ priores dividantur, prodibunt istæ formulæ

$$\begin{aligned}
 \frac{\sin. \frac{m\pi}{2n}}{\sin. \frac{k\pi}{2n}} &= \frac{m}{k} \cdot \frac{2n-m}{2n-k} \cdot \frac{2n+m}{2n+k} \cdot \frac{4n-m}{4n-k} \cdot \frac{4n+m}{4n+k} \cdot \&c. \\
 \frac{\sin. \frac{m\pi}{2n}}{\cos. \frac{k\pi}{2n}} &= \frac{m}{n-k} \cdot \frac{2n-m}{n+k} \cdot \frac{2n+m}{3n-k} \cdot \frac{4n-m}{3n+k} \cdot \frac{4n+m}{5n-k} \cdot \&c. \\
 \frac{\cos. \frac{m\pi}{2n}}{\cos. \frac{k\pi}{2n}} &= \left(\frac{n-m}{n-k}\right) \left(\frac{n+m}{n+k}\right) \left(\frac{3n-m}{3n-k}\right) \left(\frac{3n+m}{3n+k}\right) \left(\frac{5n-m}{5n-k}\right) \cdot \&c. \\
 \frac{\cos. \frac{m\pi}{2n}}{\sin. \frac{k\pi}{2n}} &= \left(\frac{n-m}{k}\right) \left(\frac{n+m}{2n-k}\right) \left(\frac{3n-m}{2n+k}\right) \left(\frac{3n+m}{4n-k}\right) \left(\frac{5n-m}{4n+k}\right) \cdot \&c.
 \end{aligned}$$

Sumto ergo pro $\frac{k\pi}{2n}$ ejusmodi Angulo cujus Sinus & Cofinus dentur, per hos licebit alius cujuscunque Anguli $\frac{m\pi}{2n}$ Sinum & Cofinum determinare.

188. Vicissim igitur hujusmodi expressionum, quæ ex Factoribus

etoribus infinitis constant, valores veri vel per Circuli Peripheriam, vel per Sinus & Cofinus Angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiam nunc aliæ methodi non constant, quarum ope hujusmodi productorum infinitorum valores exhiberi queant. Ceterum vero hujusmodi expressiones parum utilitatis afferunt, ad valores cum ipsius π tum Sinuum Cofinumve Angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti Factores $-\frac{\pi}{2} =$

$2(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras justum invenire vellemus.

189. Præcipuus autem usus hujusmodi expressionum, etsi infinitarum, in inventione Logarithmorum versatur, in quo negotio Factorum utilitas tanta est, ut sine illis Logarithmorum supputatio esset difficillima. Ac primo quidem, cum sit $\pi =$

$4(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., erit, sumendis Logarithmis, $l\pi = l4 + l(1 - \frac{1}{9}) + l(1 - \frac{1}{25}) + l(1 - \frac{1}{49}) +$
 &c., vel $l\pi = l2 - l(1 - \frac{1}{4}) - l(1 - \frac{1}{16}) - l(1 - \frac{1}{36})$

— &c., sive Logarithmi communes sive hyperbolici sumantur. Quoniam vero ex Logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad Logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur, Logarithmis hyperbolicis sumendis, sit $l(1 - x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ &c., si hoc modo singuli termini evolvantur, erit

LIB. I.

$$l\pi = l_4 \left\{ \begin{array}{l} -\frac{1}{9} - \frac{1}{2 \cdot 9^2} - \frac{1}{3 \cdot 9^3} - \frac{1}{4 \cdot 9^4} - \&c. \\ -\frac{1}{25} - \frac{1}{2 \cdot 25^2} - \frac{1}{3 \cdot 25^3} - \frac{1}{4 \cdot 25^4} - \&c. \\ -\frac{1}{49} - \frac{1}{2 \cdot 49^2} - \frac{1}{3 \cdot 49^3} - \frac{1}{4 \cdot 49^4} - \&c. \\ \&c. \end{array} \right.$$

In his Seriebus numero infinitis verticaliter descendendo ejusmodi prodeunt Series, quarum summas supra jam invenimus, quare si brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$

$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.$$

$$\text{erit } l\pi = l_4 - (A - 1) - \frac{1}{2}(B - 1) - \frac{1}{3}(C - 1) - \frac{1}{4}(D - 1) - \&c.$$

Est vero, summis supra inventis proxime exprimendis,

$$A = 1, 23370055013616982735431$$

$$B = 1, 01467803160419205454625$$

$$C = 1, 00144707664094212190647$$

$$D = 1, 00015517902529611930298$$

$$E = 1, 00001704136304482550816$$

$$F = 1, 00000188584858311957590$$

$$G = 1, 00000020924051921150010$$

$$H = 1, 00000002323715737915670$$

I =

<i>I</i>	=	1, 00000000258143755665977
<i>K</i>	=	1, 00000000028680769745558
<i>L</i>	=	1, 00000000003186677514044
<i>M</i>	=	1, 00000000000354072294392
<i>N</i>	=	1, 00000000000039341246691
<i>O</i>	=	1, 00000000000004371244859
<i>P</i>	=	1, 00000000000000485693682
<i>Q</i>	=	1, 00000000000000053965957
<i>R</i>	=	1, 00000000000000005996217
<i>S</i>	=	1, 00000000000000000666246
<i>T</i>	=	1, 00000000000000000074027
<i>V</i>	=	1, 00000000000000000008225
<i>W</i>	=	1, 00000000000000000000913
<i>X</i>	=	1, 000000000000000000000101

Hinc sine tædioſo calculo reperitur Logarithmus hyperbolicus ipſius $\pi = 1, 14472288584940017414342$, qui ſi multiplicetur per $0, 43429$ &c., prodit Logarithmus vulgaris ipſius $\pi = 0, 49714987269413385435126$.

191. Quia porro tam Sinum quam Cofinum Anguli $\frac{m\pi}{2n}$ expreſſum habemus per Factores numero infinitos, utriusque Logarithmum commode exprimere poterimus. Erit autem ex formulis primo inventis

$$l \sin. \frac{m\pi}{2n} = l \pi + l \frac{m}{2n} + l \left(1 - \frac{mm}{4nn} \right) + l \left(1 - \frac{mm}{16nn} \right) + l \left(1 - \frac{mm}{36nn} \right) \&c.$$

$$l \cos. \frac{m\pi}{2n} = l \left(1 - \frac{mm}{nn} \right) + l \left(1 - \frac{mm}{9nn} \right) + l \left(1 - \frac{mm}{25nn} \right) + l \left(1 - \frac{mm}{49nn} \right) + \&c.$$

Hinc primum Logarithmi hyperbolici, ut ante, per Series maxime convergentes facile exprimuntur. Ne autem præter
neceſſi-

LIB. I. necessitatem Series infinitas multiplicemus, terminos priores
 — actu in Logarithmis involutos relinquamus, eritque

$$\begin{aligned}
 l \sin. \frac{m\pi}{2n} &= l\pi + lm + l(2n - m) + l(2n + m) - l8 - 3ln \\
 &\quad - \frac{mm}{16nn} - \frac{m^4}{2 \cdot 16^2 n^4} - \frac{m^6}{3 \cdot 16^3 n^6} - \frac{m^8}{4 \cdot 16^4 n^8} - \&c. \\
 &\quad - \frac{mm}{36nn} - \frac{m^4}{2 \cdot 36^2 n^4} - \frac{m^6}{3 \cdot 36^3 n^6} - \frac{m^8}{4 \cdot 36^4 n^8} - \&c. \\
 &\quad - \frac{mm}{64nn} - \frac{m^4}{2 \cdot 64^2 n^4} - \frac{m^6}{3 \cdot 64^3 n^6} - \frac{m^8}{4 \cdot 64^4 n^8} - \&c. \\
 &\quad \&c.
 \end{aligned}$$

$$\begin{aligned}
 l \cos. \frac{m\pi}{2n} &= l(n - m) + l(n + m) - 2ln \\
 &\quad - \frac{mm}{9nn} - \frac{m^4}{2 \cdot 9^2 n^4} - \frac{m^6}{3 \cdot 9^3 n^6} - \frac{m^8}{4 \cdot 9^4 n^8} - \&c. \\
 &\quad - \frac{mm}{25nn} - \frac{m^4}{2 \cdot 25^2 n^4} - \frac{m^6}{3 \cdot 25^3 n^6} - \frac{m^8}{4 \cdot 25^4 n^8} - \&c. \\
 &\quad - \frac{mm}{49nn} - \frac{m^4}{2 \cdot 49^2 n^4} - \frac{m^6}{3 \cdot 49^3 n^6} - \frac{m^8}{4 \cdot 49^4 n^8} - \&c. \\
 &\quad \&c.
 \end{aligned}$$

192. Occurrunt ergo in his Seriebus singulae Potestates pares ipsius $\frac{m}{n}$, quae sunt multiplicatae per Series, quarum summas jam supra assignavimus. Erit nempe

$$\begin{aligned}
 l \sin. \frac{m\pi}{2n} &= lm + l(2n - m) + l(2n + m) - 3ln + l\pi - l8 \\
 &\quad - \frac{mm}{nn} \left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \&c. \right) \\
 &\quad - \frac{m^4}{2n^4} \left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \&c. \right) \\
 &\quad - \frac{m^6}{3n^6} \left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \&c. \right) \\
 &\quad - \frac{m^8}{4n^8} \left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \&c. \right) \\
 &\quad \&c.
 \end{aligned}$$

$l \cos.$

$$l \cos. \frac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$

$$\begin{aligned} & - \frac{mm}{nn} \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c. \right) \\ & - \frac{m^4}{2n^4} \left(\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c. \right) \\ & - \frac{m^6}{3n^6} \left(\frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c. \right) \\ & - \frac{m^8}{4n^8} \left(\frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c. \right) \\ & \&c. \end{aligned}$$

Serierum posteriorum modo ante (§. 190) summæ sunt exhibitæ; priores Series quidem ex his derivari possent, at, quo facilius ad usum transferri queant, earum summas pariter hic adjiciam.

193. Quod si ergo, brevitatis gratia, ponamus

$$\begin{aligned} \alpha &= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \&c. \\ \epsilon &= \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \&c. \\ \gamma &= \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \&c. \\ \delta &= \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \&c. \\ &\&c. \end{aligned}$$

erunt summæ in numeris proxime expressæ hæ :

$$\begin{aligned} \alpha &= 0, 41123351671205660911810 \\ \epsilon &= 0, 06764520210694613696975 \\ \gamma &= 0, 01589598534350701780804 \\ \delta &= 0, 00392217717264822007570 \\ \epsilon &= 0, 00097753376477325984898 \\ \xi &= 0, 00024420070472492872274 \end{aligned}$$

Euleri *Introduct. in Anal. infin. parv.*

V

$\eta =$

π	\equiv	0, 00006103889453949332915
θ	\equiv	0, 00001525902225127269977
ι	\equiv	0, 00000381471182744318008
κ	\equiv	0, 00000095367522617534053
λ	\equiv	0, 00000023841863595259154
μ	\equiv	0, 00000005960464832831555
ν	\equiv	0, 00000001490116141589813
ζ	\equiv	0, 00000000372529031233986
\bullet	\equiv	0, 00000000093132257548284
π	\equiv	0, 00000000023283064370807
ρ	\equiv	0, 00000000005820766091685
σ	\equiv	0, 00000000001455191522858
τ	\equiv	0, 00000000000363797880710
υ	\equiv	0, 00000000000090949470177
ϕ	\equiv	0, 000000000000022737367544
χ	\equiv	0, 000000000000005684341886
ψ	\equiv	0, 000000000000001421085471
ω	\equiv	0, 000000000000000355271367

reliquæ summæ in ratione quadrupla descrescunt.

194. His ergo in subsidium vocatis, erit

$$\begin{aligned}
 l \sin. \frac{m\pi}{2n} &= lm + l(2n - m) + l(2n + m) - 3ln + l\pi - l8 \\
 &- \frac{mm}{nn} \left(\alpha - \frac{1}{2^2} \right) - \frac{m^3}{2n^3} \left(\epsilon - \frac{1}{2^4} \right) - \frac{m^5}{3n^5} \left(\gamma - \frac{1}{2^6} \right) \\
 &- \&c.
 \end{aligned}$$

$$\begin{aligned}
 l \cos. \frac{m\pi}{2n} &= l(n - m) + l(n + m) - 2ln \\
 &- \frac{mm}{nn} (A - 1) - \frac{m^3}{2n^3} (B - 1) - \frac{m^5}{3n^5} (C - 1) - \&c.
 \end{aligned}$$

quoniam igitur Logarithmi $l\pi$ & $l8$ dantur, erit

Logarith-

Logarithmus hyperbolicus Sinus Anguli $\frac{m}{n} 90^\circ =$

CAP. XI.

$$lm + l(2n - m) + l(2n + m) - 3ln$$

—		0, 93471165583043575410
—	$\frac{m^2}{n^2}$	0, 16123351671205660911
—	$\frac{m^4}{n^4}$	0, 00257260105347306848
—	$\frac{m^6}{n^6}$	0, 00009032844783567260
—	$\frac{m^8}{n^8}$	0, 00000398179316205501
—	$\frac{m^{10}}{n^{10}}$	0, 00000019425295465196
—	$\frac{m^{12}}{n^{12}}$	0, 00000001001328748812
—	$\frac{m^{14}}{n^{14}}$	0, 00000000053404135618
—	$\frac{m^{16}}{n^{16}}$	0, 00000000002914859658
—	$\frac{m^{18}}{n^{18}}$	0, 00000000000161797979
—	$\frac{m^{20}}{n^{20}}$	0, 00000000000009097690
—	$\frac{m^{22}}{n^{22}}$	0, 00000000000000516827
—	$\frac{m^{24}}{n^{24}}$	0, 00000000000000029607
—	$\frac{m^{26}}{n^{26}}$	0, 00000000000000001708
—	$\frac{m^{28}}{n^{28}}$	0, 00000000000000000099
—	$\frac{m^{30}}{n^{30}}$	0, 00000000000000000005

LIB. I.

At Logarithmus hyperbolicus Cofinus Ang. $\frac{m}{n}$ $90^\circ =$
 $l(n - m) + l(n + m) - 2ln$

$\frac{m^2}{n^2}$	0, 23370055013616982735
$\frac{m^4}{n^4}$	0, 00733901580209602727
$\frac{m^6}{n^6}$	0, 00048235888031404063
$\frac{m^8}{n^8}$	0, 00003879475632402982
$\frac{m^{10}}{n^{10}}$	0, 00000340827260896510
$\frac{m^{12}}{n^{12}}$	0, 00000031430809718659
$\frac{m^{14}}{n^{14}}$	0, 00000002989150274450
$\frac{m^{16}}{n^{16}}$	0, 00000000290464467239
$\frac{m^{18}}{n^{18}}$	0, 00000000028682639518
$\frac{m^{20}}{n^{20}}$	0, 00000000002868076974
$\frac{m^{22}}{n^{22}}$	0, 00000000000289697956
$\frac{m^{24}}{n^{24}}$	0, 00000000000029506024
$\frac{m^{26}}{n^{26}}$	0, 00000000000003026249
$\frac{m^{28}}{n^{28}}$	0, 00000000000000312232
$\frac{m^{30}}{n^{30}}$	0, 00000000000000032379
$\frac{m^{32}}{n^{32}}$	0, 00000000000000003373
$\frac{m^{34}}{n^{34}}$	0, 00000000000000000352

$$\begin{aligned} & \frac{m^{36}}{n^{36}} \cdot 0, 00000000000000000037 \\ & \frac{m^{38}}{n^{38}} \cdot 0, 00000000000000000004 \end{aligned}$$

195. Si isti Sinuum & Cofinum Logarithmi hyperbolici multiplicentur per 0, 4342944819 &c., prodibunt eorundem Logarithmi vulgares ad Radium = 1 relati. Quoniam vero in Tabulis Logarithmis Sinus totius statui solet = 10, quo Logarithmi tabulares Sinuum & Cofinum obtineantur, post multiplicationem addi debet 10. Hinc erit

$$\begin{aligned} & \text{Logarithmus tabularis Sinus Anguli } \frac{m}{n} 90^\circ = \\ & l m + l(2n - m) + l(2n + m) - 3 l n \end{aligned}$$

$$\begin{aligned} & + 9, 594059885702190 \\ & \frac{m^2}{n^2} \cdot 0, 070022826605901 \\ & \frac{m^4}{n^4} \cdot 0, 001117266441661 \\ & \frac{m^6}{n^6} \cdot 0, 000039229146453 \\ & \frac{m^8}{n^8} \cdot 0, 000001729270798 \\ & \frac{m^{10}}{n^{10}} \cdot 0, 000000084362986 \\ & \frac{m^{12}}{n^{12}} \cdot 0, 000000004348715 \\ & \frac{m^{14}}{n^{14}} \cdot 0, 000000000231931 \\ & \frac{m^{16}}{n^{16}} \cdot 0, 000000000012659 \\ & \frac{m^{18}}{n^{18}} \cdot 0, 000000000000702 \\ & \frac{m^{20}}{n^{20}} \cdot 0, 000000000000039 \end{aligned}$$

$$\text{Logarithmus tabularis Cofinus Anguli } \frac{m}{n} 90^\circ = \\ l(n - m) + l(n + m) - 2l n$$

+	10, 0000000000000000
— $\frac{m^2}{n^2}$	0, 101494859341892
— $\frac{m^4}{n^4}$	0, 003187294065451
— $\frac{m^6}{n^6}$	0, 000209485800017
— $\frac{m^8}{n^8}$	0, 000016848348597
— $\frac{m^{10}}{n^{10}}$	0, 000001480193986
— $\frac{m^{12}}{n^{12}}$	0, 000000136502272
— $\frac{m^{14}}{n^{14}}$	0, 000000012981715
— $\frac{m^{16}}{n^{16}}$	0, 000000001261471
— $\frac{m^{18}}{n^{18}}$	0, 000000000124567
— $\frac{m^{20}}{n^{20}}$	0, 000000000012456
— $\frac{m^{22}}{n^{22}}$	0, 000000000001258
— $\frac{m^{24}}{n^{24}}$	0, 000000000000128
— $\frac{m^{26}}{n^{26}}$	0, 000000000000013

196. Harum ergo formularum ope inveniri possunt Logarithmi Sinuum & Cofinuum quorumvis Angulorum tam hyperbolici quam vulgares, etiam ignoratis ipsis Sinibus & Cofinibus. Ex Logarithmis autem Sinuum & Cofinuum per solam subtractionem inveniuntur Logarithmi Tangentium, Cofinuum tangen-

tangentium, & Secantium, Cofecantiumque, quamobrem pro CAP. XI. his peculiaribus formulis non erit opus. Ceterum notandum est numerorum $m, n, n - m, n + m, \&c.$ Logarithmos hyperbolicos accipi oportere, cum Logarithmi hyperbolici Sinuum Cofinumque quærantur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Præterea $m : n$ denotat rationem, quam Angulus propositus habet ad Angulum rectum; sicque, cum Sinus Angulorum semirecto majorum æquantur Cofinibus Angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam major accipienda erit quam $\frac{1}{2}$, hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum Tangentes & Secantes quorumvis Angulorum inveniendi, quam Caput præcedens suppeditat. Quamquam enim Tangentes & Secantes per Sinus & Cofinus determinantur; tamen hoc sit per divisionem, quæ operatio in tantis numeris nimis est operosa. Ac Tangentes quidem & Cotangentes jam supra (§. 136.) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic Capiti reservavimus.

198. Ex §. 181. ergo primum expressionem pro Tangente Anguli $\frac{m}{2n} \pi$ elicimus. Cum enim sit $\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. = \frac{\varpi}{4mn} \cdot \text{tang. } \frac{m}{2n} \varpi$ erit $\text{tang. } \frac{m}{2n} \varpi = \frac{4mn}{\varpi} \left(\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. \right)$. Cum deinde sit $\frac{1}{nn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + \&c. = \frac{1}{2mm} - \frac{\varpi}{2mn} \cdot \text{cot. } \frac{m}{n} \varpi$, si pro n scribamus $2n$ erit

$$\text{cot. } \frac{m}{2n} \pi = \frac{2n}{m\varpi} - \frac{4mn}{\varpi} \left(\frac{1}{4nn - mm} + \frac{1}{16nn - mm} + \frac{1}{36nn - mm} + \&c. \right)$$

LIB. I. $\frac{1}{36nn - mm} + \&c.$). Convertantur hæ fractiones, præter primas, quippe quæ facile in computum ducuntur, in Series infinitas, erit

$$\begin{aligned} \text{tang. } \frac{m}{2n} \pi &= \frac{mn}{nn - mm} \cdot \frac{4}{\pi} \\ &+ \frac{4}{\pi} \left(\frac{m}{3^2 n} + \frac{m^3}{3^4 n^3} + \frac{m^5}{3^6 n^5} + \&c. \right) \\ &+ \frac{4}{\pi} \left(\frac{m}{5^2 n} + \frac{m^3}{5^4 n^3} + \frac{m^5}{5^6 n^5} + \&c. \right) \\ &+ \frac{4}{\pi} \left(\frac{m}{7^2 n} + \frac{m^3}{7^4 n^3} + \frac{m^5}{7^6 n^5} + \&c. \right) \\ &\&c. \end{aligned}$$

$$\begin{aligned} \text{cot. } \frac{m}{2n} \varpi &= \frac{n}{m} \cdot \frac{2}{\varpi} - \frac{mn}{4nn - mm} \cdot \frac{4}{\varpi} \\ &- \frac{4}{\pi} \left(\frac{m}{4^2 n} + \frac{m^3}{4^4 n^3} + \frac{m^5}{4^6 n^5} + \&c. \right) \\ &- \frac{4}{\pi} \left(\frac{m}{6^2 n} + \frac{m^3}{6^4 n^3} + \frac{m^5}{6^6 n^5} + \&c. \right) \\ &- \frac{4}{\pi} \left(\frac{m}{8^2 n} + \frac{m^3}{8^4 n^3} + \frac{m^5}{8^6 n^5} + \&c. \right) \\ &\&c. \end{aligned}$$

198. At ex valore ipsius ϖ cognito reperitur

$\frac{1}{\pi} = 0, 318309886183790671537767926745028724,$
deinde hic eadem Series occurrunt, quas supra litteris A, B, C, D, &c., & $\alpha, \beta, \gamma, \delta, \&c.$, indicavimus. His ergo notatis, erit

$$\begin{aligned} \text{tang. } \frac{m}{2n} \pi &= \frac{mn}{nn - mm} \cdot \frac{4}{\pi} + \frac{m}{n} \cdot \frac{4}{\pi} (A - 1) + \frac{m^3}{n^3} \times \\ &\frac{4}{\pi} (B - 1) + \frac{m^5}{n^5} \cdot \frac{4}{\pi} (C - 1) + \frac{m^7}{n^7} \cdot \frac{4}{\pi} (D - 1) \&c. \end{aligned}$$

Deinde erit pro Cotangente

cot.

$$\cot. \frac{m}{2n} \varpi = \frac{n}{m} \cdot \frac{2}{\varpi} - \frac{4mn}{4m^3 - m^3} \cdot \frac{1}{\varpi} - \frac{m}{n} \cdot \frac{4}{\pi} \left(\alpha - \frac{1}{2^2} \right) \text{CAP. XI.}$$

$$- \frac{m^3}{n^3} \cdot \frac{4}{\varpi} \left(\zeta - \frac{1}{2^4} \right) - \frac{m^5}{n^5} \cdot \frac{4}{\pi} \left(\gamma - \frac{1}{2^6} \right) - \&c.,$$

atque ex his formulis natæ sunt expressiones, quas supra (§. 135.) pro Tangente & Cotangente dedimus; simul vero (§. 137.) ostendimus, quomodo ex Tangentibus & Cotangentibus inventis per solam additionem & subtractionem Secantes & Cosecantes reperiantur. Harum ergo regularum ope universus Canon Sinuum, Tangentium & Secantium, eorumque Logarithmorum multo facilius supputari posset, quam quidem hoc a primis conditoribus est factum.

C A P U T X I I.

De reali Functionum fractarum evolutione.

199. **J** Am supra, in Capite secundo, methodus est tradita Functionem quamcunque fractam in tot partes resolvendi quot ejus denominator habeat Factores simplices; hi enim præbent denominatores fractionum illarum partialium. Ex quo manifestum est, si denominator quos habeat Factores simplices imaginarios, fractiones quoque inde ortas fore imaginarias: his ergo casibus parum juvabit fractionem realem in imaginarias resolvissse. Cum igitur ostendissem omnem Functionem integram, qualis est denominator cujusvis fractionis, quantumvis Factoribus simplicibus imaginariis scateat, tamen in Factores duplices, seu secundæ dimensionis, reales semper resolveri posse; hoc modo in resolutione fractionum quantitates imaginariæ evitari poterunt, si pro denominatoribus fractionum partialium non Factores denominatoris principalis simplices, sed duplices reales assumamus.

LIB. I.

200. Sit igitur proposita hæc Functio fracta $\frac{M}{N}$, ex qua tot fractiones simplices secundum methodum supra expositam eliciantur, quot denominator N habuerit Factores simplices reales. Sit autem, loco imaginariorum, hæc expressio $pp - 2pqz \cos. \phi + qqzz$ Factor ipsius N ; &, quoniam in hoc negotio numeratorem & denominatorem in forma evoluta contemplari oportet, sit hæc fractio proposita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.}{(pp - 2pqz \cos. \phi + qqzz)(\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.)},$$

ac ponatur fractio partialis ex denominatoris Factore $pp - 2pqz \cos. \phi + qqzz$ oriunda hæc: $\frac{A + Az}{pp - 2pqz \cos. \phi + qqzz}$,

quoniam enim variabilis z in denominatore duas habet dimensiones, in numeratore unam habere poterit, non vero plures; alias enim integra Functio contineretur, quam scorsim elici oportet.

201. Sit, brevitatis gratia, numerator $A + Bz + Cz^2 + \&c. = M$ & alter denominatoris Factor. $\alpha + \beta z + \gamma z^2 + \&c. = Z$; ponatur altera pars ex denominatoris Factore Z oriunda $= \frac{r}{Z}$, eritque $r = \frac{M - AZ - AZz}{pp - 2pqz \cos. \phi + qqzz}$, quæ expressio Functio integra ipsius z esse debet, ideoque necesse est ut $M - AZ - AZz$ divisibile sit per $pp - 2pqz \cos. \phi + qqzz$. Evanescet ergo $M - AZ - AZz$, si ponatur $pp - 2pqz \cos. \phi + qqzz = 0$, hoc est si ponatur tam $z = \frac{p}{q} (\cos. \phi + \sqrt{-1} \sin. \phi)$ quam $z = \frac{p}{q} (\cos. \phi - \sqrt{-1} \sin. \phi)$; sit $\frac{p}{q} = f$, eritque $z^n = f^n (\cos. n\phi \pm \sqrt{-1} \sin. n\phi)$. Duplex ergo hic valor pro z substitutus duplicem dabit æquationem, unde ambas incognitas constantes A & A definire licet.

202. Facta ergo hac substitutione, æquatio $M = AZ + A.Zz$ evoluta hanc duplicem dabit æquationem

$$A +$$

$$\left. \begin{aligned} A + Bf.\cos.\phi + Cff.\cos.2\phi + Df^3.\cos.3\phi + \&c. \\ + (Bf.\sin.\phi + Cff.\sin.2\phi + Df^3.\sin.3\phi + \&c.)\sqrt{-1} \end{aligned} \right\} = \frac{\text{CAP. XII.}}{1}$$

$$\left\{ \begin{aligned} A(\alpha + \epsilon f.\cos.\phi + \gamma ff.\cos.2\phi + \delta f^3.\cos.3\phi + \&c.) \\ \pm A(\epsilon f.\sin.\phi + \gamma ff.\sin.2\phi + \delta f^3.\sin.3\phi + \&c.)\sqrt{-1} \\ + A(\alpha f.\cos.\phi + \epsilon ff.\cos.2\phi + \gamma f^3.\cos.3\phi + \&c.) \\ \pm A(\alpha f.\sin.\phi + \epsilon ff.\sin.2\phi + \gamma f^3.\sin.3\phi + \&c.)\sqrt{-1} \end{aligned} \right.$$

Sit, ad calculum abbreviandum,

$$\begin{aligned} A + Bf.\cos.\phi + Cff.\cos.2\phi + Df^3.\cos.3\phi + \&c. &= P \\ Bf.\sin.\phi + Cff.\sin.2\phi + Df^3.\sin.3\phi + \&c. &= P \\ \alpha + \epsilon f.\cos.\phi + \gamma ff.\cos.2\phi + \delta f^3.\cos.3\phi + \&c. &= Q \\ \epsilon f.\sin.\phi + \gamma ff.\sin.2\phi + \delta f^3.\sin.3\phi + \&c. &= Q \\ \alpha f.\cos.\phi + \epsilon ff.\cos.2\phi + \gamma f^3.\cos.3\phi + \&c. &= R \\ \alpha f.\sin.\phi + \epsilon ff.\sin.2\phi + \gamma f^3.\sin.3\phi + \&c. &= R \end{aligned}$$

eritque, his positis,

$$P \pm P\sqrt{-1} = A Q \pm A Q\sqrt{-1} + A R \pm A R\sqrt{-1}.$$

203. Ob signorum ambiguitatem hæ duæ oriuntur æquationes,

$$\begin{aligned} P &= A Q + A R \\ P &= A Q + A R \end{aligned}$$

ex quibus incognitæ A & A ita definiuntur, ut sit

$$A = \frac{P R - P R}{Q R - Q R} \quad \& \quad A = \frac{P Q - P Q}{Q R - Q R}.$$

Proposita ergo fractione $\frac{M}{(pp - 2pqz.\cos.\phi + qqzz)Z}$
per sequentem regulam fractio partialis ex ea oriunda
 $\frac{A + Az}{pp + 2pqz.\cos.\phi + qqzz}$ definietur. Posito $f = \frac{p}{q}$, & evo-
lutis singulis terminis, fiat ut sequitur,

X 2

posito

LIB. I. posito $z^n = f^n \cdot \cos. n \phi$, sit $M = P$
 $z^n = f^n \cdot \sin. n \phi$, sit $M = P$
 $z^n = f^n \cdot \cos. n \phi$, sit $Z = Q$
 $z^n = f^n \cdot \sin. n \phi$, sit $Z = Q$
 $z^n = f^n \cdot \cos. n \phi$, sit $z Z = R$
 $z^n = f^n \cdot \sin. n \phi$, sit $z Z = R$

Inventis hoc modo valoribus P, Q, R, P, Q, R erit

$$A = \frac{P_R - P R}{Q_R - Q R}, \text{ \& } A = \frac{P Q - P Q}{Q_R - Q R}.$$

EXEMPLUM I.

Si fuerit proposita hæc Functio fracta $\frac{z z}{(1 - z + z z)(1 + z^4)}$ ex qua partem a denominatoris Factore $1 - z + z z$ oriundam definire oporteat, quæ sit $\frac{A + A z}{1 - z + z z}$. Ac primo quidem hic Factor, cum forma generali $p p - 2 p q z \cdot \cos. \phi + q q z z$ comparatus, dat $p = 1, q = 1$ & $\cos. \phi = \frac{1}{2}$, unde fit $\phi = 60^\circ = \frac{\pi}{3}$. Quia itaque est $M = z z; Z = 1 + z^4$ & $f = 1$ erit

$$P = \cos. \frac{2}{3} \pi = -\frac{1}{2}; \quad P = \frac{\sqrt{3}}{2}$$

$$Q = 1 + \cos. \frac{4}{3} \pi = -\frac{1}{2}; \quad Q = -\frac{\sqrt{3}}{2}$$

$$R = \cos. \frac{\pi}{3} + \cos. \frac{5\pi}{3} = 1; \quad R = 0.$$

Ex his invenitur $A = -1$; & $A = 0$, ideoque fractio quæsitæ est $\frac{-1}{1 - z + z z}$, hujusque complementum erit

$1 + z$

$\frac{1+z+zz}{1+z^4}$, cujus denominator $1+z^4$ cum habeat Factores CAP.
XII.
 $1+z\sqrt{2+zz}$ & $1-z\sqrt{2+zz}$, resolutio denuo suscipi
 potest; fit autem $\phi = \frac{\pi}{4}$ & priori casu $f = -1$ poste-
 riori $f = +1$.

E X E M P L U M. II.

Sit igitur proposita hæc fractio resolvenda

$$\frac{1+z+zz}{(1+z\sqrt{2+zz})(1-z\sqrt{2+zz})}$$
 & erit $M = 1+z+zz$; & pro priore Factore habebitur
 $f = -1$; $\phi = \frac{\pi}{4}$, & $Z = 1-z\sqrt{2+zz}$, unde erit

$$P = 1 - \cos. \frac{\pi}{4} + \cos. \frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$P = - \sin. \frac{\pi}{4} + \sin. \frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$Q = 1 + \sqrt{2} \cdot \cos. \frac{\pi}{4} + \cos. \frac{2\pi}{4} = 2$$

$$Q = + \sqrt{2} \cdot \sin. \frac{\pi}{4} + \sin. \frac{2\pi}{4} = 2$$

$$R = - \cos. \frac{\pi}{4} - \sqrt{2} \cdot \cos. \frac{2\pi}{4} - \cos. \frac{3\pi}{4} = 0$$

$$R = - \sin. \frac{\pi}{4} - \sqrt{2} \cdot \sin. \frac{2\pi}{4} - \sin. \frac{3\pi}{4} = -2\sqrt{2}$$

Ex his reperitur $QR - QR = -4\sqrt{2}$: &

$A = \frac{\sqrt{2}-1}{2\sqrt{2}}$, & $A = 0$ unde ex denominatoris Factore

$1+z\sqrt{2+zz}$ hæc orietur fractio partialis $\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2+zz}}$,

alter autem Factor dabit simili modo hanc $\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2+zz}}$.

Hinc Functio primum proposita $\frac{zz}{(1-z\sqrt{2+zz})(1+z\sqrt{2+zz})}$ resol-
X 3
vitur

LIB. I. vitur in has $\frac{-1}{1-2+2z} + \frac{(\sqrt{2}-1):2\sqrt{2}}{1+2\sqrt{2}+2z} + \frac{(\sqrt{2}+1):2\sqrt{2}}{1-2\sqrt{2}+2z}$

EXEMPLUM III.

Sit proposita hæc fractio resolvenda

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$$

Pro Factore denominatoris $1-\frac{8}{5}z+zz$ oriatur ista fractio

$\frac{A+Az}{1-\frac{8}{5}z+zz}$; eritque $p=1$; $q=i$; $\cos. \phi = \frac{4}{5}$, unde $f=1$; $M=1+2z+zz$; $Z=1+2z+3zz$. Quia vero hic ratio Anguli ϕ ad rectum non constat, Sinus & Cosinus ejus multiplo- rum seorsim debent investigari. Cum sit

$$\begin{aligned} \cos. \phi &= \frac{4}{5}; \text{ erit } \sin. \phi = \frac{3}{5} \\ \cos. 2\phi &= \frac{7}{25}; \quad \sin. 2\phi = \frac{24}{25} \\ \cos. 3\phi &= \frac{-44}{125}; \quad \sin. 3\phi = \frac{117}{125}; \end{aligned}$$

hinc fit

$$\begin{aligned} P &= 1 + 2 \cdot \frac{4}{5} + \frac{7}{25} = \frac{72}{25} \\ P &= \quad \quad 2 \cdot \frac{3}{5} + \frac{24}{25} = \frac{54}{25} \\ Q &= 1 + 2 \cdot \frac{4}{5} + 3 \cdot \frac{7}{25} = \frac{86}{25} \\ Q &= \quad \quad 2 \cdot \frac{3}{5} + 3 \cdot \frac{24}{25} = \frac{102}{25} \\ R &= \frac{4}{5} + 2 \cdot \frac{7}{25} - 3 \cdot \frac{44}{125} = \frac{38}{125} \\ R &= \frac{3}{5} + 2 \cdot \frac{24}{25} + 3 \cdot \frac{117}{125} = \frac{666}{125} \end{aligned}$$

$$\text{ideoque } QR - Q\bar{R} = \frac{53400}{25 \cdot 125} = \frac{2136}{125}. \text{ Ergo}$$

A =

$$A = \frac{1836}{2136} = \frac{153}{178}; \quad A = -\frac{540}{2136} = -\frac{45}{178}.$$

 C A P.
XII.

Quare fractio ex Factore $1 - \frac{8}{5}z + zz$ oriunda erit

$$\frac{9(17 - 5z) : 178}{1 - \frac{8}{5}z + zz}.$$

Quæramus simili modo fractionem alteri Factori respondentem;

erit $p = 1$, $q = -\sqrt{3}$ & $\cos. \Phi = \frac{1}{\sqrt{3}}$, ergo $f = -\frac{1}{\sqrt{3}}$,

$M = 1 + 2z + zz$ & $Z = 1 - \frac{8}{5}z + zz$. Fiet

autem, ob $\cos. \Phi = \frac{1}{\sqrt{3}}$, $\sin. \Phi = \frac{\sqrt{2}}{\sqrt{3}}$

$$\cos. 2\Phi = -\frac{1}{3}, \quad \sin. 2\Phi = \frac{2\sqrt{2}}{3}$$

$$\cos. 3\Phi = -\frac{5}{3\sqrt{3}}, \quad \sin. 3\Phi = \frac{\sqrt{3}}{3\sqrt{3}}$$

confequenter

$$P = 1 - \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot -\frac{1}{3} = \frac{2}{9}$$

$$P = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9}$$

$$Q = 1 + \frac{8}{5\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot -\frac{1}{3} = \frac{64}{45}$$

$$Q = +\frac{8}{5\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45}$$

$$R = -\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} - \frac{8}{5 \cdot 3} \cdot -\frac{1}{3} - \frac{1}{3\sqrt{3}} \cdot -\frac{5}{3\sqrt{3}} = \frac{4}{135}$$

$$R = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5 \cdot 3} \cdot \frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}} \cdot \frac{\sqrt{2}}{3\sqrt{3}} = -\frac{98\sqrt{2}}{135}$$

ideoque $QR - Q\bar{R} = -\frac{712\sqrt{2}}{675}$; fiet ergo

$$A = \frac{100}{712} = \frac{25}{178}; \quad A = \frac{540}{712} = \frac{135}{178}.$$

Fractio

LIB. I.

Fractio ergo proposita $\frac{1 + 2z + 3z^2}{(1 - \frac{8}{5}z + 2z^2)(1 + 2z + 3z^2)}$ re-
solvitur in $\frac{9(17 - 5z):178}{1 - \frac{8}{5}z + 2z^2} + \frac{5(5 + 27z):178}{1 + 2z + 3z^2}$.

204. Possunt autem valores litterarum R & R ex litteris Q & Q definiri, cum enim sit

$$Q = \alpha + 6f \cdot \cos. \Phi + \gamma f^2 \cdot \cos. 2\Phi + \delta f^3 \cdot \cos. 3\Phi + \&c.$$

$$Q = 6f \cdot \sin. \Phi + \gamma f^2 \cdot \sin. 2\Phi + \delta f^3 \cdot \sin. 3\Phi + \&c.$$

erit

$$Q \cdot \cos. \Phi - Q \cdot \sin. \Phi = \alpha \cdot \cos. \Phi + 6f \cdot \cos. 2\Phi + \gamma f^2 \cdot \cos. 3\Phi + \&c.$$

$$\text{ideoque } R = f(Q \cdot \cos. \Phi - Q \cdot \sin. \Phi)$$

deinde erit

$$Q \cdot \sin. \Phi + Q \cdot \cos. \Phi = \alpha \cdot \sin. \Phi + 6f \cdot \sin. 2\Phi + \gamma f^2 \cdot \sin. 3\Phi + \&c.$$

$$\text{ergo } R = f(Q \cdot \sin. \Phi + Q \cdot \cos. \Phi)$$

Ex his porro fit

$$Q_R - Q_R = (Q_Q + Q_Q) f \cdot \sin. \Phi$$

$$P_R - P_R = (P_Q + P_Q) f \cdot \sin. \Phi + (P_Q - P_Q) f \cdot \cos. \Phi$$

eritque consequenter

$$A = \frac{P_Q + P_Q}{Q_Q + Q_Q} + \frac{P_Q - P_Q}{Q_Q + Q_Q} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$A = - \frac{P_Q + P_Q}{(Q_Q + Q_Q) f \cdot \sin. \Phi}$$

Quare ex denominatoris Factore $pp - 2pqz \cdot \cos. \Phi + qqzz$ nascitur ista fractio partialis

$$\frac{(P_Q + P_Q) f \cdot \sin. \Phi + (P_Q - P_Q) (f \cdot \cos. \Phi - z)}{(pp - 2pqz \cdot \cos. \Phi + qqzz) (Q_Q + Q_Q) f \cdot \sin. \Phi}$$

seu, ob $f = \frac{p}{q}$, hæc

$$\frac{(P_Q + P_Q) p \cdot \sin. \Phi + (P_Q - P_Q) (p \cdot \cos. \Phi - qz)}{(pp - 2pqz \cdot \cos. \Phi + qqzz) (Q_Q + Q_Q) p \cdot \sin. \Phi}$$

205. Oritur ergo hæc fractio partialis ex Functionis propositæ $\frac{M}{(pp - 2pqz \cdot \cos. \Phi + qqzz) Z}$ Factore denominatoris $pp - 2pqz \cdot \cos. \Phi + qqzz$, atque litteræ P, p, Q & Q sequenti modo ex Functionibus M & Z inveniuntur:

posito

$$\text{posito } z^n = \frac{p^n}{q^n} \cdot \cos. n\phi, \text{ fit } M = P,$$

$$\& Z = Q;$$

$$\& \text{posito } z^n = \frac{p^n}{q^n} \cdot \sin. n\phi, \text{ fit } M = P,$$

$$\& Z = Q:$$

ubi notandum est Functiones M & Z , antequam hæc substitutio fiat, omnino evolvi debere, ut hujusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$

$$\& Z = \alpha + \epsilon z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.;$$

eritque ideo

$$P = A + B \frac{p}{q} \cdot \cos. \phi + C \frac{p^2}{q^2} \cdot \cos. 2\phi + D \frac{p^3}{q^3} \cdot \cos. 3\phi + \&c.$$

$$P = B \frac{p}{q} \cdot \sin. \phi + C \frac{p^2}{q^2} \cdot \sin. 2\phi + D \frac{p^3}{q^3} \cdot \sin. 3\phi + \&c.$$

$$Q = \alpha + \epsilon \frac{p}{q} \cdot \cos. \phi + \gamma \frac{p^2}{q^2} \cdot \cos. 2\phi + \delta \frac{p^3}{q^3} \cdot \cos. 3\phi + \&c.$$

$$Q = \epsilon \frac{p}{q} \cdot \sin. \phi + \gamma \frac{p^2}{q^2} \cdot \sin. 2\phi + \delta \frac{p^3}{q^3} \cdot \sin. 3\phi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc resolutionem locum habere non posse, si Functio Z eundem Factorem $pp - 2pqz \cdot \cos. \phi + qqzz$ adhuc in se complectatur; hoc enim casu in æquatione $M = AZ + A Zz$ facta substitutione $z^n = f^n (\cos. n\phi \pm \sqrt{-1} \cdot \sin. \phi)$, ipsa quantitas Z evanesceret, nihilque propterea colligi posset. Quamobrem, si Functionis fractæ $\frac{M}{N}$ denominator habeat Factorem $(pp - 2pqz \cdot \cos. \phi + qqzz)^2$ vel altiore Potestatem, peculiari opus erit resolutione. Sit igitur $N = (pp - 2pqz \cdot \cos. \phi + qqzz)^2 Z$; atque ex denominatoris Factore $(pp - 2pqz \cdot \cos. \phi + qqzz)^2$ orientur hujusmodi duæ fractiones partiales

Euleri *Introduct. in Anal. infin. parv.*

Y

A +

LIB. I.
$$\frac{A + Az}{(pp - 2pqz \cdot \cos. \phi + qqzz)^2} + \frac{B + Bz}{pp - 2pqz \cdot \cos. \phi + qqzz},$$

ubi litteras constantes A, A, B, B determinari oportet.

207. His positis, debeat ista expressio

$$\frac{M - (A + Az)Z - (B + Bz)Z(pp - 2pqz \cdot \cos. \phi + qqzz)}{(pp - 2pqz \cdot \cos. \phi + qqzz)^2}$$

esse Functio integra, & hanc ob rem numerator divisibilis erit per denominatorem. Primum ergo hac expressio $M - AZ - AzZ$ divisibilis esse debet per $pp - 2pqz \cdot \cos. \phi + qqzz$; qui cum sit casus præcedens, eodem quoque modo litteræ A & A determinabuntur.

Quare, posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\phi$, sit $M = P$,
& $Z = N$:

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\phi$, sit $M = P$,
& $Z = N$.

Hisque factis secundum regulam supra datam, erit

$$A = \frac{PN + PN}{N^2 + N^2} + \frac{PN - PN}{N^2 + N^2} \cdot \frac{\cos. \phi}{\sin. \phi}$$

$$A = - \frac{PN + PN}{N^2 + N^2} \cdot \frac{q}{p \sin. \phi}$$

208. Inventis ergo hoc modo A & A , fiet

$\frac{M - (A + Az)Z}{pp - 2pqz \cdot \cos. \phi + qqzz}$ Functio integra, quæ sit $= P$; atque
superest ut $P - BZ - BzZ$ divisibile evadat per $pp - 2pqz \cdot \cos. \phi + qqzz$, quæ expressio cum similis sit præcedenti, si

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\phi$, vocetur $P = R$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\phi$, vocetur $P = R$; erit

$$B =$$

$$B = \frac{RN + RN}{N^2 + N^2} + \frac{RN - RN}{N^2 + N^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$B = - \frac{RN + RN}{N^2 + N^2} \cdot \frac{q}{p \sin. \Phi}$$

209. Hinc jam generaliter concludere licet quomodo resolutio institui debeat, si denominator Functionis propositæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz \cdot \cos. \Phi + qqzz)^k$: fit enim

$N = (pp - 2pqz \cdot \cos. \Phi + qqzz)^k Z$, ita ut hæc resolvenda sit Functio fracta

$$\frac{M}{(pp - 2pqz \cdot \cos. \Phi + qqzz)^k Z}$$

Præbeat ergo Factor denominatoris $(pp - 2pqz \cdot \cos. \Phi + qqzz)^k$ has partes:

$$\frac{A + Az}{(pp - 2pqz \cdot \cos. \Phi + qqzz)^k} + \frac{B + Bz}{(pp - 2pqz \cdot \cos. \Phi + qqzz)^{k-1}} + \frac{C + Cz}{(pp - 2pqz \cdot \cos. \Phi + qqzz)^{k-2}} + \frac{D + Dz}{(pp - 2pqz \cdot \cos. \Phi + qqzz)^{k-3}} + \&c.$$

Jam, posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $M = M$,

& $Z = N$;

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $M = M$;

& $Z = N$;

$$\text{erit} \\ A = \frac{MN + MN}{N^2 + N^2} + \frac{MN - MN}{N^2 + N^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$A = - \frac{MN + MN}{N^2 + N^2} \cdot \frac{q}{p \sin. \Phi}$$

Deinde vocetur $\frac{M - (A + Az)Z}{pp - 2ppz \cdot \cos. \Phi + qqzz} = P$; atque, posito

LIB. I.

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\phi$, fit $P = P$,

& posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\phi$, fit $P = P$;

$$B = \frac{PN + PN}{N^2 + N^2} + \frac{PN - PN}{N^2 + N^2} \cdot \frac{\cos. \phi}{\sin. \phi}$$

$$B = - \frac{PN + PN}{N^2 + N^2} \cdot \frac{q}{p \sin. \phi}$$

Tum vocetur $\frac{P - (B + B_2)Z}{pp - 2pqz \cdot \cos. \phi + qqz^2} = Q$, atque

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\phi$, fit $Q = Q$,

& posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\phi$, fit $Q = Q$;

$$C = \frac{QN + QN}{N^2 + N^2} + \frac{QN - QN}{N^2 + N^2} \cdot \frac{\cos. \phi}{\sin. \phi}$$

$$C = - \frac{QN + QN}{N^2 + N^2} \cdot \frac{q}{p \sin. \phi}$$

Porro vocetur $\frac{Q - (C + C_2)Z}{pp - 2pqz \cdot \cos. \phi + qqz^2} = R$, atque

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\phi$, fit $R = R$,

& posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\phi$, fit $R = R$;

$$D = \frac{RN + RN}{N^2 + N^2} + \frac{RN - RN}{N^2 + N^2} \cdot \frac{\cos. \phi}{\sin. \phi}$$

$$D = - \frac{RN + RN}{N^2 + N^2} \cdot \frac{q}{p \sin. \phi}$$

hocque

hocque modo progrediendum est donec ultimæ fractionis, CAP.
cujus denominator est $pp - 2pqz \cdot \cos. \Phi + qqzz$, nume- XII.
rator fuerit determinatus.

E X E M P L U M.

Sit ista proposita Functio fracta

$$\frac{z - z^3}{(1 + zz)^4 (1 + z^4)}$$

ex cujus denominatoris Factore $(1 + zz)^4$ oriantur hæ fra-
ctiones partiales,

$$\frac{A + Az}{(1 + zz)^4} + \frac{B + Bz}{(1 + zz)^3} + \frac{C + Cz}{(1 + zz)^2} + \frac{D + Dz}{1 + zz}.$$

Comparatione ergo instituta, erit $p = 1$, $q = 1$, $\cos. \Phi = 0$;

ideoque $\Phi = \frac{1}{2} \pi$, porroque $M = z - z^3$ & $Z = 1 + z^4$.

Hinc erit $M = 0$; $M = 2$; $N = 2$; $N = 0$, & $\sin. \Phi = 1$.

Hinc itaque invenitur

$$A = -\frac{4}{4} \cdot 0 = 0, \text{ \& } A = 1.$$

ergo $A + Az = z$; hincque $P = \frac{z - z^3 - z - z^5}{1 + zz} = -z^3$,

& $P = 0$, $P = 1$: unde reperitur

$$B = 0, \text{ \& } B = \frac{1}{2}.$$

Ergo $B + Bz = \frac{1}{2}z$, & $Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1 + zz} = -$

$$\frac{1}{2}z - \frac{1}{2}z^3;$$

unde $Q = 0$ & $Q = 0$, ergo

$C = 0$ & $C = 0$, hincque $R = -$

$$\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z,$$

ergo $R = 0$; $R = -\frac{1}{2}$; unde fit

$$D = 0 \text{ \& } D = -\frac{1}{4}.$$

Y 3

Quam-

LIB. I. Quamobrem fractiones quæsitæ sunt hæ

$$\frac{z}{(1+zz)^4} + \frac{z}{2(1+zz)^3} - \frac{z}{4(1+zz)}. \text{ Reliquæ vero fractionis numerator est } = S = \frac{R - (D + Dz)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3, \text{ quæ ergo erit } = \frac{-z + z^3}{4(1+zz^2)}.$$

210. Hac ergo methodo simul innotescit fractio complementi, quæ cum inventis conjuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp - 2pqz \cdot \cos. \Phi + qqzz)^k Z}$$

inventæ fuerint omnes fractiones partiales ex Factore $(pp - 2pqz \cdot \cos. \Phi + qqzz)^k$ oriundæ, pro quibus formati sunt valores Functionum P, Q, R, S, T , si harum litterarum Series ulterius continetur, erit ea, quæ ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquæ fractionis denominatorem Z habentis; nempe, si $k=1$, erit reliqua fractio $\frac{P}{Z}$; si $k=2$, erit reliqua fractio $\frac{Q}{Z}$; si $k=3$, erit ea $\frac{R}{Z}$, & ita porro. Inventæ autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius resolvi poterit.

CAPUT XIII.

De Seriebus recurrentibus.

211. **A**D hoc Serierum genus, quas MOIVRÆUS *recurrentes* vocare solet, hic refero omnes Series quæ ex evolutione Functionis cujusque fractæ per divisionem actualem instituta nascuntur. Supra enim jam ostendimus has Series ita esse comparatas, ut quivis terminus ex aliquot præcedentibus secundum legem quandam constantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias simplices resolvere docuerim, hinc Series quoque recurrens in alias simplices resolveretur. In hoc igitur Capite propositum est Serierum recurrentium cujusvis gradus resolutionem in simplices exponere.

212. Sit proposita ista Functio fracta genuina

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - az - bz^2 - \gamma z^3 - d^2 z^4 - \&c.}$$

quæ per divisionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, supra est ostensum. Quod si jam Functio illa fracta resolvatur in fractiones suas simplices, & unaquæque in Seriem recurrentem evolvatur, manifestum est summam omnium harum Serierum ex fractionibus partialibus ortarum æqualem esse debere Seriei recurrenti.

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas supra invenire docuimus, dabunt

LIB. I. bunt Series partiales, quarum indoles ob simplicitatem facile perspicitur; omnes autem Series partiales junctim sumtæ producent Seriem recurrentem propositam; unde & hujus natura penitus cognoscetur.

213. Sint Series recurrentes ex singulis fractionibus partialibus ortæ hæ.

$$\begin{aligned} a + bz + czz + dz^3 + ez^4 + \&c. \\ a' + b'z + c'zz + d'z^3 + e'z^4 + \&c. \\ a'' + b''z + c''zz + d''z^3 + e''z^4 + \&c. \\ a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \&c. \\ \&c. \end{aligned}$$

Quoniam hæ Series junctim sumtæ æquales esse debent huic

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$

necesse est ut sit

$$\begin{aligned} A &= a + a' + a'' + a''' + \&c. \\ B &= b + b' + b'' + b''' + \&c. \\ C &= c + c' + c'' + c''' + \&c. \\ D &= d + d' + d'' + d''' + \&c. \\ \&c. \end{aligned}$$

Hinc, si singularum Serierum ex fractionibus partialibus ortarum definiri queant coëfficientes Potestatis z^n , horum summa dabit coëfficientem Potestatis z^n in Serie recurrente $A + Bz + Cz^2 + Dz^3 + \&c.$

214. Dubium hic suboriri posset, an, si duæ hujusmodi Series fuerint inter se æquales

$$A + Bz + Cz^2 + Dz^3 + \&c. = A + Bz + Cz^2 + Dz^3 + \&c.,$$

necessario inde sequatur, coëfficientes similium Potestatum ipsius z inter se esse æquales; seu an sit $A = A$; $B = B$; $C = C$;

$C = C; D = D; \&c.$. Hoc autem dubium facile tolletur, si perpendamus hanc æqualitatem subsistere debere quemcunque valorem obtineat variabilis z . Sit igitur $z = 0$, atque manifestum est fore $A = A$. His ergo terminis æqualibus utrinque sublatis, ac reliqua æquatione per z divisa, habebitur

$$B + Cz + Dz^2 + \&c. = B + Cz + Dz^2 + \&c.;$$

unde sequitur fore $B = B$: simili autem modo ostendetur esse $C = C; D = D$, & ita porro in infinitum.

215. Contemplemur ergo Series, quæ ex fractionibus partialibus, in quas fractio quæpiam proposita resolvitur, oriuntur. Ac primo quidem patet fractionem $\frac{A}{1-pz}$ dare Seriem $A + Apz + Ap^2z^2 + Ap^3z^3 + \&c.$, cujus terminus generalis est $Ap^n z^n$; hæc enim expressio vocari solet *terminus generalis*, quoniam ex ea, loco n numeros omnes successive substituendo, omnes Seriei termini nascuntur. Deinde ex fractione $\frac{A}{(1-pz)^2}$ oritur Series $A + 2Apz + 3Ap^2z^2 + 4Ap^3z^3 + \&c.$, cujus terminus generalis est $(n+1)Ap^n z^n$.

Tum ex fractione $\frac{A}{(1-pz)^3}$ oritur Series $A + 3Apz + 6Ap^2z^2 + 10Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n+1)(n+2)}{1 \cdot 2} Ap^n z^n$. Generatim autem fractio

$\frac{A}{(1-pz)^k}$ præbet Seriem hanc $A + kApz + \frac{k(k+1)}{1 \cdot 2} Ap^2z^2 + \frac{k(k+1)(k+2)}{1 \cdot 2 \cdot 3} Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n+1)(n+2)(n+3) \dots (n+k-1)}{1 \cdot 2 \cdot 3 \dots (k-1)} Ap^n z^n$. Ex ipsa autem Seriei progressionem colligitur hic idem terminus $= \frac{k(k+1)(k+2) \dots (k+n-1)}{1 \cdot 2 \cdot 3 \dots n} Ap^n z^n$: hæc vero

Euleri *Introduct. in Anal. infin. parv.*

Z

expressio

LIB. I. expressio illi est æqualis, id quod multiplicatione per crucem
 — instituta patebit, fiet enim,

$$1.2.3....n(n+1)....(n+k-1) = 1.2.3....(k-1)k....(k+n-1)$$

quæ est æquatio identica.

216. Quoties ergo in resolutione Functionum fractarum ad hujusmodi fractiones partiales $\frac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 + \&c.$, terminus generalis assignari poterit, quippe qui erit summa terminorum generalium Serierum, quæ ex fractionibus partialibus nascuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\frac{1-z}{1-z-2zz}$ nascitur.

Series hinc nata est $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 + \&c.$. Ad coefficientem potestatis generalis z^n inveniendum, fractio $\frac{1-z}{1-z-2zz}$ resolvatur in $\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z}$, unde oritur terminus generalis quæsitus $(\frac{2}{3}(-1)^n + \frac{1}{3} \cdot 2^n)z^n = \frac{2^n + 2}{3} z^n$, ubi signum + valet si n sit numerus par, signum — si n sit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\frac{1-z}{1-5z+6zz}$, seu Seriei hujus $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 + \&c.$

Ob

Ob denominatorem $= (1 - 2z)(1 - 3z)$ resolvitur CAP. XIII.
 fractio in has $\frac{1}{1-2z} + \frac{2}{1-3z}$, ex quibus fit terminus ge-
 neralis $2 \cdot 3^n z^n - 2^n z^n = (2 \cdot 3^n - 2^n) z^n$.

EXEMPLUM III.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 + \&c.$, qua oritur ex evolutione fractionis $\frac{1+2z}{1-z-zz}$.

Ob denominatoris Factores $1 - (\frac{1+\sqrt{5}}{2})z$ & $1 - (\frac{1-\sqrt{5}}{2})z$, per resolutionem prodeunt $\frac{\frac{\sqrt{5}+1}{2}}{1 - (\frac{1+\sqrt{5}}{2})z} + \frac{\frac{1-\sqrt{5}}{2}}{1 - (\frac{1-\sqrt{5}}{2})z}$, unde erit terminus generalis $= (\frac{1+\sqrt{5}}{2})^{n+1} z^n + (\frac{1-\sqrt{5}}{2})^{n+1} z^n$.

EXEMPLUM IV.

Invenire terminum generalem Seriei hujus $z + (\alpha a + b)z + (\alpha^2 a + \alpha b + \beta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b)z^3 + \&c.$, qua oritur ex evolutione fractionis $\frac{a+bz}{1-\alpha z-\beta zz}$.

Per resolutionem oriuntur hæ duæ fractiones:

$Z \quad 2$

$(a$

LIB. I.

$$\frac{(a(\alpha + \sqrt{(\alpha\alpha + 4\beta)}) + 2b) : 2\sqrt{(\alpha\alpha + 4\beta)}}{1 - \left(\frac{\alpha + \sqrt{(\alpha\alpha + 4\beta)}}{2}\right)z} +$$

$$\frac{(a(\sqrt{(\alpha\alpha + 4\beta)} - \alpha) - 2b) : 2\sqrt{(\alpha\alpha + 4\beta)}}{1 - \left(\frac{\alpha - \sqrt{(\alpha\alpha + 4\beta)}}{2}\right)z}, \text{ hinc}$$

$$\text{terminus generalis erit } \frac{a(\sqrt{(\alpha\alpha + 4\beta)} + \alpha) + 2b}{2\sqrt{(\alpha\alpha + 4\beta)}}$$

$$\left(\frac{\alpha + \sqrt{(\alpha\alpha + 4\beta)}}{2}\right)^n z^n + \frac{a(\sqrt{(\alpha\alpha + 4\beta)} - \alpha) - 2b}{2\sqrt{(\alpha\alpha + 4\beta)}}$$

$$\left(\frac{\alpha - \sqrt{(\alpha\alpha + 4\beta)}}{2}\right)^n z^n; \text{ ex quo omnium Serierum re-}$$

currentium, quarum quisque terminus per duos præcedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM V.

Invenire terminum generalem hujus Seriei $1 + z + 2z^2 + 2z^3 + 3z^4 + 3z^5 + 4z^6 + 4z^7 + \&c.$, *qua oritur ex fractione*

$$\frac{1}{1 - z - zz + z^3} = \frac{1}{(1 - z)^2 (1 + z)}.$$

Quanquam lex progressionis primo intuitu ita est manifesta ut explicatione non indigeat, tamen fractiones per resolutionem ortæ

$$\frac{\frac{1}{2}}{(1 - z)^2} + \frac{\frac{1}{4}}{1 - z} + \frac{\frac{1}{4}}{1 + z} \text{ dant hunc termi-}$$

$$\text{num generalem } \frac{1}{2}(n + 1)z^n + \frac{1}{4}z^n + \frac{1}{4}(-1)^n z^n =$$

$$\frac{2n + 3 \pm 1}{4} z^n, \text{ ubi signum superius valet si } n \text{ fuerit numerus}$$

par, inferius si n fuerit impar.

217. Hoc pacto omnium Serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in hujusmodi fractiones partiales simplices resolvere licet. Quod si autem expressiones imaginarias vitare velimus, sæpenumero ad hujusmodi fractiones partiales pervenietur

A +

$$\frac{A + Bpz}{1 - 2pz \cos \phi + p^2 z^2}; \quad \frac{A + Bpz}{(1 - 2pz \cos \phi + p^2 z^2)^2}; \quad \& \quad \frac{A + Bpz}{(1 - 2pz \cos \phi + p^2 z^2)^k}, \text{ ex quarum evolutione cujusmodi}$$

Series nascentur videndum est. Ac primo quidem, ob $\cos. n \phi = 2 \cos. \phi \cos. (n-1) \phi - \cos. (n-2) \phi$, fractio

$$\frac{A}{1 - 2pz \cos \phi + p^2 z^2} \text{ evoluta dabit}$$

$$A + 2Apz \cos \phi + 2Appz^2 \cos 2\phi + 2Ap^3 z^3 \cos 3\phi + 2Ap^4 z^4 \cos 4\phi + \dots \\ + Ap^2 z^2, \quad + 2Ap^3 z^3 \cos \phi + 2Ap^4 z^4 \cos 2\phi + Ap^4 z^4. \quad \&c.$$

cujus Seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas Series

$$Ppz \sin \phi + Pp^2 z^2 \sin 2\phi + Pp^3 z^3 \sin 3\phi + Pp^4 z^4 \sin 4\phi + \dots \\ Q + Qpz \cos \phi + Qp^2 z^2 \cos 2\phi + Qp^3 z^3 \cos 3\phi + Qp^4 z^4 \cos 4\phi + \dots$$

quæ duæ Series utique nascuntur ex evolutione fractionis, cujus denominator est $1 - 2pz \cos \phi + p^2 z^2$. Ac prior quidem oritur ex hac fractione $\frac{Ppz \sin \phi}{1 - 2pz \cos \phi + p^2 z^2}$, posterior

vero ex hac $\frac{Q - Qpz \cos \phi}{1 - 2pz \cos \phi + p^2 z^2}$. Addantur hæ duæ fractiones, atque summa $\frac{Q + Ppz \sin \phi - Qpz \cos \phi}{1 - 2pz \cos \phi + p^2 z^2}$ dabit

Seriem cujus terminus generalis erit $= (P \sin n\phi + Q \cos n\phi) p^n z^n$. Fiat autem hæc fractio propositæ $\frac{A + Bpz}{1 - 2pz \cos \phi + p^2 z^2}$

æqualis, erit $Q = A$, & $P = A \cot \phi + B \operatorname{cosec} \phi$. Seriei ergo ex hac fractione $\frac{A + Bpz}{1 - 2pz \cos \phi + p^2 z^2}$ ortæ terminus generalis erit $= \frac{A \cos \phi \sin n\phi + B \sin n\phi + A \sin \phi \cos n\phi}{\sin \phi} p^n z^n =$

$$\frac{A \sin (n+1) \phi + B \sin n\phi}{\sin \phi} p^n z^n.$$

LIB. I.

219. Ad terminum generalem inveniendum, si denominator fractionis fuerit Potestas, ut $(1 - 2pz \cdot \cos \Phi + ppz^2)^k$, conveniet hanc fractionem resolvere in duas etsi imaginarias

$$\frac{a}{(1 - (\cos \Phi + \sqrt{-1} \sin \Phi) pz)^k} + \frac{b}{(1 - (\cos \Phi - \sqrt{-1} \sin \Phi) pz)^k}$$

quarum simul sumtarum terminus generalis Seriei ex ipsis ortæ erit

$$\frac{(n+1)(n+2)(n+3) \dots (n+k-1)}{1. 2. 3 \dots (k-1)} (\cos n\Phi + \sqrt{-1} \sin n\Phi) ap^n z^n +$$

$$\frac{(n+1)(n+2)(n+3) \dots (n+k-1)}{1. 2. 3 \dots (k-1)} (\cos n\Phi - \sqrt{-1} \sin n\Phi) bp^n z^n.$$

Sit $a + b = f$; $a - b = \frac{g}{\sqrt{-1}}$, ut sit $a = \frac{f\sqrt{-1} + g}{2\sqrt{-1}}$ &

$b = \frac{f\sqrt{-1} - g}{2\sqrt{-1}}$, eritque hæc expressio

$$\frac{(n+1)(n+2)(n+3) \dots (n+k-1)}{1. 2. 3 \dots (k-1)} (f \cos n\Phi + g \sin n\Phi) p^n z^n$$

terminus generalis Seriei, quæ oritur ex his fractionibus

$$\frac{\frac{1}{2}f + \frac{1}{2\sqrt{-1}}g}{(1 - (\cos \Phi + \sqrt{-1} \sin \Phi) pz)^k} + \frac{\frac{1}{2}f - \frac{1}{2\sqrt{-1}}g}{(1 - (\cos \Phi - \sqrt{-1} \sin \Phi) pz)^k},$$

feu quæ oritur ex hac fractione una

$$\frac{f - kfpz \cos \Phi + \frac{k(k-1)}{1. 2} fp^2 z^2 \cos 2\Phi - \frac{k(k-1)(k-2)}{1. 2. 3} fp^3 z^3 \cos 3\Phi}{(1 - 2pz \cos \Phi + ppz^2)^k}$$

$$+ \frac{kfpz \sin \Phi - \frac{k(k-1)}{1. 2} gp^2 z^2 \sin 2\Phi + \frac{k(k-1)(k-2)}{1. 2. 3} gp^3 z^3 \sin 3\Phi}{(1 - 2pz \cos \Phi + ppz^2)^k} \text{ \&c.}$$

220. Posito ergo $k = 2$, erit Seriei ex hac fractione

$$\frac{f - 2pz (f \cos \Phi - g \sin \Phi) + ppz^2 (f \cos 2\Phi - g \sin 2\Phi)}{(1 - 2pz \cos \Phi + ppz^2)^2}$$

ortæ terminus generalis $= (n+1) (f \cos n\Phi + g \sin n\Phi) p^n z^n$.

At Seriei ex hac fractione $\frac{a}{1 - 2pz \cos \Phi + ppz^2}$, seu hac

$a =$

$a = \frac{2apz \cdot \cos \phi + appz^2}{(1 - 2pz \cdot \cos \phi + ppz^2)^2}$ ortæ terminus generalis est $= \frac{a \sin. (n+1) \phi}{\sin. \phi} p^n z^n$. Addantur hæ fractiones invicem, ac

ponatur $a + f = A$; $2a \cdot \cos \phi + 2f \cdot \cos \phi - 2g \cdot \sin \phi = B$
 & $a + f \cdot \cos 2\phi - g \cdot \sin 2\phi = 0$, hinc erit $g =$
 $\frac{B + 2A \cos \phi}{2 \sin \phi}$, $a = \frac{A + B \cos \phi}{1 - \cos 2\phi} = \frac{A + B \cos \phi}{2 (\sin \phi)^2}$ & $f =$
 $\frac{A \cos 2\phi - B \cos \phi}{2 (\sin \phi)^2}$, & $g = \frac{B \sin \phi + A \sin 2\phi}{2 (\sin \phi)^2}$. Hanc ob

rem Seriei ex hac fractione $\frac{A + B \cos \phi}{(1 - 2pz \cdot \cos \phi + ppz^2)^2}$, ortæ ter-
 minus generalis est $\frac{A + B \cos \phi}{2 (\sin \phi)^3} \sin. (n+1) \phi \cdot p^n z^n + (n+1)$
 $\frac{(B \sin \phi \cdot \sin n\phi + A \sin 2\phi \cdot \sin n\phi - B \cos \phi \cdot \cos n\phi - A \cos 2\phi \cdot \cos n\phi)}{2 (\sin \phi)^2}$

$p^n z^n = \frac{(n+1)(A \cos (n+2)\phi + B \cos (n+1)\phi)}{2 (\sin \phi)^2} p^n z^n +$

$\frac{(A + B \cos \phi) \sin. (n+1) \phi}{2 (\sin \phi)^3} p^n z^n =$

$\frac{(\frac{1}{2}(n+3) \sin. (n+1) \phi - \frac{1}{2}(n+1) \sin. (n+3) \phi)}{2 (\sin \phi)^3} A p^n z^n +$

$\frac{(\frac{1}{2}(n+2) \sin. n\phi - \frac{1}{2}n \sin. (n+2) \phi)}{2 (\sin \phi)^3} B p^n z^n$. Est ergo

iste terminus generalis quæsitus =

$\frac{(n+3) \sin. (n+1) \phi - (n+1) \sin. (n+3) \phi}{4 (\sin \phi)^3} A p^n z^n +$

$\frac{(n+2) \sin. n\phi - n \sin. (n+2) \phi}{4 (\sin \phi)^3} B p^n z^n$: Seriei quæ oritur ex

fractione $\frac{A + B \cos \phi}{(1 - 2pz \cdot \cos \phi + ppz^2)^2}$.

221. Sit $k = 3$, eritque Seriei ex hac fractione ortæ

$f = \frac{3pz \cdot (f \cdot \cos \phi - g \cdot \sin \phi) + 3ppz^2 \cdot (f \cdot \cos 2\phi - g \cdot \sin 2\phi) - p^3 z^3 \cdot (f \cdot \cos 3\phi - g \cdot \sin 3\phi)}{(1 - 2pz \cdot \cos \phi + ppz^2)^3}$

terminus generalis = $\frac{(n+1)(n+2)}{1 \cdot 2} (f \cdot \cos n\phi + g \cdot \sin n\phi) p^n z^n$.

Deinde

LIB. I.

Deinde Seriei ex fractione $\frac{a + bpz}{(1 - 2pz \cdot \cos \Phi + p^2 z^2)}$, seu ex hac

$$\frac{a - 2apz \cdot \cos \Phi + appz^2 + bpz - 2bppz \cdot \cos \Phi + bp^2 z^2}{(1 - 2pz \cdot \cos \Phi + p^2 z^2)^3} \text{ ortæ terminus ge-}$$

$$\text{neralis est } \frac{(n+3) \sin. (n+1) \Phi - (n+1) \sin. (n+3) \Phi}{4 (\sin. \Phi)^3} p^n z^n + \frac{(n+2) \sin. n\Phi - n \sin. (n+2) \Phi}{4 (\sin. \Phi)^3} bp^n z^n. \text{ Addantur hæ fra-}$$

$$\text{ctiones ac ponatur numerator} = A, \text{ erit } a + f = A \\ 3f \cdot \cos \Phi - 3g \cdot \sin \Phi + 2a \cdot \cos \Phi - b = 0, \quad 3f \cdot \cos 2\Phi - 3g \cdot \sin 2\Phi + a - 2b \cdot \cos \Phi = 0; \quad \& b = f \cdot \cos 3\Phi - g \cdot \sin 3\Phi, \text{ hinc erit } a = \\ \frac{f \cdot \cos 3\Phi - g \cdot \sin 3\Phi - 3f \cdot \cos \Phi + 3g \cdot \sin \Phi}{2 \cos \Phi} = 2g \cdot (\sin \Phi)^2 \tan \Phi -$$

$$f - 2f \cdot (\sin \Phi)^2. \text{ Deinde reperitur } \frac{f}{g} = \frac{\sin 5\Phi - 2\sin 3\Phi + \sin \Phi}{\cos 5\Phi - 2\cos 3\Phi + \cos \Phi}$$

$$\& a + f = A = 2g \cdot (\sin \Phi)^2 \tan \Phi - 2f \cdot (\sin \Phi)^2; \text{ ergo}$$

$$\frac{A}{2 (\sin \Phi)^2} = \frac{g \sin \Phi - f \cos \Phi}{\cos \Phi}; \text{ ex quibus tandem oritur}$$

$$f = \frac{A (\sin \Phi - 2\sin 3\Phi + \sin 5\Phi)}{16 (\sin \Phi)^5}, \quad g = \frac{A (\cos \Phi - 2\cos 3\Phi + \cos 5\Phi)}{16 (\sin \Phi)^5},$$

$$\text{ob } 16 (\sin \Phi)^5 = \sin 5\Phi - 5 \sin 3\Phi + 10 \sin \Phi, \text{ erit } a = \frac{A (9 \sin \Phi - 3 \sin 3\Phi)}{16 (\sin \Phi)^5} \quad \& b = \frac{A (-\sin 2\Phi + \sin 2\Phi)}{16 (\sin \Phi)^5} = 0. \text{ Est}$$

$$\text{autem } 3 \sin \Phi - \sin 3\Phi = 4 (\sin \Phi)^3; \text{ ergo } a = \frac{3A}{4 (\sin \Phi)^2}. \text{ Quo-}$$

$$\text{circa erit terminus generalis } \frac{(n+1)(n+2)}{1 \cdot 2} p^n z^n$$

$$A \frac{(\sin. (n+1) \Phi - 2 \sin. (n+2) \Phi + \sin. (n+3) \Phi)}{16 (\sin. \Phi)^5} +$$

$$3Ap^n z^n \cdot \frac{((n+3) \sin. (n+1) \Phi - (n+1) \sin. (n+3) \Phi)}{16 (\sin. \Phi)^5} =$$

$$\frac{Ap^n z^n}{16 (\sin. \Phi)^5} \left(\frac{(n+4)(n+5)}{1 \cdot 2} \sin. (n+1) \Phi - \frac{2(n+1)(n+5)}{1 \cdot 2} \right.$$

$$\left. \sin. (n+3) \Phi + \frac{(n+1)(n+2)}{1 \cdot 2} \sin. (n+5) \Phi \right).$$

222. Se-

222. Seriei ergo quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz \cos \phi + p^2 z^2)^3}$$

terminus generalis erit hic

$$\begin{aligned} & \frac{Ap^n z^n}{16 (\sin \phi)^5} \left(\frac{(n+5)(n+4)}{1 \cdot 2} \sin(n+1)\phi - \frac{2(n+1)(n+5)}{1 \cdot 2} \times \right. \\ & \quad \left. \sin(n+3)\phi + \frac{(n+1)(n+2)}{1 \cdot 2} \sin(n+5)\phi \right) \\ & + \frac{Bp^n z^n}{16 (\sin \phi)^5} \left(\frac{(n+4)(n+3)}{1 \cdot 2} \sin n\phi - \frac{2n(n+4)}{1 \cdot 2} \sin(n+2)\phi + \right. \\ & \quad \left. \frac{n(n+1)}{1 \cdot 2} \sin(n+4)\phi \right). \end{aligned}$$

Atque, ulterius progrediendo, Seriei, quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz \cos \phi + p^2 z^2)^4}$$

terminus generalis erit hic

$$\begin{aligned} & + \frac{Ap^n z^n}{64 (\sin \phi)^7} \left(\frac{(n+7)(n+6)(n+5)}{1 \cdot 2 \cdot 3} \sin(n+1)\phi - \right. \\ & \quad \frac{3(n+1)(n+7)(n+6)}{1 \cdot 2 \cdot 3} \sin(n+3)\phi + \frac{3(n+1)(n+2)(n+7)}{1 \cdot 2 \cdot 3} \times \\ & \quad \left. \sin(n+5)\phi - \frac{(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3} \sin(n+7)\phi \right) \\ & + \frac{Bp^n z^n}{64 (\sin \phi)^7} \left(\frac{(n+6)(n+5)(n+4)}{1 \cdot 2 \cdot 3} \sin n\phi - \right. \\ & \quad \frac{3n(n+6)(n+5)}{1 \cdot 2 \cdot 3} \sin(n+2)\phi + \frac{3n(n+1)(n+6)}{1 \cdot 2 \cdot 3} \\ & \quad \left. \sin(n+4)\phi - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} \sin(n+6)\phi \right). \end{aligned}$$

Ex his autem expressionibus facile intelligitur, quemadmodum formæ terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expressionum penitus inspiciendam notari convenit esse

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A a

$\sin \phi$

LIB. I.

$$\sin. \Phi = \sin. \Phi$$

$$4(\sin. \Phi)^3 = 3\sin. \Phi - \sin. 3\Phi$$

$$16(\sin. \Phi)^5 = 10\sin. \Phi - 5\sin. 3\Phi + \sin. 5\Phi$$

$$64(\sin. \Phi)^7 = 35\sin. \Phi - 21\sin. 3\Phi + 7\sin. 5\Phi - \sin. 7\Phi$$

$$256(\sin. \Phi)^9 = 126\sin. \Phi - 84\sin. 3\Phi + 36\sin. 5\Phi - 9\sin. 7\Phi + \sin. 9\Phi$$

&c.

223. Cum igitur hoc pacto omnes functiones fractæ in fractiones partiales reales resolvi queant, simul omnium Serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adjuncta sunt.

EXEMPLUM I.

Ex fractione $\frac{1}{(1-z)(1-zz)(1-z^3)} =$
 $\frac{1}{1-z-zz+z^4+z^5-z^6}$, oritur ista Series recurrens
 $1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\&c.$;
 cujus terminus generalis desideratur. Fractio proposita secundum Factores ordinata fit $= \frac{1}{(1-z)^3(1+z)(1+z+zz)}$, quæ
 resolvitur in has fractiones $\frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} +$
 $\frac{17}{72(1-z)} + \frac{1}{8(1+z)} + \frac{(2+zz)}{9(1+z+zz)}$. Harum prima
 $\frac{1}{6(1-z)^3}$ dat terminum generalem $\frac{(n+1)(n+2)}{1 \cdot 2 \cdot 6} \cdot \frac{1}{6} z^n =$
 $\frac{nn+3n+2}{12} z^n$; secunda $\frac{1}{4(1-z)^2}$ dat $\frac{n+1}{4} z^n$; tertia
 $\frac{17}{72(1-z)}$ dat $\frac{17}{72} z^n$; quarta $\frac{1}{8(1+z)}$ dat $\frac{1}{8} (-1)^n z^n$.
 Quinta

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma
 $\frac{A+Bpz}{1-2pz\cos\phi+ppzz}$ (218) dat $p=1$, $\phi=\frac{\pi}{3}=60^\circ$;

$A=+\frac{2}{9}$; & $B=-\frac{1}{9}$, unde oritur terminus generalis
 $+ \frac{2\sin.(n+1)\phi - \sin.n\phi}{9\sin.\phi} (-1)^n z^n = + \frac{4\sin.(n+1)\phi - 2\sin.n\phi}{9\sqrt{3}}$

$(-1)^n z^n = + \frac{4\sin.(n+1)\frac{\pi}{3} - 2\sin.n\frac{\pi}{3}}{9\sqrt{3}} (-1)^n z^n$. Col-

ligantur hæ expressiones omnes in unam summam, ac prodibit
 Seriei propositæ terminus generalis quæsitus $= (\frac{nn}{12} + \frac{n}{2} +$

$\frac{47}{72})z^n \pm \frac{1}{8} z^n \pm \frac{4\sin.(n+1)\frac{\pi}{3} - 2\sin.n\frac{\pi}{3}}{9\sqrt{3}} z^n$, ubi fig-

na superiora valent si n numerus par, inferiora sin impar.
 Ubi notandum est si fuerit n numerus formæ $3m$ fore

$\frac{4\sin.\frac{1}{3}(n+1)\pi - 2\sin.\frac{1}{3}n\pi}{9\sqrt{3}} = \pm \frac{2}{9}$; si fuerit $n =$

$3m+1$ erit hæc expressio $= \mp \frac{1}{9}$; at si $n = 3m+2$ erit

ista expressio $= \mp \frac{1}{9}$, prout n fuerit numerus vel par vel im-
 par. Ex his natura Seriei ita explicari potest, ut

si fuerit

terminus generalis futurus fit

$n = 6m + 0$	$(\frac{nn}{12} + \frac{n}{2} + 1) z^n$
$n = 6m + 1$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}) z^n$
$n = 6m + 2$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}) z^n$
$n = 6m + 3$	$(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4}) z^n$
$n = 6m + 4$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}) z^n$
$n = 6m + 5$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}) z^n$

Sic, si fuerit $n = 50$, valet forma $n = 6m + 2$, eritque terminus Seriei $= 234z^{50}$

EXEMPLUM II.

Ex fractione $\frac{1+z+zz}{1-z-z^2+z^3}$, oritur hæc Series recurrens
 $1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \&c.$,
 cujus terminum generalem invenire oportet. Fractio propo-
 sita ad hanc formam reducitur $\frac{1+z+zz}{(1-z)^2(1+z)(1+zz)}$,
 quæ propterea resolvitur in has fractiones partiales
 $\frac{3}{4(1-z)^2} + \frac{3}{8(1-z)} + \frac{1}{8(1+z)} - \frac{1+z}{4(1+zz)}$. Ha-
 rum prima $\frac{3}{4(1-z)^2}$ dat terminum generalem $\frac{3(n+1)}{4} z^n$;
 secunda $\frac{3}{8(1-z)}$ dat $\frac{3}{8} z^n$; tertia dat $\frac{1}{8} (-1)^n z^n$; &
 quarta $-\frac{1+z}{4(1+zz)}$ comparata cum forma $\frac{A+Bpz}{1-2pz\cos\phi+ppzz}$
 dat $p=1$; $\cos\phi=0$; & $\phi=\frac{1}{2}\pi$; $A=-\frac{1}{4}$;
 $B=$

$B = +\frac{1}{4}$, unde fit terminus generalis $= (-\frac{1}{4} \sin. \frac{1}{2} \text{CAP. XIII.}$
 $(n+1) \varpi + \frac{1}{4} \sin. \frac{1}{2} n \varpi) z^n$. Quare colligendo erit ter-
 minus generalis quæsitus $= (\frac{3}{4} n + \frac{9}{8}) z^n \pm \frac{1}{8} z^n -$
 $\frac{1}{4} (\sin. \frac{1}{2} (n+1) \varpi - \sin. \frac{1}{2} n \varpi) z^n$. Hinc

fi fuerit	erit terminus generalis
$n = 4m + 0$	$(\frac{3}{4} n + 1) z^n$
$n = 4m + 1$	$(\frac{3}{4} n + \frac{5}{4}) z^n$
$n = 4m + 2$	$(\frac{3}{4} n + \frac{3}{2}) z^n$
$n = 4m + 3$	$(\frac{3}{4} n + \frac{3}{4}) z^n$

Ita, si $n = 50$, valebit $n = 4m + 2$, eritque terminus $= 39z^{50}$.

224. Proposita ergo Serie recurrente, quoniam illa fractio unde oritur, facile cognoscitur, ejus terminus generalis secundum præcepta data reperietur. Ex lege autem Seriei recurrentis, qua quisque terminus ex præcedentibus definitur, statim innotescit denominator fractionis, hujusque Factores præbebunt formam termini generalis, per numeratorem enim tantum coefficientes determinantur. Sit nempe proposita hæc Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus lex progressionis, qua unusquisque terminus ex aliquot præcedentibus determinatur, præbeat hunc fractionis denominatorem $1 - \alpha z - \epsilon z^2 - \gamma z^3$. Ita ut fit $D = \alpha C + \epsilon B + \gamma A$; $E = \alpha D + \epsilon C + \gamma B$; $F = \alpha E + \epsilon D + \gamma C$; $\&c.$,

LIB. I. &c., qui multiplicatores $\alpha, + \epsilon, + \gamma$ a MOIVRÆO *scalam relationis* constituere dicuntur. Lex ergo progressionis posita est in scala relationis, atque scala relationis statim præbet denominatorem fractionis, ex cujus resolutione proposita Series recurrens oritur.

225. Ad terminum ergo generalem, seu coëfficientem Potestatis indefinitæ z^n , inveniendum, quæri debent denominatoris $1 - \alpha z - \epsilon z^2 - \gamma z^3$ Factores vel simplices vel duplices, si imaginarios vitare velimus. Sint primo Factores simplices omnes inter se inæquales & reales hi $(1 - pz)(1 - qz)(1 - rz)$; atque fractio generans Seriem propositam resolvetur in $\frac{A}{1 - pz} + \frac{B}{1 - qz} + \frac{C}{1 - rz}$; unde Seriei terminus generalis erit $(Ap^n + Bq^n + Cr^n)z^n$. Si duo Factores fuerint æquales nempe $q = p$, tum terminus generalis hujusmodi erit $((An + B)p^n + Cr^n)z^n$, &, si insuper fuerit $r = q = p$, erit terminus generalis $(An^2 + Bn + C)p^n z^n$. Quod si vero denominator $1 - \alpha z - \epsilon z^2 - \gamma z^3$ duplicem habeat Factorem, ut sit $= (1 - pz)(1 - 2qz \cos \Phi + qqz)$ tum terminus generalis erit $= (Ap^n + \frac{B \sin(n+1)\Phi + C \sin n\Phi}{\sin \Phi} q^n)z^n$. Cum igitur, positis pro n successive numeris 0, 1, 2, prodire debeant termini A, Bz, Cz^2 , hinc valores litterarum A, B, C determinabuntur.

226. Sit scala relationis bimembris, seu determinetur quisque terminus per duos præcedentes, ita ut sit

$$C = \alpha B - \epsilon A; D = \alpha C - \epsilon B; E = \alpha D - \epsilon C, \text{ \&c.},$$

atque manifestum est Seriem hanc recurrentem, quæ sit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots + Pz^n + Qz^{n+1} + \text{\&c.},$$

oriri ex fractione cujus denominator sit $1 - \alpha z + \epsilon z^2$. Sint hujus denominatoris Factores $(1 - pz)(1 - qz)$ erit $p + q =$

$q = \alpha$ & $p q = \epsilon$: atque Seriei terminus generalis erit CAP. XIII.
 $(A p^n + B q^n) z^n$. Hinc facto $n = 0$, erit $A = A + B$;
 & facto $n = 1$ erit $A = A p + B q$; unde fit $A q - B =$
 $A (q - p)$ & $A = \frac{A q - B}{q - p}$; & $B = \frac{A p - B}{p - q}$. Inventis au-
 tem valoribus A & B , erit $P = A p^n + B q^n$ & $Q = A p^{n+1} +$
 $B q^{n+1}$. Tum vero erit $AB = \frac{BB - \alpha AB + \epsilon AA}{4\epsilon - \alpha\alpha}$.

227. Hinc deduci potest modus quemvis terminum ex
 unico præcedente formandi, cum ad hoc per legem progref-
 sionis duo requirantur. Cum enim fit

$$P = A p^n + B q^n \text{ \& } Q = A p \cdot p^n + B q \cdot q^n$$

erit

$P q - Q = A (q - p) p^n$ & $P p - Q = B (p - q) q^n$:
 multiplicentur hæ expressiones in se invicem ; eritque
 $P^2 p q - (p + q) P Q + Q Q - A B (p - q)^2 p^n q^n = 0$.

At est

$$p + q = \alpha ; p q = \epsilon ; (p - q)^2 = (p + q)^2 - 4 p q =$$

$$\alpha\alpha - 4\epsilon \text{ \& } p^n q^n = \epsilon^n. \text{ Quibus substitutis habebitur}$$

$$\epsilon P^2 - \alpha P Q + Q Q = (\epsilon A A - \alpha A B + B B) \epsilon^n, \text{ seu}$$

$$\frac{Q Q - \alpha P Q + \epsilon P P}{B B - \alpha A B + \epsilon A A} = \epsilon^n ; \text{ quæ est insignis proprietas Se-}$$

rierum recurrentium, quarum quisque terminus per duos præ-
 cedentes determinatur. At cognito quovis termino P , erit se-
 quens $Q = \frac{1}{2} \alpha P + \sqrt{((\frac{1}{4} \alpha^2 - \epsilon) P^2 + (B^2 - \alpha A B +$
 $\epsilon A A) \epsilon^n)}$, quæ expressio, etsi speciem irrationalitatis præ se
fert,

LIB. I. fert, tamen semper est rationalis, propterea quod termini irrationales in Serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibuscumque Pz^n & Qz^{n+1} commodè assignari potest terminus multo magis remotus Xz^{2n} Ponatur enim

$$X = fP^2 + gPQ - hAB\epsilon^n \quad \text{Quoniam est}$$

$$P = Ap^n + Bq^n \text{ \& } Q = Ap.p^n + Bq.q^n \text{ atque}$$

$$X = Ap^{2n} + Bq^{2n}; \text{ erit ut sequitur}$$

$$fP^2 = fA^2p^{2n} + fB^2q^{2n} + 2fAB\epsilon^n$$

$$gPQ = gA^2p.p^{2n} + gB^2q.q^{2n} + gAB\alpha\epsilon^n$$

$$-hAB\epsilon^n = \quad \quad \quad -hAB\epsilon^n$$

$$X = Ap^{2n} + Bq^{2n}$$

$$\text{Fiet ergo } f + gp = \frac{1}{A}; f + gq = \frac{1}{B} \text{ \& } h = 2f + g\alpha,$$

$$\text{unde } g = \frac{B - A}{AB(p - q)} \text{ \& } f = \frac{Ap - Bq}{AB(p - q)}. \text{ At est } B - A =$$

$$\frac{\alpha A - 2B}{p - q}; Ap - Bq = \frac{\alpha B - 2A\epsilon}{p - q}. \text{ Ergo } f = \frac{\alpha B - 2A\epsilon}{AB(\alpha\alpha - 4\epsilon)}$$

$$\text{ \& } g = \frac{\alpha A - 2B}{AB(\alpha\alpha - 4\epsilon)} \text{ seu } f = \frac{2A\epsilon - \alpha B}{BB - \alpha AB + \epsilon AA} \text{ \& }$$

$$g = \frac{2B - \alpha A}{BB - \alpha AB + \epsilon AA}; \text{ ideoque } h = \frac{(4\epsilon - \alpha\alpha)A}{BB - \alpha AB + \epsilon AA}.$$

Eritque ergo

$$X = \frac{(2A\epsilon - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \epsilon AA} - A\epsilon^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\epsilon A - (\alpha\alpha - 2\epsilon)B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \epsilon AA)} - \frac{2B\epsilon^n}{\alpha}.$$

His conjungendis per eliminationem termini ϵ^n reperitur

$$X = \frac{(\epsilon A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \epsilon AA}.$$

229. Si

229. Simili modo, si statuantur termini sequentes C A P.
XIII.

$$A + Bz + Cz^2 + \dots + Pz^n + Qz^{n+1} + Rz^{n+2} + \dots +$$

$$Xz^{2n} + Yz^{2n+1} + Zz^{2n+2},$$

erit

$$Z = \frac{(\zeta A - \alpha B) Q^2 + 2BQR - ARR}{BB - \alpha AB + \zeta AA}, \text{ \& ob } R = \alpha Q - \zeta P.$$

erit

$$Z = \frac{-\zeta \zeta A P^2 + 2\zeta(\alpha A - B) PQ + (\alpha B - (\alpha\alpha - \zeta)A) Q^2}{BB - \alpha AB + \zeta AA}$$

At est

$$Z = \alpha Y - \zeta X, \text{ ergo } Y = \frac{Z + \zeta X}{\alpha}; \text{ unde fit}$$

$$Y = \frac{-\zeta B P^2 + 2\zeta A P Q + \alpha(B - \alpha A) Q Q}{BB - \alpha AB + \zeta AA}.$$

Sic igitur porro ex X & Y definiri poterunt simili modo coëfficientes potestatum z^{4n} , & z^{4n+1} ; hincque ipsarum z^{8n} , z^{8n+1} , & ita porro.

E X E M P L U M.

Sit proposita ista Series recurrens

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \dots + Pz^n + Qz^{n+1} + \&c.,$$

cujus cum quilibet coëfficiens sit summa duorum præcedentium, erit denominator fractionis hanc Seriem producentis $1 - z - zz$; ideoque $\alpha = 1$; $\zeta = -1$; & $A = 1$; $B = 3$; unde fit $BB - \alpha AB + \zeta AA = 5$, ex quo

$$\text{oriatur primum } Q = \frac{P + \sqrt{(5PP + 20(-1)^n)}}{2} =$$

$$\frac{P + \sqrt{(5PP \pm 20)}}{2}, \text{ ubi signum superius valet, si } n \text{ sit nu-}$$

merus par, inferius si impar. Sic, si $n = 4$, ob $P = 11$, erit

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$$B b \quad Q =$$

LIB. I. $Q = \frac{11 + \sqrt{(5 \cdot 121 + 20)}}{2} = \frac{11 + 25}{2} = 18$. Si porro coëfficiens termini z^{2n} sit X , erit $X = \frac{4PP + 6PQ - QQ}{5}$; ergo Potestatis z^5 coëfficiens erit $= \frac{4 \cdot 121 + 6 \cdot 198 - 324}{5} = 76$. Cum autem sit $Q = \frac{P + \sqrt{(5PP + 20)}}{2}$ erit $QQ = \frac{3PP + 10 + P\sqrt{(5PP + 20)}}{2}$; ideoque $X = \frac{PP + 2 + P\sqrt{(5PP + 20)}}{2}$. Ex termino ergo Seriei quocunque Pz^n , obtinentur hi $\frac{P + \sqrt{(5PP + 20)}}{2} z^{n+1}$, & $\frac{PP + 2 + P\sqrt{(5PP + 20)}}{2} z^{2n}$.

230. Simili modo in Seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri potest. Sit enim Series hujusmodi recurrens

$A + Bz + Cz^2 + Dz^3 + \dots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.$,
cujus scala relationis sit α , $-\epsilon$, $+\gamma$, seu quæ oriatur ex fractione cujus denominator $= 1 - \alpha z + \epsilon z^2 - \gamma z^3$. Quod si jam termini P, Q, R eodem modo per Factores hujus denominatoris, qui sint $(1 - pz)(1 - qz)(1 - rz)$ exprimantur, ut sit $P = Ap^n + Bq^n + Cr^n$; $Q = Ap \cdot p^n + Bq \cdot q^n + Cr \cdot r^n$; & $R = Ap^2 \cdot p^n + Bq^2 \cdot q^n + Cr^2 \cdot r^n$; ob $p + q + r = \alpha$; $pq + pr + qr = \epsilon$ & $pqr = \gamma$, reperietur hæc proportio

$$\left. \begin{matrix} R^3 - 2\alpha Q \\ + \epsilon P \end{matrix} \right\} \left. \begin{matrix} R^2 + (\alpha\alpha + \epsilon) Q^2 \\ - (\alpha\epsilon + 3\gamma) PQ \\ + \alpha\gamma P^2 \end{matrix} \right\} \left. \begin{matrix} R + (\alpha\gamma + \epsilon\epsilon) PQ^2 \\ - 2\epsilon\gamma P^2 Q \\ + \gamma\gamma P^3 \end{matrix} \right\} : c^n =$$

C³

$$C^1 \frac{-2\alpha B}{+6A} \left\{ C^2 \frac{+(a^2+6)B^2}{-(a6+3\gamma)AB} \right\} C \frac{-(a6-\gamma)B^3}{+(a\gamma+66)AB^2} : 1. \quad \text{CAP. XIII.}$$

Pendet ergo inventio termini R ex duobus præcedentibus P & Q a resolutione æquationis cubicæ.

231. His de terminis generalibus Serierum recurrentium notatis, superest ut earundem Serierum summas investigemus. Ac primo quidem manifestum est summam Seriei recurrentis in infinitum extensæ æqualem esse fractioni ex qua oritur: cuius fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est ut numeratorem definiamus. Sit itaque proposita hæc Series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \&c.,$$

cujus lex progressionis præbeat hunc denominatorem $1 - az + 6z^2 - \gamma z^3 + d z^4$. Sumamus fractionem summæ Seriei in infinitum æqualem esse $= \frac{a + bz + cz^2 + dz^3}{1 - az + 6z^2 - \gamma z^3 + d z^4}$, ex qua cum Series proposita oriri debeat, erit comparando

$$\begin{aligned} a &= A \\ b &= B - aA \\ c &= C - aB + 6A \\ d &= D - aC + 6B - \gamma A. \end{aligned}$$

Hinc erit summa quæsitæ

$$\frac{A + (B - aA)z + (C - aB + 6A)z^2 + (D - aC + 6B - \gamma A)z^3}{1 - az + 6z^2 - \gamma z^3 + d z^4}.$$

232. Hinc facile intelligitur quemadmodum Seriei recurrentis summa ad datum terminum ulque inveniri debeat.

B b 2

Quæ-

LIB. I. Quærat^{ur} scilicet Seriei modo assumptæ summa ad terminum Pz^n , atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots + Pz^n,$$

quoniam hujus Seriei summa in infinitum constat, quærat^{ur} summa terminorum ultimum Pz^n in infinitum sequentium, qui sint

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \&c.,$$

hæc Series per z^{n+1} divisa dat Seriem recurrentem propositæ æqualem, cujus propterea summa erit $t =$

$$\frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} +$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

Unde orietur summa quæsitæ $s =$

$$+ \frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3 - Qz^{n+1} -$$

$$\frac{(R - \alpha Q)z^{n+2} - (S - \alpha R + \beta Q)z^{n+3} -$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

233. Quod si ergo scala relationis fuerit bimembris
 $\alpha, -$

$\alpha, -\beta$; Seriei $A + Bz + Cz^2 + Dz^3 + \dots + Pz^n$, CAP.
XIII.
quæ oritur ex fractione $\frac{A + (B - \alpha A)z}{1 - \alpha z + \beta z^2}$, summa erit

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \beta z^2}$$

At est, ex natura Seriei, $R = \alpha Q - \beta P$, unde prodibit summa

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \beta Pz^{n+2}}{1 - \alpha z + \beta z^2}$$

EXEMPLUM.

Sit proposita Series $1 + 3z + 4z^2 + 7z^3 + \dots + Pz^n$
ubi est $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; erit hujus summa

$$\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - z^2} \quad . \quad \text{Posito vero } z = 1,$$

erit summa Seriei $1 + 3 + 4 + 7 + 11 + \dots + P = P + Q - 3$. Summa ergo termini ultimi & sequentis ternario excedit summam Seriei. Quia vero est $Q =$

$\frac{P + \sqrt{(5PP \pm 20)}}{2}$ erit summa Seriei $1 + 3 + 4 + 7 + 11 + \dots +$
 $P = \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}$. Ex solo ergo termino ultimo summa potest exhiberi.

CAPUT XIV.

De multiplicatione ac divisione Angulorum.

234. **S**It Angulus, vel Arcus, in Circulo cujus Radius $= 1$, quicunque $= z$; ejus Sinus $= x$; Cofinus $= y$, & Tangens $= t$; erit $xx + yy = 1$ & $t = \frac{x}{y}$. Cum igitur, uti supra vidimus; tam Sinus quam Cofinus Angulorum z ; $2z$; $3z$; $4z$; $5z$; &c., constituent Seriem recurrentem cujus scala relationis est $2y$, -1 ; primum Sinus horum Arcuum ita se habebunt:

$$\begin{aligned} \sin. 0z &= 0 \\ \sin. 1z &= x \\ \sin. 2z &= 2xy \\ \sin. 3z &= 4xy^2 - x \\ \sin. 4z &= 8xy^3 - 4xy \\ \sin. 5z &= 16xy^4 - 12xy^2 + x \\ \sin. 6z &= 32xy^5 - 32xy^3 + 6xy \\ \sin. 7z &= 64xy^6 - 80xy^4 + 24xy^2 - x \\ \sin. 8z &= 128xy^7 - 192xy^5 + 80xy^3 - 0xy \end{aligned}$$

hinc concluditur fore

$$\begin{aligned} \sin. nz &= x (2^{n-1} y^{n-1} - (n-2) 2^{n-3} y^{n-3} + \\ &\quad \frac{(n-3)(n-4)}{1. 2} 2^{n-5} y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1. 2. 3} 2^{n-7} y^{n-7} + \\ &\quad \frac{(n-5)(n-6)(n-7)(n-8)}{1. 2. 3. 4} 2^{n-9} y^{n-9} - \&c.) \end{aligned}$$

235. Si ponamus Arcum $nz = s$; erit $\sin. nz = \sin. s = \sin. (\varpi - s) = \sin. (2\varpi + s) = \sin. (3\varpi - s)$ &c., hi enim

enim Sinus omnes sunt inter se æquales. Hinc obtinemus
plures valores pro x , qui erunt

C A P.
XIV.

$$\sin. \frac{s}{n}; \sin. \frac{\pi - s}{n}; \sin. \frac{2\pi + s}{n}; \sin. \frac{3\pi - s}{n}; \sin. \frac{4\pi + s}{n}; \&c.,$$

qui ergo omnes æquationi inventæ æque conveniunt. Tot autem prodibunt diverſi pro x valores, quot numerus n continet unitates, qui propterea erunt radices æquationis inventæ. Cavendum ergo est, ne valores æquales pro iisdem habeantur, quod fiet dum alternæ tantum expressiones assumantur. Cognitis igitur radicibus æquationis a posteriori, earum comparatio cum terminis æquationis notatu dignas præbebit proprietates. Quoniam autem ad hoc æquatio, in qua tantum x tanquam incognita insit, requiritur, pro y suus valor $\sqrt{(1 - xx)}$ substitui debet; unde duplex operatio instituenta erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar, quia Arcuum $-z, +z, +3z, +5z; \&c.$, differentia est $2z$, huiusque Cofinus $= 1 - 2xx$, erit progressionis Sinuum scala relationis hæc $2 - 4xx, -1$. Hinc erit

$$\sin. -z = -x$$

$$\sin. z = x$$

$$\sin. 3z = 3x - 4x^3$$

$$\sin. 5z = 5x - 20x^3 + 16x^5$$

$$\sin. 7z = 7x - 56x^3 + 112x^5 - 64x^7$$

$$\sin. 9z = 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9$$

ergo

$$\sin. nz = nx - \frac{n(nn-1)}{1.2.3} x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5} x^5 -$$

$$\frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7} x^7 + \&c.,$$

si quidem n fuerit numerus impar. Huiusque æquationis radices
sunt

LIB. I. sunt $\sin. z$; $\sin. (\frac{2\pi}{n} + z)$; $\sin. (\frac{4\pi}{n} + z)$; $\sin. (\frac{6\pi}{n} + z)$; $\sin. (\frac{8\pi}{n} + z)$; &c., quarum numerus est n .

237. Hujus ergo æquationis

$$0 = 1 - \frac{nx}{\sin. nz} + \frac{n(n-1)x^2}{1. 2. 3 \sin. nz} - \frac{n(n-1)(n-2)x^3}{1. 2. 3. 4. 5 \sin. nz} \dots \pm \frac{2^{n-1} x^n}{\sin. nz},$$

(ubi signum superius valet si n unitate deficiat a multiplo quaternarii, contra inferius,) Factores sunt $(1 - \frac{x}{\sin. z})$

$$(1 - \frac{x}{\sin. (\frac{2\pi}{n} + z)}) (1 - \frac{x}{\sin. (\frac{4\pi}{n} + z)}) \&c., \text{ ex quibus con-}$$

cluditur fore

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin. (\frac{2\pi}{n} + z)} + \frac{1}{\sin. (\frac{4\pi}{n} + z)} + \frac{1}{\sin. (\frac{6\pi}{n} + z)} +$$

&c., donec habeantur n termini. Tum vero productum omnium

$$\text{erit } \mp \frac{2^{n-1}}{\sin. nz} = \frac{1}{\sin. z. \sin. (\frac{2\pi}{n} + z). \sin. (\frac{4\pi}{n} + z). \sin. (\frac{6\pi}{n} + z) \&c.}$$

$$\text{feu } \sin. nz = \mp 2^{n-1} \sin. z. \sin. (\frac{2\pi}{n} + z). \sin. (\frac{4\pi}{n} + z) \times \sin. (\frac{6\pi}{n} + z) \&c.. \text{ Et, quia terminus penultimus deest, erit}$$

$$0 = \sin. z + \sin. (\frac{2\pi}{n} + z) + \sin. (\frac{4\pi}{n} + z) + \sin. (\frac{6\pi}{n} + z) \&c.$$

EXEMPLUM I.

Si ergo fuerit $n = 3$, prodibunt hæ æqualitates

$$0 = \sin. z + \sin. (120^\circ + z) + \sin. (240^\circ + z) = \sin. z + \sin. (60 - z) - \sin. (60 + z).$$

$$\frac{3}{\sin. 3z}$$

$$\frac{3}{\sin. 3z} = \frac{1}{\sin. z} + \frac{1}{\sin. (120 + z)} + \frac{1}{\sin. (240 + z)} = \frac{1}{\sin. z} + \frac{\text{CAP. XIV.}}{\sin. z}$$

$$\sin. 3z = \frac{\sin. (60 - z)}{4 \sin. z \sin. (120 + z) \sin. (240 + z)} = \frac{\sin. (60 + z)}{4 \sin. z \sin. (60 - z) \sin. (60 + z)}.$$

Erit ergo, uti jam supra notavimus,

$$\sin. (60 + z) = \sin. z + \sin. (60 - z), \&$$

$$3 \operatorname{cosec}. 3z = \operatorname{cosec}. z + \operatorname{cosec}. (60 - z) - \operatorname{cosec}. (60 + z).$$

EXEMPLUM II.

Ponamus esse $n = 5$, atque prodibunt hæ æquationes:

$$0 = \sin. z + \sin. \left(\frac{2}{5} \pi + z \right) + \sin. \left(\frac{4}{5} \pi + z \right) + \sin. \left(\frac{6}{5} \pi + z \right) + \sin. \left(\frac{8}{5} \pi + z \right)$$

$$\text{feu } 0 = \sin. z + \sin. \left(\frac{2}{5} \pi + z \right) + \sin. \left(\frac{1}{5} \pi - z \right) - \sin. \left(\frac{1}{5} \pi + z \right) - \sin. \left(\frac{2}{5} \pi - z \right)$$

$$\text{feu } 0 = \sin. z + \sin. \left(\frac{1}{5} \pi - z \right) - \sin. \left(\frac{1}{5} \pi + z \right) + \sin. \left(\frac{2}{5} \pi + z \right) - \sin. \left(\frac{2}{5} \pi - z \right)$$

deinde erit

$$\frac{5}{\sin. 5z} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{1}{5} \pi - z \right)} - \frac{1}{\sin. \left(\frac{1}{5} \pi + z \right)} - \frac{1}{\sin. \left(\frac{2}{5} \pi - z \right)} + \frac{1}{\sin. \left(\frac{2}{5} \pi + z \right)}$$

$$\sin. 5z = 16 \sin. z \sin. \left(\frac{1}{5} \pi - z \right) \sin. \left(\frac{1}{5} \pi + z \right) \times \sin. \left(\frac{2}{5} \pi - z \right) \sin. \left(\frac{2}{5} \pi + z \right)$$

EXEMPLUM III.

Hoc modo, si ponamus $n = 2m + 1$, erit

$$0 = \sin. z + \sin. \left(\frac{\pi}{n} - z \right) - \sin. \left(\frac{\pi}{n} + z \right) - \sin. \left(\frac{2\pi}{n} - z \right) + \\ \sin. \left(\frac{2\pi}{n} + z \right) + \sin. \left(\frac{3\pi}{n} - z \right) - \sin. \left(\frac{3\pi}{n} + z \right) - \dots \pm \\ \sin. \left(\frac{m\pi}{n} - z \right) \mp \sin. \left(\frac{m\pi}{n} + z \right)$$

ubi signa superiora valent si m fit numerus impar, inferiora si fit par. Altera æquatio erit hæc.

$$\frac{n}{\sin. n z} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{\pi}{n} - z \right)} - \frac{1}{\sin. \left(\frac{\pi}{n} + z \right)} - \\ \frac{1}{\sin. \left(\frac{2\pi}{n} - z \right)} + \frac{1}{\sin. \left(\frac{2\pi}{n} + z \right)} + \frac{1}{\sin. \left(\frac{3\pi}{n} - z \right)} - \\ \frac{1}{\sin. \left(\frac{3\pi}{n} + z \right)} - \dots \pm \frac{1}{\sin. \left(\frac{m\pi}{n} - z \right)} \mp \frac{1}{\sin. \left(\frac{m\pi}{n} + z \right)}$$

quæ ad Cosecantes commode transfertur. Tertio habetur hoc productum:

$$\sin. n z = 2^{2m} \sin. z. \sin. \left(\frac{\pi}{n} - z \right). \sin. \left(\frac{\pi}{n} + z \right). \sin. \left(\frac{2\pi}{n} - z \right) \times \\ \sin. \left(\frac{2\pi}{n} + z \right). \sin. \left(\frac{3\pi}{n} - z \right). \sin. \left(\frac{3\pi}{n} + z \right). \dots \times \\ \sin. \left(\frac{m\pi}{n} - z \right). \sin. \left(\frac{m\pi}{n} + z \right).$$

238. Sit n nunc numerus par, & quoniam est $y = \sqrt{(1 - xx)}$ & $\cos. 2z = 1 - 2xx$, ita ut Seriei Sinuum fit scala relationis, ut ante, $2 - 4xx, -1$, erit

sin.

$$\sin. 0 z = 0$$

$$\sin. 2 z = 2 x \sqrt{(1 - xx)}$$

$$\sin. 4 z = (4x - 8x^3) \sqrt{(1 - xx)}$$

$$\sin. 6 z = (6x - 32x^3 + 32x^5) \sqrt{(1 - xx)}$$

$$\sin. 8 z = (8x - 80x^3 + 192x^5 - 128x^7) \sqrt{(1 - xx)}$$

& generaliter

$$\sin. nz = \left(nx - \frac{n(n-4)}{1 \cdot 2 \cdot 3} x^3 + \frac{n(n-4)(n-16)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^5 - \frac{n(n-4)(n-16)(n-36)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^7 + \dots \pm 2^{n-1} x^{n-1} \right) \sqrt{(1 - xx)}$$

denotante n numerum quemcunque parem.

239. Ad æquationem hanc rationalem efficiendam sumantur utrinque quadrata, ac prodibit hujusmodi æquatio

$$(\sin. nz)^2 = nnxx + Px^4 + Qx^6 + \dots - 2^{2n-2} x^{2n}$$

$$\text{feu } x^{2n} \dots - \frac{nn}{2^{2n-2}} xx + \frac{1}{2^{2n-2}} (\sin. nz)^2 = 0$$

cujus æquationis radices erunt tam affirmativæ quam negativæ;

Scilicet $\pm \sin. z$; $\pm \sin. \left(\frac{\pi}{n} - z \right)$; $\pm \sin. \left(\frac{2\pi}{n} + z \right)$;

$\pm \sin. \left(\frac{3\pi}{n} - z \right)$; $\pm \sin. \left(\frac{4\pi}{n} + z \right)$ &c. Sumendo omnino

n hujusmodi expressiones. Cum igitur ultimus terminus sit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\sin. nz = \pm 2^{n-1} \sin. z \sin. \left(\frac{\pi}{n} - z \right) \sin. \left(\frac{2\pi}{n} + z \right) \times$$

$\sin. \left(\frac{3\pi}{n} - z \right) \dots$; ubi, quibus casibus utrumvis signum valeat, ex casibus particularibus erit dispiciendum.

EXEMPLUM.

Substituendo autem pro n successive numeros 2, 4, 6, &c. & eligendo n Sinus diversos erit.

C c 2

$\sin.$

LIB. I.

$$\sin. 2z = 2 \sin. z. \sin. \left(\frac{\pi}{2} - z \right)$$

$$\sin. 4z = 8 \sin. z. \sin. \left(\frac{\pi}{4} - z \right). \sin. \left(\frac{\pi}{4} + z \right). \sin. \left(\frac{\pi}{2} - z \right)$$

$$\sin. 6z = 32 \sin. z. \sin. \left(\frac{\pi}{6} - z \right). \sin. \left(\frac{\pi}{6} + z \right). \sin. \left(\frac{2\pi}{6} - z \right) \times \\ \sin. \left(\frac{2\pi}{6} + z \right). \sin. \left(\frac{3\pi}{6} - z \right)$$

$$\sin. 8z = 128 \sin. z. \sin. \left(\frac{\pi}{8} - z \right). \sin. \left(\frac{\pi}{8} + z \right). \sin. \left(\frac{2\pi}{8} - z \right) \times$$

$$\sin. \left(\frac{2\pi}{8} + z \right). \sin. \left(\frac{3\pi}{8} - z \right). \sin. \left(\frac{3\pi}{8} + z \right). \sin. \left(\frac{4\pi}{8} - z \right)$$

240. Patet ergo fore generatim

$$\sin. n z = 2^{n-1} \sin. z. \sin. \left(\frac{\pi}{n} - z \right). \sin. \left(\frac{\pi}{n} + z \right). \sin. \left(\frac{2\pi}{n} - z \right) \times$$

$$\sin. \left(\frac{2\pi}{n} + z \right). \sin. \left(\frac{3\pi}{n} - z \right). \sin. \left(\frac{3\pi}{n} + z \right). \dots \sin. \left(\frac{1}{2} \pi - z \right)$$

si n fuerit numerus par. Quod si autem hæc cum superiori, ubi n erat numerus impar, comparatur tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, five n fuerit numerus par five impar,

$$\sin. n z = 2^{n-1} \sin. z. \sin. \left(\frac{\pi}{n} - z \right). \sin. \left(\frac{\pi}{n} + z \right). \sin. \left(\frac{2\pi}{n} - z \right) \times \\ \sin. \left(\frac{2\pi}{n} + z \right). \sin. \left(\frac{3\pi}{n} - z \right). \sin. \left(\frac{3\pi}{n} + z \right) \&c.$$

donec tot habeantur Factores, quot numerus n continet unitates.

241. Expressiones istæ, quibus Sinus Angulorum multiplo-
rum per Factores exponuntur, non parum utilitatis afferre pos-
sunt ad Logarithmos Sinuum Angulorum multiplo-
rum inveniendos, itemque ad plures expressiones Sinuum per Factores, quales
supra (§. 184) dedimus, reperiendas. Erit autem

sin.

$$\sin. z = 1 \sin. z$$

$$\sin. 2 z = 2 \sin. z. \sin. \left(\frac{\pi}{2} - z \right)$$

$$\sin. 3 z = 4 \sin. z. \sin. \left(\frac{\pi}{3} - z \right). \sin. \left(\frac{\pi}{3} + z \right)$$

$$\sin. 4 z = 8 \sin. z. \sin. \left(\frac{\pi}{4} - z \right). \sin. \left(\frac{\pi}{4} + z \right). \sin. \left(\frac{2\pi}{4} - z \right)$$

$$\sin. 5 z = 16 \sin. z. \sin. \left(\frac{\pi}{5} - z \right). \sin. \left(\frac{\pi}{5} + z \right). \sin. \left(\frac{2\pi}{5} - z \right) \times \\ \sin. \left(\frac{2\pi}{5} + z \right)$$

$$\sin. 6 z = 32 \sin. z. \sin. \left(\frac{\pi}{6} - z \right). \sin. \left(\frac{\pi}{6} + z \right). \sin. \left(\frac{2\pi}{6} - z \right) \times \\ \sin. \left(\frac{2\pi}{6} + z \right). \sin. \left(\frac{3\pi}{6} - z \right) \\ \&c.$$

242. Cum deinde fit $\frac{\sin. 2nz}{\sin. nz} = 2 \cos. nz$, Cofinus Angulorum multiplo-
rum multiplorum simili modo per Factores exprimentur.

$$\cos. z = 1 \sin. \left(\frac{\pi}{2} - z \right)$$

$$\cos. 2z = 2 \sin. \left(\frac{\pi}{4} - z \right). \sin. \left(\frac{\pi}{4} + z \right)$$

$$\cos. 3z = 4 \sin. \left(\frac{\pi}{6} - z \right). \sin. \left(\frac{\pi}{6} + z \right). \sin. \left(\frac{3\pi}{6} - z \right)$$

$$\cos. 4z = 8 \sin. \left(\frac{\pi}{8} - z \right). \sin. \left(\frac{\pi}{8} + z \right). \sin. \left(\frac{3\pi}{8} - z \right) \times \\ \sin. \left(\frac{3\pi}{8} + z \right)$$

$$\cos. 5z = 16 \sin. \left(\frac{\pi}{10} - z \right). \sin. \left(\frac{\pi}{10} + z \right). \sin. \left(\frac{3\pi}{10} - z \right) \times \\ \sin. \left(\frac{3\pi}{10} + z \right). \sin. \left(\frac{5\pi}{10} - z \right)$$

& generaliter

C c 3

cos.

LIB. I. $\cos. nz = 2^{n-1} \sin. \left(\frac{\pi}{2n} - z \right) \cdot \sin. \left(\frac{\pi}{2n} + z \right) \cdot \sin. \left(\frac{3\pi}{2n} - z \right) \times$
 $\sin. \left(\frac{3\pi}{2n} + z \right) \cdot \sin. \left(\frac{5\pi}{2n} - z \right) \&c.,$

quoad tot habeantur Factores quot numerus n continet unitates.

243. Eadem expressiones prodibunt ex consideratione Continuum Arcuum multiplo-
 rum, si enim fuerit $\cos. z = y$, erit
 ut sequitur

$$\begin{aligned} \cos. 0z &= 1 \\ \cos. 1z &= y \\ \cos. 2z &= 2yy - 1 \\ \cos. 3z &= 4y^3 - 3y \\ \cos. 4z &= 8y^4 - 8yy + 1 \\ \cos. 5z &= 16y^5 - 20y^3 + 5y \\ \cos. 6z &= 32y^6 - 48y^4 + 18yy - 1 \\ \cos. 7z &= 64y^7 - 112y^5 + 56y^3 - 7y \\ &\& \text{ generaliter.} \end{aligned}$$

$$\begin{aligned} \cos. nz &= 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1 \cdot 2} \\ &\quad 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1 \cdot 2 \cdot 3} 2^{n-7} y^{n-6} + \\ &\quad \frac{n(n-5)(n-6)(n-7)}{1 \cdot 2 \cdot 3 \cdot 4} 2^{n-9} y^{n-8} - \&c., \end{aligned}$$

cujus æquationis, cum sit $\cos. nz = \cos. (2\pi - nz) =$
 $\cos. (2\pi + nz) = \cos. (4\pi \pm nz) = \cos. (6\pi \pm nz) \&c.,$
 erunt radices ipsius y hæ: $\cos. z$; $\cos. \left(\frac{2\pi}{n} \pm z \right)$; $\cos. \left(\frac{4\pi}{n} \pm z \right)$;
 $\cos. \left(\frac{6\pi}{n} \pm z \right) \&c.,$ quarum formularum tot diversæ sunt pro
 y eligendæ quot dantur; dantur autem tot, quot n continet
 unitates.

244. Pri-

244. Primum igitur patet, ob terminum secundum deficientem excepto casu $n = 1$, fore summam harum radicum omnium $= 0$. Erit ergo

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XIV.

$$0 = \cos z + \cos\left(\frac{2\pi}{n} - z\right) + \cos\left(\frac{2\pi}{n} + z\right) + \cos\left(\frac{4\pi}{n} - z\right) + \cos\left(\frac{4\pi}{n} + z\right) + \&c.,$$

sumendo tot terminos quot n continet unitates: Hæc autem æqualitas sponte se offert si n sit numerus par, cum quivis terminus ab alio sui negativo destruat. Contemplemur ergo numeros impares, unitate exclusa, eritque, ob $\cos v = -\cos(\pi - v)$

$$0 = \cos z - \cos\left(\frac{\pi}{3} - z\right) - \cos\left(\frac{\pi}{3} + z\right)$$

$$0 = \cos z - \cos\left(\frac{\pi}{5} - z\right) - \cos\left(\frac{\pi}{5} + z\right) + \cos\left(\frac{2\pi}{5} - z\right) + \cos\left(\frac{2\pi}{5} + z\right)$$

$$0 = \cos z - \cos\left(\frac{\pi}{7} - z\right) - \cos\left(\frac{\pi}{7} + z\right) + \cos\left(\frac{2\pi}{7} - z\right) + \cos\left(\frac{2\pi}{7} + z\right) - \cos\left(\frac{3\pi}{7} - z\right) - \cos\left(\frac{3\pi}{7} + z\right)$$

& generaliter, si fuerit n numerus impar quicunque, erit

$$0 = \cos z - \cos\left(\frac{\pi}{n} - z\right) - \cos\left(\frac{\pi}{n} + z\right) + \cos\left(\frac{2\pi}{n} - z\right) + \cos\left(\frac{2\pi}{n} + z\right) - \cos\left(\frac{3\pi}{n} - z\right) - \cos\left(\frac{3\pi}{n} + z\right) + \cos\left(\frac{4\pi}{n} - z\right) + \cos\left(\frac{4\pi}{n} + z\right) - \&c.,$$

sumendo tot terminos, quot numerus n continet unitates: oportet autem n esse numerum imparem unitate majorem, uti jam monuimus.

245. Quod

LIB. I. 245. Quod ad productum ex omnibus attinet, variae quidem prodeunt expressiones, prout n fuerit numerus vel impar, vel impariter par, vel pariter par: omnes autem comprehenduntur in expressione generali (§. 242.) inventa, si singuli Sinus in Cofinus transmutentur: Erit scilicet

$$\cos. z = 1 \cos. z$$

$$\cos. 2z = 2 \cos. \left(\frac{\pi}{4} + z \right) \cos. \left(\frac{\pi}{4} - z \right)$$

$$\cos. 3z = 4 \cos. \left(\frac{2\pi}{6} + z \right) \cos. \left(\frac{2\pi}{6} - z \right) \cos. z$$

$$\cos. 4z = 8 \cos. \left(\frac{3\pi}{8} + z \right) \cos. \left(\frac{3\pi}{8} - z \right) \cos. \left(\frac{\pi}{8} + z \right) \times \\ \cos. \left(\frac{\pi}{8} - z \right)$$

$$\cos. 5z = 16 \cos. \left(\frac{4\pi}{8} + z \right) \cos. \left(\frac{4\pi}{8} - z \right) \cos. \left(\frac{2\pi}{8} + z \right) \times \\ \cos. \left(\frac{2\pi}{8} - z \right) \cos. z$$

& generaliter

$$\cos. nz = 2^{n-1} \cos. \left(\frac{n-1}{n} \pi + z \right) \cos. \left(\frac{n-1}{n} \pi - z \right) \times \\ \cos. \left(\frac{n-3}{n} \pi + z \right) \cos. \left(\frac{n-3}{n} \pi - z \right) \times \\ \cos. \left(\frac{n-5}{n} \pi + z \right) \cos. \left(\frac{n-5}{n} \pi - z \right) \times \\ \cos. \left(\frac{n-7}{n} \pi + z \right) \&c.,$$

suntis tot Factoribus, quot numerus n continet unitates.

246. Sit n numerus impar, atque æquatio incipiatur ab unitate, erit

$$0 = 1 \mp \frac{ny}{\cos. nz} + \&c., \text{ ubi signum superius valet si } n \text{ fuerit} \\ \text{numerus impar formæ } 4m + 1, \text{ inferius si } n = 4m - 1. \\ \text{Hinc erit}$$

+

$$\begin{aligned}
+ \frac{1}{\cos z} &= \frac{1}{\cos z} \\
- \frac{3}{\cos 3z} &= \frac{1}{\cos z} - \frac{1}{\cos(\frac{\pi}{3} - z)} - \frac{1}{\cos(\frac{\pi}{3} + z)} \\
+ \frac{5}{\cos 5z} &= \frac{1}{\cos z} - \frac{1}{\cos(\frac{\pi}{5} - z)} - \frac{1}{\cos(\frac{\pi}{5} + z)} + \\
&\quad \frac{1}{\cos(\frac{2\pi}{5} - z)} + \frac{1}{\cos(\frac{2\pi}{5} + z)}
\end{aligned}$$

& generaliter, posito $n = 2m + 1$, erit

$$\begin{aligned}
\frac{n}{\cos nz} &= \frac{2m+1}{\cos(2m+1)z} = \frac{1}{\cos(\frac{m}{n}\pi + z)} + \frac{1}{\cos(\frac{m}{n}\pi - z)} - \\
&\quad \frac{1}{\cos(\frac{m-1}{n}\pi + z)} - \frac{1}{\cos(\frac{m-1}{n}\pi - z)} + \\
&\quad \frac{1}{\cos(\frac{m-2}{n}\pi + z)} + \frac{1}{\cos(\frac{m-2}{n}\pi - z)} - \\
&\quad \frac{1}{\cos(\frac{m-3}{n}\pi + z)} - \&c.
\end{aligned}$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{\cos v} = \sec v$, hinc pro Secantibus insignes proprietates deducuntur, erit nempe

$$\sec z = \sec z.$$

$$3\sec 3z = \sec(\frac{\pi}{3} + z) + \sec(\frac{\pi}{3} - z) - \sec(\frac{\pi}{3} + z)$$

$$\begin{aligned}
5\sec 5z &= \sec(\frac{2\pi}{5} + z) + \sec(\frac{2\pi}{5} - z) - \sec(\frac{\pi}{5} + z) - \\
&\quad \sec(\frac{\pi}{5} - z) + \sec(\frac{\pi}{5} + z)
\end{aligned}$$

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D d

7sec.

LIB. I. $7 \sec. 7z = \sec. \left(\frac{3\pi}{7} + z \right) + \sec. \left(\frac{3\pi}{7} - z \right) - \sec. \left(\frac{2\pi}{7} + z \right) -$
 $\sec. \left(\frac{2\pi}{7} - z \right) + \sec. \left(\frac{\pi}{7} + z \right) + \sec. \left(\frac{\pi}{7} - z \right) -$
 $\sec. \left(\frac{0\pi}{7} + z \right)$

& generaliter, posito $n = 2m + 1$, erit

$$\begin{aligned} n \sec. nz &= \sec. \left(\frac{m}{n} \pi + z \right) + \sec. \left(\frac{m}{n} \pi - z \right) - \\ &\sec. \left(\frac{m-1}{n} \pi + z \right) - \sec. \left(\frac{m-1}{n} \pi - z \right) + \\ &\sec. \left(\frac{m-2}{n} \pi + z \right) + \sec. \left(\frac{m-2}{n} \pi - z \right) - \\ &\sec. \left(\frac{m-3}{n} \pi + z \right) - \sec. \left(\frac{m-3}{n} \pi - z \right) + \\ &\sec. \left(\frac{m-4}{n} \pi + z \right) + \dots \pm \sec. z. \end{aligned}$$

248. Pro Cosecantibus autem erit ex §. 237.

$$\begin{aligned} \operatorname{cosec}. z &= \operatorname{cosec}. z \\ 3 \operatorname{cosec}. 3z &= \operatorname{cosec}. z + \operatorname{cosec}. \left(\frac{\pi}{3} - z \right) - \operatorname{cosec}. \left(\frac{\pi}{3} + z \right) \\ 5 \operatorname{cosec}. 5z &= \operatorname{cosec}. z + \operatorname{cosec}. \left(\frac{\pi}{5} - z \right) - \operatorname{cosec}. \left(\frac{\pi}{5} + z \right) - \\ &\operatorname{cosec}. \left(\frac{2\pi}{5} - z \right) + \operatorname{cosec}. \left(\frac{2\pi}{5} + z \right) \\ 7 \operatorname{cosec}. 7z &= \operatorname{cosec}. z + \operatorname{cosec}. \left(\frac{\pi}{7} - z \right) - \operatorname{cosec}. \left(\frac{\pi}{7} + z \right) - \\ &\operatorname{cosec}. \left(\frac{2\pi}{7} - z \right) + \operatorname{cosec}. \left(\frac{2\pi}{7} + z \right) + \\ &\operatorname{cosec}. \left(\frac{3\pi}{7} - z \right) - \operatorname{cosec}. \left(\frac{3\pi}{7} + z \right) \end{aligned}$$

& generaliter, ponendo $n = 2m + 1$, erit

$$n \operatorname{cosec}.$$

$$\begin{aligned}
 n. \operatorname{cosec}. nz &= \operatorname{cosec}. z + \operatorname{cosec}. \left(\frac{\pi}{n} - z \right) - \operatorname{cosec}. \left(\frac{\pi}{n} + z \right) - \\
 &\quad \operatorname{cosec}. \left(\frac{2\pi}{n} - z \right) + \operatorname{cosec}. \left(\frac{2\pi}{n} + z \right) + \\
 &\quad \operatorname{cosec}. \left(\frac{3\pi}{n} - z \right) - \operatorname{cosec}. \left(\frac{3\pi}{n} + z \right) - \\
 &\quad \dots \mp \operatorname{cosec}. \left(\frac{m\pi}{n} - z \right) \pm \operatorname{cosec}. \left(\frac{m\pi}{n} + z \right)
 \end{aligned}$$

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ubi signa superiora valent si m fuerit numerus par, inferiora si m sit impar.

249. Cum sit, uti supra vidimus, $\operatorname{cosec}. nz \pm \sqrt{-1} \cdot \sin. nz = (\operatorname{cosec}. z \pm \sqrt{-1} \cdot \sin. z)^n$, erit $\operatorname{cosec}. nz =$

$$\frac{(\operatorname{cosec}. z + \sqrt{-1} \cdot \sin. z)^n + (\operatorname{cosec}. z - \sqrt{-1} \cdot \sin. z)^n}{2}, \& \sin. nz =$$

$$\frac{(\operatorname{cosec}. z + \sqrt{-1} \cdot \sin. z)^n - (\operatorname{cosec}. z - \sqrt{-1} \cdot \sin. z)^n}{2\sqrt{-1}}, \text{ erit}$$

$$\operatorname{tang}. nz = \frac{(\operatorname{cosec}. z + \sqrt{-1} \cdot \sin. z)^n - (\operatorname{cosec}. z - \sqrt{-1} \cdot \sin. z)^n}{(\operatorname{cosec}. z + \sqrt{-1} \cdot \sin. z)^n \sqrt{-1} + (\operatorname{cosec}. z - \sqrt{-1} \cdot \sin. z)^n \sqrt{-1}}.$$

Ponamus $\operatorname{tang}. z = \frac{\sin. z}{\operatorname{cosec}. z} = t$, erit $\operatorname{tang}. nz =$

$$\frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n \sqrt{-1} + (1 - t\sqrt{-1})^n \sqrt{-1}}, \text{ unde oriun-}$$

tur Tangentes Angulorum multiplo- rum sequentes

$$\operatorname{tang}. z = t$$

$$\operatorname{tang}. 2z = \frac{2t}{1 - t^2}$$

$$\operatorname{tang}. 3z = \frac{3t - t^3}{1 - 3t^2}$$

$$\operatorname{tang}. 4z = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

$$\operatorname{tang}. 5z = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

$$\text{tang. } nz = \frac{nt - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} t^5 - \&c.}{1 - \frac{n(n-1)}{1 \cdot 2} t^2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} t^4 - \&c.}$$

Cum jam fit $\text{tang. } nz = \text{tang. } (\pi + nz) = \text{tang. } (2\pi + nz) = \text{tang. } (3\pi + nz) \&c.$, erunt valores ipsius t , seu radices æquationis, hæ, $\text{tang. } z$; $\text{tang. } (\frac{\pi}{n} + z)$; $\text{tang. } (\frac{2\pi}{n} + z)$; $\text{tang. } (\frac{3\pi}{n} + z)$; $\&c.$, quarum numerus est n .

250. Quod si æquatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{\text{tang. } nz} - \frac{n(n-1)t^2}{1 \cdot 2} + \frac{n(n-1)(n-2)t^3}{1 \cdot 2 \cdot 3 \text{ tang. } nz} + \&c..$$

Ex comparatione ergo coefficientium cum radicibus, erit

$$n \cdot \cot. nz = \cot. z + \cot. (\frac{\pi}{n} + z) + \cot. (\frac{2\pi}{n} + z) + \cot. (\frac{3\pi}{n} + z) + \cot. (\frac{4\pi}{n} + z) + \dots + \cot. (\frac{n-1}{n} \pi + z)$$

deinde erit summa quadratorum harum Cotangentium omnium $= \frac{nn}{(\sin. nz)^2} - n$, similique modo ultiores Potestates possunt definiri. Ponendo autem loco n numeros definitos, erit.

$$\cot. z = \cot. z$$

$$2 \cot. 2z = \cot. z + \cot. (\frac{\pi}{2} + z)$$

$$3 \cot. 3z = \cot. z + \cot. (\frac{\pi}{3} + z) + \cot. (\frac{2\pi}{3} + z)$$

$$4 \cot. 4z = \cot. z + \cot. (\frac{\pi}{4} + z) + \cot. (\frac{2\pi}{4} + z) + \cot. (\frac{3\pi}{4} + z)$$

5 cot.

$$5 \cot. 5z = \cot. z + \cot. \left(\frac{\pi}{5} + z \right) + \cot. \left(\frac{2\pi}{5} + z \right) + \cot. \left(\frac{3\pi}{5} + z \right) + \cot. \left(\frac{4\pi}{5} + z \right).$$

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251. Quia vero est $\cot. v = - \cot. (\pi - v)$, erit

$$\cot. z = \cot. z$$

$$2 \cot. 2z = \cot. z - \cot. \left(\frac{\pi}{2} - z \right)$$

$$3 \cot. 3z = \cot. z - \cot. \left(\frac{\pi}{3} - z \right) + \cot. \left(\frac{2\pi}{3} + z \right)$$

$$4 \cot. 4z = \cot. z - \cot. \left(\frac{\pi}{4} - z \right) + \cot. \left(\frac{\pi}{4} + z \right) - \cot. \left(\frac{2\pi}{4} - z \right)$$

$$5 \cot. 5z = \cot. z - \cot. \left(\frac{\pi}{5} - z \right) + \cot. \left(\frac{\pi}{5} + z \right) - \cot. \left(\frac{2\pi}{5} - z \right) + \cot. \left(\frac{2\pi}{5} + z \right)$$

& generaliter

$$\begin{aligned} n \cot. nz &= \cot. z - \cot. \left(\frac{\pi}{n} - z \right) + \cot. \left(\frac{\pi}{n} + z \right) - \\ &\quad \cot. \left(\frac{2\pi}{n} - z \right) + \cot. \left(\frac{2\pi}{n} + z \right) - \\ &\quad \cot. \left(\frac{3\pi}{n} - z \right) + \cot. \left(\frac{3\pi}{n} + z \right) - \\ &\quad \&c. \end{aligned}$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus æquationem inventam a Potestate summa, ubi primum distingendi sunt casus, quibus n est vel numerus par, vel impar. Sit n numerus impar, $n = 2m + 1$ erit

LIB. I. $t - \text{tang. } z = 0$

$$t^3 - 3t \cdot \text{tang. } 3z - 3t + \text{tang. } 3z = 0$$

$$t^5 - 5t^4 \cdot \text{tang. } 5z - 10t^3 + 10t \cdot \text{tang. } 5z + 5t - \text{tang. } 5z = 0$$

& generaliter

$$t^n - n t^{n-1} \text{tang. } nz - \dots \mp \text{tang. } nz = 0$$

ubi signum superius — valet, si m fit numerus par, inferius + si m fit numerus impar. Erit ergo ex coëfficiente secundi termini

$$\text{tang. } z = \text{tang. } z$$

$$3\text{tang. } 3z = \text{tang. } z + \text{tang. } \left(\frac{\pi}{3} + z\right) + \text{tang. } \left(\frac{2\pi}{3} + z\right)$$

$$5\text{tang. } 5z = \text{tang. } z + \text{tang. } \left(\frac{\pi}{5} + z\right) + \text{tang. } \left(\frac{2\pi}{5} + z\right) + \text{tang. } \left(\frac{3\pi}{5} + z\right) + \text{tang. } \left(\frac{4\pi}{5} + z\right).$$

&c.

253. Cum igitur fit $\text{tang. } v = -\text{tang. } (\pi - v)$, Anguli recto majores ad Angulos recto minores reducuntur, eritque

$$\text{tang. } z = \text{tang. } z$$

$$3\text{tang. } 3z = \text{tang. } z - \text{tang. } \left(\frac{\pi}{3} - z\right) + \text{tang. } \left(\frac{\pi}{3} + z\right)$$

$$5\text{tang. } 5z = \text{tang. } z - \text{tang. } \left(\frac{\pi}{5} - z\right) + \text{tang. } \left(\frac{\pi}{5} + z\right) - \text{tang. } \left(\frac{2\pi}{5} - z\right) + \text{tang. } \left(\frac{2\pi}{5} + z\right)$$

$$7\text{tang. } 7z = \text{tang. } z - \text{tang. } \left(\frac{\pi}{7} - z\right) + \text{tang. } \left(\frac{\pi}{7} + z\right) - \text{tang. } \left(\frac{2\pi}{7} - z\right) + \text{tang. } \left(\frac{2\pi}{7} + z\right) - \text{tang. } \left(\frac{3\pi}{7} - z\right) + \text{tang. } \left(\frac{3\pi}{7} + z\right)$$

& gene-

& generaliter, si $n = 2m + 1$, erit

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$$\begin{aligned} n.tang.nz &= tang.z - tang.\left(\frac{\pi}{n} - z\right) + tang.\left(\frac{\pi}{n} + z\right) - \\ & tang.\left(\frac{2\pi}{n} - z\right) + tang.\left(\frac{2\pi}{n} + z\right) - \\ & tang.\left(\frac{3\pi}{n} - z\right) + \dots\dots\dots - \\ & tang.\left(\frac{m\pi}{n} - z\right) + tang.\left(\frac{m\pi}{n} + z\right). \end{aligned}$$

254. Tum vero productum ex his Tangentibus omnibus erit $= tang.nz$, propterea quod per signorum negativo-
rum numerum alternatim parem & imparem, superior signo-
rum ambiguitas tollitur. Sic erit

$$tang.z = tang.z$$

$$tang.3z = tang.z.tang.\left(\frac{\pi}{3} - z\right).tang.\left(\frac{\pi}{3} + z\right)$$

$$\begin{aligned} tang.5z &= tang.z.tang.\left(\frac{\pi}{5} - z\right).tang.\left(\frac{\pi}{5} + z\right).tang.\left(\frac{2\pi}{5} - z\right) \times \\ & tang.\left(\frac{2\pi}{5} + z\right) \end{aligned}$$

& generaliter, si $n = 2m + 1$, erit

$$\begin{aligned} tang.nz &= tang.z.tang.\left(\frac{\pi}{n} - z\right).tang.\left(\frac{\pi}{n} + z\right).tang.\left(\frac{2\pi}{n} - z\right) \times \\ & tang.\left(\frac{2\pi}{n} + z\right).tang.\left(\frac{3\pi}{n} - z\right) \dots\dots\dots \times \\ & tang.\left(\frac{m\pi}{n} - z\right).tang.\left(\frac{m\pi}{n} + z\right). \end{aligned}$$

255. Sit jam n numerus par, atque, incipiendo a Potestate
summa, erit

$$tt + 2t.cot.2z - 1 = 0$$

$$t^4 + 4t^3.cot.4z - 6tt - 4t.cot.4z + 1 = 0$$

&

& generaliter, si $n = 2m$, erit

$$x^n + nx^{n-1} \cot. nz - \dots - 1 = 0$$

ubi signum superius — valet si m sit numerus impar, inferius + si m sit par. Comparando ergo radices cum coefficiente secundi termini, erit

$$- 2 \cot. 2z = \text{tang. } z + \text{tang. } \left(\frac{\pi}{2} + z \right)$$

$$- 4 \cot. 4z = \text{tang. } z + \text{tang. } \left(\frac{\pi}{4} + z \right) + \text{tang. } \left(\frac{2\pi}{4} + z \right) + \text{tang. } \left(\frac{3\pi}{4} + z \right)$$

$$- 6 \cot. 6z = \text{tang. } z + \text{tang. } \left(\frac{\pi}{6} + z \right) + \text{tang. } \left(\frac{2\pi}{6} + z \right) + \text{tang. } \left(\frac{3\pi}{6} + z \right) + \text{tang. } \left(\frac{4\pi}{6} + z \right) + \text{tang. } \left(\frac{5\pi}{6} + z \right).$$

&c.

256. Cum sit $\text{tang. } v = -\text{tang. } (\pi - v)$, sequentes formabuntur æquationes

$$2 \cot. 2z = -\text{tang. } z + \text{tang. } \left(\frac{\pi}{2} - z \right)$$

$$4 \cot. 4z = -\text{tang. } z + \text{tang. } \left(\frac{\pi}{4} - z \right) - \text{tang. } \left(\frac{\pi}{4} + z \right) + \text{tang. } \left(\frac{2\pi}{4} - z \right)$$

$$6 \cot. 6z = -\text{tang. } z + \text{tang. } \left(\frac{\pi}{6} - z \right) - \text{tang. } \left(\frac{\pi}{6} + z \right) + \text{tang. } \left(\frac{2\pi}{6} - z \right) - \text{tang. } \left(\frac{2\pi}{6} + z \right) + \text{tang. } \left(\frac{3\pi}{6} - z \right)$$

&

& generaliter, si $n = 2m$, erit

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$$\begin{aligned} n.\cot.nz = & -\tan.z + \tan.\left(\frac{\varpi}{n} - z\right) - \tan.\left(\frac{\varpi}{n} + z\right) + \\ & \tan.\left(\frac{2\varpi}{n} - z\right) - \tan.\left(\frac{2\varpi}{n} + z\right) + \\ & \tan.\left(\frac{3\varpi}{n} - z\right) - \tan.\left(\frac{3\varpi}{n} + z\right) + \\ & \dots\dots\dots + \tan.\left(\frac{m\varpi}{n} - z\right). \end{aligned}$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus destruitur; eritque idcirco

$$\begin{aligned} 1 &= \tan.z.\tan.\left(\frac{\varpi}{2} - z\right) \\ 1 &= \tan.z.\tan.\left(\frac{\varpi}{4} - z\right).\tan.\left(\frac{\varpi}{4} + z\right).\tan.\left(\frac{2\varpi}{4} - z\right) \\ 1 &= \tan.z.\tan.\left(\frac{\varpi}{6} - z\right).\tan.\left(\frac{\varpi}{6} + z\right).\tan.\left(\frac{2\varpi}{6} - z\right) \times \\ &\quad \tan.\left(\frac{2\varpi}{6} + z\right).\tan.\left(\frac{3\varpi}{6} - z\right). \end{aligned}$$

&c.

Harum vero æquationum ratio statim sponte in oculos incurrit, cum perpetuo bini Anguli reperiantur, quorum alter est alterius complementum ad rectum. Hujusmodi ergo binorum Angulorum Tangentes productum dant $= 1$; ideoque omnium productum unitati debet esse æquale.

258. Quoniam Sinus & Cosinus Angulorum progressionem arithmeticam constituentium Seriem recurrentem præbent, per Caput præcedens summa hujusmodi Sinuum & Cosinuum quotcunque exhiberi poterit. Sint Anguli in arithmetica progressionem

$$a, a + b, a + 2b, a + 3b, a + 4b, a + 5b, \&c.$$

& quæraturno summa Sinuum horum Angulorum in infinitum progredientium; ponatur ergo

$$\text{Euleri Introduct. in Anal. infin. parv.} \quad Ee \quad s =$$

LIB. I. $s = \sin. a + \sin. (a + b) + \sin. (a + 2b) + \sin. (a + 3b) + \&c.$

& quia hæc Series est recurrens, cujus scala relationis est $2 \cos. b$, — 1, orietur hæc Series ex evolutione fractionis, cujus denominator est $1 - 2z. \cos. b + zz$, posito $z = 1$. Ipsa

vero fractio erit $= \frac{\sin. a + z(\sin. (a + b) - 2 \sin. a. \cos. b)}{1 - 2z. \cos. b + zz}$, quare,

facto $z = 1$, erit $s = \frac{\sin. a + \sin. (a + b) - 2 \sin. a. \cos. b}{2 - 2 \cos. b} =$

$\frac{\sin. a - \sin. (a - b)}{2(1 - \cos. b)}$, ob $2 \sin. a. \cos. b = \sin. (a + b) + \sin. (a - b)$.

Cum autem sit $\sin. f - \sin. g = 2 \cos. \frac{f+g}{2} \cdot \sin. \frac{f-g}{2}$, erit

$\sin. a - \sin. (a - b) = 2 \cos. (a - \frac{1}{2}b) \cdot \sin. \frac{1}{2}b$; & $1 - \cos. b = 2 (\sin. \frac{1}{2}b)^2$, unde erit $s = \frac{\cos. (a - \frac{1}{2}b)}{2 \sin. \frac{1}{2}b}$.

259. Hinc itaque summa quocunque Sinuum, quorum Arcus in arithmetica progressionem incedunt, assignari poterit; quæraturnempe summa hujus progressionis

$\sin. a + \sin. (a + b) + \sin. (a + 2b) + \sin. (a + 3b) + \dots + \sin. (a + nb)$.

Quia summa hujus progressionis in infinitum continuatæ est $\frac{\cos. (a - \frac{1}{2}b)}{2 \sin. \frac{1}{2}b}$, considerentur termini ultimum sequentes in infinitum hi

$\sin. (a + (n+1)b) + \sin. (a + (n+2)b) + \sin. (a + (n+3)b) + \&c.$

quia horum Sinuum summa est $= \frac{\cos. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2}b}$; si hæc

a priori subtrahatur, remanebit summa quæsitæ. Scilicet, si fuerit $s = \sin. a + \sin. (a + b) + \sin. (a + 2b) + \dots + \sin. (a + nb)$,

erit $s = \frac{\cos. (a - \frac{1}{2}b) - \cos. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2}b} =$
 $\frac{\sin. (a + \frac{1}{2}nb) \sin. \frac{1}{2}(n+1)b}{\sin. \frac{1}{2}b}$

260. Pari

260. Pari modo, si consideretur summa Cofinuum, atque ponatur

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$$s = \cos. a + \cos. (a + b) + \cos. (a + 2b) + \cos. (a + 3b) + \&c.$$

in infinitum, erit $s = \frac{\cos. a + z(\cos. (a + b) - 2\cos. a \cos. b)}{1 - 2z \cos. b + z^2}$, posito $z = 1$. Quare, ob $2\cos. a \cos. b = \cos. (a - b) + \cos. (a + b)$, fiet $s = \frac{\cos. a - \cos. (a - b)}{2(1 - \cos. b)}$. At est $\cos. f - \cos. g = 2 \sin. \frac{f + g}{2} \times \sin. \frac{g - f}{2}$; unde erit $\cos. a - \cos. (a - b) = -2 \sin. (a - \frac{1}{2}b) \times \sin. \frac{1}{2}b$, & ob $1 - \cos. b = 2(\sin. \frac{1}{2}b)^2$, erit $s = -\frac{\sin. (a - \frac{1}{2}b)}{2 \sin. \frac{1}{2}b}$. Quare, cum simili modo fit hujus Seriei

$$\cos. (a + (n + 1)b) + \cos. (a + (n + 2)b) + \cos. (a + (n + 3)b) + \&c.$$

summa fit $= -\frac{\sin. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2}b}$, si hæc ab illa subtrahatur, relinquetur summa hujus Seriei

$$s = \cos. a + \cos. (a + b) + \cos. (a + 2b) + \cos. (a + 3b) + \dots + \cos. (a + nb):$$

$$\text{eritque } s = -\frac{\sin. (a - \frac{1}{2}b) + \sin. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2}b} = \frac{\cos. (a + \frac{1}{2}nb) \sin. \frac{1}{2}(n + 1)b}{\sin. \frac{1}{2}b}.$$

261. Plurimæ aliæ quæstiones circa Sinus & Tangentes ex principiis allatis resolvi possent; cujusmodi sunt, si quadrata, altioresve Potestates Sinuum, Tangentiumve summari deberent, verum quia hæc ex reliquis æquationum superiorum coefficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque Sinuum Cofinusque Potestatem per singulos Sinus Cofinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

E e 2

262. Ad

LIB. I. 262. Ad hoc expediendum juvabit ex precedentibus hæc
 ——— Lemmata depromisse

$$\begin{aligned} 2\sin.a.\sin.z &= \cos.(a-z) - \cos.(a+z) \\ 2\cos.a.\sin.z &= \sin.(a+z) - \sin.(a-z) \\ 2\sin.a.\cos.z &= \sin.(a+z) + \sin.(a-z) \\ 2\cos.a.\cos.z &= \cos.(a-z) + \cos.(a+z) \end{aligned}$$

Hinc igitur primum Potestates Sinuum reperiuntur

$$\begin{aligned} \sin.z &= \sin.z \\ 2(\sin.z)^2 &= 1 - \cos.2z \\ 4(\sin.z)^3 &= 3\sin.z - \sin.3z \\ 8(\sin.z)^4 &= 3 - 4\cos.2z + \cos.4z \\ 16(\sin.z)^5 &= 10\sin.z - 5\sin.3z + \sin.5z \\ 32(\sin.z)^6 &= 10 - 15\cos.2z + 6\cos.4z - \cos.6z \\ 64(\sin.z)^7 &= 35\sin.z - 21\sin.3z + 7\sin.5z - \sin.7z \\ 128(\sin.z)^8 &= 35 - 56\cos.2z + 28\cos.4z - 8\cos.6z + \cos.8z \\ 256(\sin.z)^9 &= 126\sin.z - 84\sin.3z + 36\sin.5z - 9\sin.7z + \sin.9z \\ &\quad \&c. \end{aligned}$$

Lex, qua hi coëfficientes progrediuntur, ex unciis Binomii elevati intelligitur, nisi quod numerus absolutus in Potestatibus paribus semissis tantum sit ejus, quem uncia præbent.

263. Pari modo Potestates Cosinuum definiuntur

$$\begin{aligned} \cos.z &= \cos.z \\ 2(\cos.z)^2 &= 1 + \cos.2z \\ 4(\cos.z)^3 &= 3\cos.z + \cos.3z \\ 8(\cos.z)^4 &= 3 + 4\cos.2z + \cos.4z \\ 16(\cos.z)^5 &= 10\cos.z + 5\cos.3z + \cos.5z \\ 32(\cos.z)^6 &= 10 + 15\cos.2z + 6\cos.4z + \cos.6z \\ 64(\cos.z)^7 &= 35\cos.z + 21\cos.3z + 7\cos.5z + \cos.7z \\ &\quad \&c. \end{aligned}$$

Hic ratione legis progressionis eadem sunt monenda quæ circa Sinus notavimus.

CAPUT XV.

De Seriebus ex evolutione Factorum ortis.

264. **S**It propositum productum ex Factoribus, numero five finitis five infinitis, constans hujusmodi

$$(1 + az)(1 + bz)(1 + \gamma z)(1 + \delta z)(1 + \varepsilon z)(1 + \xi z) \&c.,$$

quod, si per multiplicationem actualem evolvatur, det

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.,$$

atque manifestum est coëfficientes $A, B, C, D, E, \&c.$, ita formari ex numeris $a, b, \gamma, \delta, \varepsilon, \xi, \&c.$, ut sit

$$A = a + b + \gamma + \delta + \varepsilon + \xi + \&c. = \text{summæ singulorum}$$

$$B = \text{summæ Factorum ex binis diversis}$$

$$C = \text{summæ Factorum ex ternis diversis}$$

$$D = \text{summæ Factorum ex quaternis diversis}$$

$$E = \text{summæ Factorum ex quinis diversis}$$

&c.

donec perveniatur ad productum ex omnibus.

265. Quod si ergo ponatur $z = 1$, productum hoc

$$(1 + a)(1 + b)(1 + \gamma)(1 + \delta)(1 + \varepsilon) \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $a, b, \gamma, \delta, \varepsilon, \&c.$, vel sumendis singulis, vel duobus pluribusve diversis in se multiplicandis, nascuntur. Atque si idem numerus duobus pluribusve modis resultare queat, etiam idem bis pluriesve in hac numerorum Serie occurret.

266. Si ponatur $z = \frac{1}{E e^3}$, productum hoc

$$(1 - \frac{1}{E e^3}) \&c.$$

$$(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)(1 - \varepsilon) \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \&c.$ vel sumendis singulis, vel duobus pluribusve diversis in se multiplicandis, nascuntur; ut ante quidem, verum hoc discrimine, ut ii numeri, qui vel ex singulis, vel ternis, vel quinis, vel numero imparibus nascuntur, sint negativi, illi vero, qui vel ex binis, vel quaternis, vel senis vel numero paribus resultant, sint affirmativi.

267. Scribantur pro $\alpha, \beta, \gamma, \delta, \&c.$ numeri primi omnes 2, 3, 5, 7, 11, 13, $\&c.$, atque hoc productum

$$(1 + 2)(1 + 3)(1 + 5)(1 + 7)(1 + 11)(1 + 13) \&c. = P$$

æquabitur unitati, cum Serie omnium numerorum vel primorum ipsorum, vel ex primis diversis per multiplicationem ortorum. Erit ergo

$$P = 1 + 2 + 3 + 5 + 6 + 7 + 10 + 11 + 13 + 14 + 15 + 17 + \&c.,$$

in qua Serie omnes occurrunt numeri naturales, exceptis Potestatibus, iisque qui per quamvis Potestatem sunt divisibiles. Defunt scilicet numeri 4, 8, 9, 12, 16, 18 $\&c.$, quoniam sunt vel Potestates, ut 4, 8, 9, 16, $\&c.$, vel per Potestates divisibiles ut 12, 18, $\&c.$

268. Simili modo res se habebit, si pro $\alpha, \beta, \gamma, \delta, \&c.$ Potestates quæcunque numerorum primorum substituantur. Scilicet si ponamus

$$P = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \&c.,$$

erit enim multiplicatione instituta :

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

in

in quibus fractionibus omnes occurrunt numeri præter illos qui vel ipsi sunt Potestates, vel per Potestatem quampiam divisibiles. Cum enim omnes numeri integri sint vel primi vel ex primis per multiplicationem compositi, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur.

C A P.
X V.

269. Si numeri $\alpha, \epsilon, \gamma, \delta$, &c., negative capiantur, ut ante (266.) fecimus, atque ponatur

$$P = \left(1 - \frac{1}{2^n}\right) \left(1 - \frac{1}{3^n}\right) \left(1 - \frac{1}{5^n}\right) \left(1 - \frac{1}{7^n}\right) \left(1 - \frac{1}{11^n}\right) \dots$$

&c., erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \dots$$

&c.,

ubi iterum, ut ante, omnes occurrunt numeri præter Potestates ac divisibiles per Potestates. Verum ipsi numeri primi, & qui ex ternis, quinis, numerove imparibus constant, signum habent præfixum —, qui autem ex binis, vel quaternis, vel senis, vel numero paribus formantur, signum habent +. Sic in hac Serie occurret terminus $\frac{1}{30^n}$, quia est $30 =$

$2 \cdot 3 \cdot 5$, neque adeo Potestatem complectitur, habebit vero hic terminus $\frac{1}{30^n}$ signum —, quia 30 est productum ex tribus numeris primis.

270. Consideremus jam hanc expressionem

$$\frac{1}{(1 - \alpha z)(1 - \epsilon z)(1 - \gamma z)(1 - \delta z)(1 - \varepsilon z) \dots}$$

quæ per divisionem actualem evoluta præbeat hanc Seriem:

1 +

LIB. I. $1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.$

atque manifestum est coëfficientes $A, B, C, D, E, \&c.$, sequenti modo ex numeris $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$, componi, ut sit

$$\left. \begin{array}{l} A = \text{summæ singulorum} \\ B = \text{summæ Factorum ex binis} \\ C = \text{summæ Factorum ex ternis} \\ D = \text{summæ Factorum ex quaternis} \\ \&c. \end{array} \right\} \text{non exclusis Factoribus iisdem.}$$

271. Posito ergo $z = 1$, ista expressio

$$\frac{1}{(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)(1 - \epsilon) \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c.$, vel sumendis singulis, vel duobus pluribusve in se multiplicandis, oriuntur, non exclusis æqualibus. Hoc ergo differt ista numerorum Series ab illa, quæ (§.265.) prodiit, quod ibi Factores tantum diversi sumi debebant, hic autem idem Factor bis pluriesve occurrere possit. Hic scilicet omnes numeri occurrunt, qui per multiplicationem ex his $\alpha, \beta, \gamma, \delta, \&c.$, provenire possunt.

272. Hanc ob rem Series semper ex terminorum numero infinito constat, sive Factorum numerus fuerit infinitus, sive finitus. Sic erit

$$\frac{1}{1 - \frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.,$$

ubi omnes numeri adsunt, qui ex binario solo per multiplicationem oriuntur; seu omnes binarii Potestates. Deinde erit

$$\frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \&c.,$$

ubi

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 & 3 per multiplicationem originem trahunt; seu qui alios Divisores præter 2 & 3 non habent. CAP. XV.

273. Si igitur pro $\alpha, \epsilon, \gamma, \delta$, &c., unitas per singulos omnes numeros primos scribatur, ac ponatur

$$P = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13}) \&c.,}$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi omnes numeri tam primi, quam qui ex primis per multiplicationem nascuntur, occurrunt. Cum autem omnes numeri vel sint ipsi primi, vel ex primis per multiplicationem oriundi, manifestum est, hic omnes omnino numeros integros in denominatoribus adesse debere.

274. Idem evenit, si numerorum primorum Potestates quæcunque accipiantur: si enim ponatur

$$P = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.,}$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod si autem in Factoribus ubique signum + statuatur, ut fit

$$P = \frac{1}{(1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \&c.,}$$

erit

LIB. I.
$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ubi numeri primi habent signum —; qui sunt producti ex duobus primis, five iisdem five diversis, signum habent +; & generatim, quorum numerorum numerus Factorum primorum est par, signum habent +, qui autem ex Factoribus primis numero imparibus constant, habent signum —. Sic terminus $\frac{1}{240^n}$, ob $240 =$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$, habebit signum +, cujus legis ratio percipitur ex §. 270, si ponatur $z = -1$.

275. Si hæc cum superioribus conferantur, nascentur binæ Series quarum productum unitati æquatur. Sit enim

$$P = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.,}$$

$$Q = (1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.,$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} \&c.$$

(269), atque manifestum est fore $PQ = 1$.

275. Sin autem ponatur

$$P = \frac{1}{(1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \&c.,}$$

$$Q =$$

$$Q = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \&c., \quad \text{CAP. XV.}$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{8^n} - \frac{1}{9^n} + \&c.$$

$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

similique modo habebitur $PQ = 1$. Cognita ergo alterius Seriei summa, simul alterius innotescet.

277. Vicissim porro ex cognitis summis harum Serierum, assignari poterunt valores Factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.$$

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \&c.,$$

eritque

$$M = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.}$$

$$N = \frac{1}{(1 - \frac{1}{2^{2n}})(1 - \frac{1}{3^{2n}})(1 - \frac{1}{5^{2n}})(1 - \frac{1}{7^{2n}})(1 - \frac{1}{11^{2n}}) \&c.}$$

Hinc per divisionem nascitur

$$\frac{M}{N} = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \&c.$$

denique vero erit

$$\frac{M}{N} = \frac{2^n + 1}{2^n - 1} \cdot \frac{3^n + 1}{3^n - 1} \cdot \frac{5^n + 1}{5^n - 1} \cdot \frac{7^n + 1}{7^n - 1} \cdot \frac{11^n + 1}{11^n - 1} \cdot \&c.$$

Ex cognitis ergo M & N , præter valores horum productorum, summæ harum Serierum habebuntur

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \&c.$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \&c.$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ex quarum combinatione multæ aliæ deduci possunt.

EXEMPLUM I.

Sit $n = 1$, &, quoniam supra demonstravimus esse,

$$l \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c., \text{ erit, po-}$$

$$\text{frito } x = 1, l \frac{1}{1-1} = l\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$$

At Logarithmus numeri infinite magni ∞ ipse est infinite magnus, ex quo erit

$$M =$$

$$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty. \quad \text{CAP. XV.}$$

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$, fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \&c..$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11}) \&c.,}$$

unde fit

$$\infty = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \&c.,$$

&

$$0 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \&c..$$

Deinde per summationem Serierum supra traditam erit

$$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6}, \text{ hinc obtinentur istæ summæ Serierum}$$

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \&c.$$

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \&c.$$

Denique pro Factoribus orietur

$$\text{LIB. I. } \frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.,$$

feu

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \&c.$$

$$\&c., \text{ ob } \frac{M}{N} = \infty \text{ feu } \frac{N}{M} = 0, \text{ habebitur}$$

$$\infty = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \&c.,$$

feu

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \&c.,$$

atque

$$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \&c.,$$

feu

$$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \&c.,$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, summæ autem ex numeratoribus & denominatoribus cujusque fractionis constanter præbent numeros primos, 3, 5, 7, 11, 13, 17, 19, &c.

EXEMPLUM II.

Sit $n = 2$, eritque ex superioribus

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6}$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \&c. = \frac{\pi^4}{90}$$

Hinc primo istæ Series summantur

$$\frac{6}{\pi\pi}$$

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \frac{C_{AP}}{XV.}$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \&c.$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \&c.$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} \&c.,$$

Deinde valores sequentium productorum innotescunt

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.$$

$$\frac{90}{\pi^4} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \&c.,$$

feu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \&c.,$$

&

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \&c.,$$

five

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \&c.,$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \&c..$$

In his fractionibus numeratores unitate superant denominatores, simul vero sumti præbent quadrata numerorum primorum 3^2 , 5^2 , 7^2 , 11^2 , &c.

EXEM.

Quia ex superioribus valores ipsius M tantum si n sit numerus par, assignare licet, ponamus $n = 4$, eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \&c. = \frac{\pi^4}{90}$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \&c. = \frac{\pi^8}{9450}$$

Hinc primæ sequentes Series summantur

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} + \&c.$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} + \&c.$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} + \&c.$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} + \&c..$$

Deinde etiam valores sequentium productorum obtinentur

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{5^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \&c.$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \&c.$$

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \&c.$$

feu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \&c.,$$

in

in his Factoribus numeratores unitate superant denominatores, C A P. XV.
simul vero sumti præbent bi-quadrata numerorum primorum imparium 3, 5, 7, 11, &c.

278. Quoniam hic summam Seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

ad Factores reduximus, ad Logarithmos commode progredi licebit. Nam, cum fit

$$M = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.}$$

erit

$$lM = -l(1 - \frac{1}{2^n}) - l(1 - \frac{1}{3^n}) - l(1 - \frac{1}{5^n}) - l(1 - \frac{1}{7^n}) - \&c..$$

Hinc, sumendis Logarithmis hyperbolicis, erit

$$lM = +1(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.) \\ + \frac{1}{2}(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.) \\ + \frac{1}{3}(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.) \\ + \frac{1}{4}(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.) \\ \&c.$$

Quod si insuper ponamus

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G g

N =

$$N = \frac{1}{(1 - \frac{1}{2^{2n}})(1 - \frac{1}{3^{2n}})(1 - \frac{1}{5^{2n}})(1 - \frac{1}{7^{2n}})(1 - \frac{1}{11^{2n}}) \&c.}$$

$$\begin{aligned}
 lN = & + 1 \left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c. \right) \\
 & + \frac{1}{2} \left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c. \right) \\
 & + \frac{1}{3} \left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \&c. \right) \\
 & + \frac{1}{4} \left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \&c. \right) \\
 & \&c.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.) \\
& + \frac{1}{3} \left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c. \right) \\
& + \frac{1}{5} \left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \&c. \right) \\
& + \frac{1}{7} \left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \&c. \right) \\
& \&c.
\end{aligned}$$

279. Si

279. Si $n = 1$ erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ CAP. XV.
 $= l\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l.l\infty = \frac{1}{2} l \frac{\pi\pi}{6} =$

$$\begin{aligned} &+ 1 \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c. \right) \\ &+ \frac{1}{3} \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \&c. \right) \\ &+ \frac{1}{5} \left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \&c. \right) \\ &+ \frac{1}{7} \left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \&c. \right) \\ &\&c. \end{aligned}$$

Verum hæ Series, præter primam, non solum summas habent finitas, sed etiam cunctæ simul sumtæ summam efficiunt finitam, eamque satis parvam: unde necesse est ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.$, summa sit infinite magna, quantitate scilicet satis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\pi^4}{90}$: unde fit

$$\begin{aligned} 2l\pi - l6 &= 1 \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c. \right) \\ &+ \frac{1}{2} \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c. \right) \\ &+ \frac{1}{2} \left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c. \right) \\ &\&c. \end{aligned}$$

LIB. I.

$$46\pi - 190 = + 1 \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c. \right) \\ + \frac{1}{2} \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \&c. \right) \\ + \frac{1}{3} \left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c. \right) \\ \&c.$$

$$\frac{1}{2} / \frac{5}{2} = 1 \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c. \right) \\ + \frac{1}{3} \left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c. \right) \\ + \frac{1}{5} \left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c. \right) \\ \&c.$$

281. Quamquam lex, qua numeri primi progrediuntur, non constat, tamen harum Serierum altiorum Potestatum summæ non difficulter proxime assignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c., \\ \&$$

$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c. \\ \text{erit}$$

$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c., \\ \& \text{ ob}$$

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c., \\ \text{erit}$$

$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.; \\ \text{feu}$$

$$S =$$

$$S = (M - 1) \left(1 - \frac{1}{2^n} \right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \&c.,$$

& ob

$$M \left(1 - \frac{1}{2^n} \right) \frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M - 1) \left(1 - \frac{1}{2^n} \right) \left(1 - \frac{1}{3^n} \right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam summam M , valor ipsius S commode invenitur, si quidem n fuerit numerus mediocriter magnus.

282. Inventis autem summis aliorum Potestatum, etiam summæ Potestatum minorum ex formulis inventis exhiberi possunt. Atque hac methodo sequentes prodierunt summæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

si fit

erit summa Seriei

$n = 2;$	0, 452247420041222
$n = 4;$	0, 076993139764252
$n = 6;$	0, 017070086850639
$n = 8;$	0, 004061405366515
$n = 10;$	0, 000993603573633
$n = 12;$	0, 000246026470033
$n = 14;$	0, 000061244396725
$n = 16;$	0, 000015282026219
$n = 18;$	0, 000003817278702

G g 3

$n =$

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$n = 20;$	0, 000000953961123
$n = 22;$	0, 000000238450446
$n = 24;$	0, 000000059608184
$n = 26;$	0, 000000014901555
$n = 28;$	0, 000000003725333
$n = 30;$	0, 000000000931323
$n = 32;$	0, 000000000232830
$n = 34;$	0, 000000000058207
$n = 36;$	0, 000000000014551

reliquæ summæ parium Potestatum in ratione quadrupla decrescunt.

283. Hæc autem Series $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$,
in productum infinitum conversio etiam directe institui potest
hoc modo : fit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c., \text{ subtrahe}$$

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \&c.,$$

erit

$$(1 - \frac{1}{2^n}) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \&c.$$

$= B$: sic sublati sunt omnes termini per 2 divisibiles,

$$\text{subtr. } \frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$(1 - \frac{1}{3^n}) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c. = C:$$

sic insuper sublati sunt omnes termini per 3 divisibiles,

subtr.

$$\text{subtr. } \frac{1}{5^n} C = \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \&c.,$$

erit

 C A P.
X V.

$$(1 - \frac{1}{5^n}) C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

fic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11, reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro *B*, *C*, *D*, *E*, &c., valoribus restitutis tandem orietur

$$A(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c. = 1,$$

unde Series propositæ summa erit =

$$A = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.},$$

feu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (175.) summas harum Serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

si *n* fuerit numerus impar, summa enim est = $N \omega^n$ & valores ipsius *N* loco citato dedimus. Notandum autem est
cum

LIB. I. cum hic tantum numeri impares occurrunt, eos qui sint formæ $4m+1$ habere signum $+$, reliquos formæ $4m-1$ signum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \dots$$

$$\frac{1}{3^n} A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \dots \text{ addatur,}$$

erit

$$(1 + \frac{1}{3^n}) A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \dots \text{ &c. ,}$$

$= B$

$$\frac{1}{5^n} B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} \text{ &c. , subtrahatur,}$$

erit

$$(1 - \frac{1}{5^n}) B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \dots \text{ &c. = C,}$$

ubi jam numeri per 3 & 5 divisibiles defunt,

$$\frac{1}{7^n} C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \dots \text{ addatur,}$$

erit

$$(1 + \frac{1}{7^n}) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \dots \text{ &c. = D,}$$

sic etiam numeri per 7 divisibiles sunt sublati

$$\frac{1}{11^n} D = \frac{1}{11^n} - \frac{1}{121^n} + \dots \text{ addatur,}$$

erit

$$(1 + \frac{1}{11^n}) D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \dots \text{ &c. = E}$$

sic numeri per 11 divisibiles quoque sunt sublati. Auferendis autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus, tandem prodibit

C A P.
X I V.

$$A(1 + \frac{1}{3^n})(1 - \frac{1}{5^n})(1 + \frac{1}{7^n})(1 - \frac{1}{11^n})(1 + \frac{1}{13^n}) \&c. = 1,$$

feu

$$A = \frac{3^n}{3^n + 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n + 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \times$$

$$\frac{17^n}{17^n - 1} \cdot \&c.,$$

ubi in numeratoribus occurrunt Potestates omnium numerorum primorum, quæ in denominatoribus insunt unitate five auctæ five minutæ, prout numeri primi fuerint formæ $4m - 1$, vel $4m + 1$.

285. Posito ergo $n = 1$, ob $A = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \&c.,$$

supra autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{3^2}{2 \cdot 4} \cdot \frac{5^2}{4 \cdot 6} \cdot \frac{7^2}{6 \cdot 8} \cdot \frac{11^2}{10 \cdot 12} \cdot \frac{13^2}{12 \cdot 14} \cdot \frac{17^2}{16 \cdot 18} \cdot \frac{19^2}{18 \cdot 20} \cdot \&c..$$

Dividatur secunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

feu

$$\frac{\pi}{2} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares, unitate differentes a nume-

Euleri *Introduct. in Anal. infin. parv.*

H h

ratori-

LIB. I. ratoribus. Quod si hæc denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{16}{18} \cdot \frac{20}{18} \cdot \frac{24}{22} \cdot \&c.,$$

feu

$$2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes dispescendo, & partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hæc expressiones cum *Wallisiana* comparentur

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11} \&c.,$$

feu

$$\frac{4}{\pi} = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{9 \cdot 9}{8 \cdot 10} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \&c.,$$

cum fit

$$\frac{\pi\pi}{8} = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \cdot \&c.,$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3} = \frac{9 \cdot 9}{8 \cdot 10} \cdot \frac{15 \cdot 15}{14 \cdot 16} \cdot \frac{21 \cdot 21}{20 \cdot 22} \cdot \frac{25 \cdot 25}{24 \cdot 26} \cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$ erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1} \cdot \frac{5^3}{5^3-1} \cdot \frac{7^3}{7^3+1} \cdot \frac{11^3}{11^3+1} \cdot \frac{13^3}{13^3-1} \cdot \frac{17^3}{17^3-1} \cdot \&c..$$

At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1} \cdot \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \text{CAP. XV.}$$

&c., seu

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \text{&c.,}$$

quæ per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot \text{&c.,}$$

hæc vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot \text{&c., seu}$$

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \text{&c.,}$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes dispescendo, ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novæ Series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \text{&c.,}$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\text{&c.,}}$$

unde per evolutionem hæc Series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \text{ &c.,}$$

H h 2

ubi

LIB. I. ubi ratio signorum ita est comparata, ut binarius habeat —; numeri primi formæ $4m - 1$ signum —; & numeri primi formæ $4m + 1$ signum +; numeri autem compositi ea habent signa, quæ ipsis ratione multiplicationis ex primis conveniunt. Sic patebit signum fractionis $\frac{1}{60}$, ob $60 =$

$$2 \cdot 2 \cdot 3 \cdot 5$$

quod erit —. Simili modo porro erit

$$\frac{\pi}{2} = \frac{1}{(1 - \frac{1}{2})(1 + \frac{1}{3})(1 - \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11})(1 - \frac{1}{13}) \&c.,}$$

unde orietur hæc Series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c.,$$

ubi binarius habet signum +; numeri primi formæ $4m - 1$ signum —; numeri primi formæ $4m + 1$ signum +; & numerus quisque compositus id habet signum, quod ipsi ratione compositionis ex primis convenit, secundum regulas multiplicationis.

289. Cum deinde fit

$$\frac{\pi}{2} = \frac{1}{(1 - \frac{1}{3})(1 + \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13}) \&c.,}$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} \&c.,$$

ubi tantum numeri impares occurrunt, signa autem ita sunt comparata, ut numeri primi formæ $4m - 1$ signum habeant +; numeri primi formæ $4m + 1$ signum —; unde simul numerorum compositorum signa definiuntur. Binæ porro Series hinc formari possunt, ubi omnes numeri occurrunt, erit scilicet

$$\pi =$$

$$\pi = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13}) \&c.}, \quad \text{XV.}$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$$

ubi binarius signum habet +; numeri primi formæ $4m - 1$ signum +; numeri vero primi formæ $4m + 1$ signum —. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1 + \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13}) \&c.},$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c.,$$

ubi binarius habet signum —, numeri primi formæ $4m - 1$ signum +, & numeri primi formæ $4m + 1$ signum —.

290. Possunt hinc etiam innumerabiles aliæ signorum conditiones exhiberi, ita ut Series

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \&c.,$$

summa assignari queat. Cum scilicet fit

$$\frac{\pi}{2} = \frac{1}{(1 - \frac{1}{2})(1 + \frac{1}{3})(1 - \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11}) \&c.},$$

H h 3. Multi-

LIB. I.

Multiplicetur hæc expressio per $\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11}) \&c.,}$$

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \&c.,$$

ubi binarius signum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m - 1$ signum $-$; at numeri primi formæ $4m + 1$ signum $+$; & unde pro numeris compositis ratio signorum intelligitur. Simili modo, cum fit

$$\pi = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11}) \&c.,}$$

multiplicetur per $\frac{1 + \frac{1}{5}}{1 - \frac{1}{5}} = \frac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13})(1 + \frac{1}{17}) \&c.,}$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet signum $+$; numeri primi formæ $4m - 1$ signum $+$; & numeri primi formæ $4m + 1$, præter quinarium, signum $-$.

291. Possunt etiam innumerabiles hujusmodi Series exhiberi, quarum summa fit $= 0$. Cum enim fit

C A P.
X V.

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1 + \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11})(1 + \frac{1}{13}) \&c.},$$

unde, ut supra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c.,$$

ubi omnes numeri primi signum habent $-$; compositorumque numerorum signa regulam multiplicationis sequuntur. Mul-

tiplicemus autem illam expressionem per $\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = 3$, erit

pariter

$$0 = \frac{1}{(1 - \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11})(1 + \frac{1}{13}) \&c.},$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$$

ubi binarius habet signum $+$; reliqui numeri primi omnes signum $-$. Simili modo quoque erit

$$0 =$$

LIB. I. $0 = \frac{1}{(1 + \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11})(1 + \frac{1}{13}) \&c.,}$
 unde oritur ista Series
 $0 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$

ubi omnes numeri primi, præter 3 & 5, habent signum —. In genere autem notandum est, quoties omnes numeri primi, exceptis tantum aliquibus, habeant signum —, summam Seriei fore = 0. Contra autem quoties omnes numeri primi, exceptis tantum aliquibus, habeant signum +, tum summam Seriei fore infinite magnam.

292. Supra etiam (176.) summam dedimus Seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} \&c.,$$

si fuerit n numerus impar: Erit ergo

$$\frac{1}{2^n} A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \&c.,$$

quæ addita dat

$$B = (1 + \frac{1}{2^n}) A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \&c.$$

$$\frac{1}{5^n} B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} \&c., \text{ addatur,}$$

erit

$$C =$$

$$C = \left(1 + \frac{1}{5^n}\right)B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \dots$$

CAP. XV.

$$\frac{1}{7^n}C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \dots, \text{ subtrahatur,}$$

erit

$$D = \left(1 - \frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \dots$$

Ex his tandem fiet

$$A \left(1 + \frac{1}{2^n}\right) \left(1 + \frac{1}{5^n}\right) \left(1 - \frac{1}{7^n}\right) \left(1 + \frac{1}{11^n}\right) \left(1 - \frac{1}{13^n}\right) \dots = 1,$$

ubi numeri primi unitate excedentes multipla senarii habent signum —, deficientes autem signum +. Eritque

$$A = \frac{2^n}{2^n + 1} \cdot \frac{5^n}{5^n + 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \cdot \dots$$

293. Consideremus casum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$,
eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \dots$$

ubi in numeratoribus post 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate discrepant, suntque omnes per 6 divisibiles. Cum jam fit

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \cdot \dots$$

erit, hac expressione per illam divisa,

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \dots$$

Euleri *Indroduct. in Anal. infin. parv.*

I i

ubi

LIB. I. ubi denominatores non sunt per 6 divisibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \&c.$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

quarum hæc per illam divisa dat

$$\frac{4}{3} = \frac{6}{4} \cdot \frac{6}{8} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{18}{16} \cdot \frac{18}{20} \cdot \frac{24}{22} \cdot \&c.,$$

feu

$$\frac{4}{3} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{9}{10} \cdot \frac{12}{11} \cdot \&c.,$$

ubi singulæ fractiones ex numeris primis 5, 7, 11, &c., formantur, singulos numeros primos in duas partes unitate differentes dispeciendo, & partes per 3 divisibiles constanter pro numeratoribus sumendo.

294. Quoniam vero supra vidimus esse

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \&c.,$$

feu

$$\frac{\pi}{3} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \&c.,$$

si superiores $\frac{\pi}{2\sqrt{3}}$ & $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15} \cdot \&c.$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \&c.,$$

In priori expressione fractiones formantur ex numeris primis formæ $12m + 6 + 1$, in posteriore ex numeris primis formæ $12m \pm 1$, singulos in duas partes unitate discrepantes dispeciendo, & partes pares pro numeratoribus, impares vero pro denominatoribus sumendo.

295. Contemplemur adhuc Seriem supra inventam (179), quæ ita progrediebatur.

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{\text{CAR. XV.}}{17}$$

&c. = A, erit

$$\frac{1}{3} A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} -$$

&c.: subtrahatur

$$(1 - \frac{1}{3}) A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} -$$

&c. = B

$$\frac{1}{5} B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \text{&c.: addatur, erit}$$

$$(1 + \frac{1}{5}) B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} \text{ &c.} = C:$$

ficque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}} (1 - \frac{1}{3})(1 + \frac{1}{5})(1 + \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13}) \\ (1 - \frac{1}{17})(1 - \frac{1}{19}) \text{ &c.} = 1.$$

ubi signa ita se habent, ut numerorum primorum formæ $8m+1$, vel $8m+3$, signa sint $-$; numerorum primorum vero formæ $8m+5$, vel $8m+7$, signa sint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{&c.},$$

ubi omnes denominatores vel divisibiles sunt per 8, vel tantum sunt numeri impariter pares. Cum igitur fit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \text{&c.}$$

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{&c.},$$

&

I i 2

$\frac{\pi \pi}{8}$

$$\text{LIB. I. } \frac{\pi \pi}{8} = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \cdot \&c.,$$

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \&c.,$$

quæ fractiones formantur ex numeris primis, singulos in duas partes unitate discrepantes dispescendo, & partes pares, (nisi sint pariter pares) pro numeratoribus sumendo.

296: Simili modo reliquæ Series, quas supra pro expressione arcuum circularium invenimus (179. & seqq.) in Factores transformari possunt, qui ex numeris primis constituentur. Sicque multæ aliæ insignes proprietates tam hujusmodi Factorum, quam Serierum infinitarum erui poterunt. Quoniam vero præcipuas hic jam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc Capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur..

CAPUT XVI.

De Partitione numerorum.

297. **P**roposita sit ista expressio

$(1 + x^{\alpha}z)(1 + x^{\beta}z)(1 + x^{\gamma}z)(1 + x^{\delta}z)(1 + x^{\epsilon}z) \&c.$,
quæ cujuscumque induat formam, si per multiplicationem evol-
vatur, inquiramus. Ponamus prodire

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

atque manifestum est P fore summam Potestatum

$x^{\alpha} + x^{\beta} + x^{\gamma} + x^{\delta} + x^{\epsilon} + \&c.$. Deinde Q est summa Fa-
ctorum ex binis Potestatibus diversis, seu Q erit aggregatum
plurium Potestatum ipsius x , quarum Exponentes sunt summæ
duorum terminorum diversorum hujus Seriei

$$\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta, \&c.$$

Simili modo R erit aggregatum Potestatum ipsius x , quarum
Exponentes sunt summæ trium terminorum diversorum. At-
que S erit aggregatum Potestatum ipsius x , quarum Expo-
nentes sunt summæ quatuor terminorum diversorum ejusdem
Seriei, $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$, & ita porro.

298. Singulæ hæ Potestates ipsius x , quæ in valoribus li-
terarum $P, Q, R, S, \&c.$, insunt, unitatem pro coëffi-
ciente habebunt, si quidem earum Exponentes unico modo ex

l i 3

$\alpha, \beta,$

LIB. I. $\alpha, \epsilon, \gamma, \delta, \&c.$, formari queant: fin autem ejusdem Potestatis Exponens pluribus modis possit esse summa duorum, trium, pluriumve terminorum Seriei $\alpha, \epsilon, \gamma, \delta, \epsilon, \&c.$, tum etiam Potestas illa coëfficiens habebit, qui unitatem toties in se complectatur. Sic, si in valore ipsius Q reperiatur Nx^n , indicio hoc erit numerum n esse N diversis modis summam duorum terminorum diversorum Seriei $\alpha, \epsilon, \gamma, \&c.$. Atque si in evolutione Factorum propositorum occurrat terminus $Nx^n z^m$, ejus coëfficiens N indicabit quot variis modis numerus n possit esse summa m terminorum diversorum Seriei $\alpha, \epsilon, \gamma, \delta, \epsilon, \xi, \&c.$

299. Quod si ergo productum propositum

$$(1 + x^\alpha z)(1 + x^\epsilon z)(1 + x^\gamma z)(1 + x^\delta z) \&c.,$$

per multiplicationem veram evolvatur, ex expressione resultante statim apparebit, quot variis modis datus numerus possit esse summa tot terminorum diversorum Seriei $\alpha, \epsilon, \gamma, \delta, \epsilon, \xi, \&c.$, quot quis voluerit. Scilicet, si quæratür quot variis modis numerus n possit esse summa m terminorum illius Seriei diversorum, in expressione evoluta quæri debet terminus $x^n z^m$, ejusque coëfficiens indicabit numerum quæsitum.

300. Quo hæc fiant planiora, sit propositum hoc productum ex Factoribus constans infinitis

$$(1 + xz)(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \&c.,$$

quod per multiplicationem actualem evolutum dat

$$1 + z$$

$$\begin{aligned}
 &1 + z(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \&c.) \\
 &+ z^2(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + \&c.) \\
 &+ z^3(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \&c.) \\
 &+ z^4(x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + 15x^{18} + \&c.) \\
 &+ z^5(x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + 18x^{23} + \&c.) \\
 &+ z^6(x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + 20x^{29} + \&c.) \\
 &+ z^7(x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + 21x^{36} + \&c.) \\
 &+ z^8(x^{36} + x^{37} + 2x^{38} + 3x^{39} + 5x^{40} + 7x^{41} + 11x^{42} + 15x^{43} + 22x^{44} + \&c.)
 \end{aligned}$$

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XVI.

Ex his ergo Seriebus statim definire licet quot variis modis propositus numerus ex dato terminorum diversorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c., numero oriri queat. Sic, si quærat^rur quot variis modis numerus 35 possit esse summa septem terminorum diversorum Seriei 1, 2, 3, 4, 5, 6, 7, &c., quærat^rur in Serie z^7 multiplicante Potestas x^{35} , ejusque coëfficiens 15 indicabit numerum propositum 35 quindecim variis modis esse summam septem terminorum Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c.

301. Quod si autem ponatur $z = 1$, & similes Potestates ipsius x in unam summam conjiciantur, seu, quod eodem re-
dit, si evolvatur hæc expressio infinita

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)\&c.,$$

quo facto orietur hæc Series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \&c.,$$

ubi quivis coëfficiens indicat, quot variis modis Exponens Potestatis ipsius x conjunctæ ex terminis diversis Seriei 1, 2, 3, 4, 5, 6, 7, &c., per additionem emergere possit. Sic apparet numerum 8 sex modis per additionem diversorum numerorum produci, qui sunt

$$8 = 8$$

$$8 = 7 + 1$$

$$8 = 6 + 2$$

$$8 = 5 + 3$$

$$8 = 5 + 2 + 1$$

$$8 = 4 + 3 + 1$$

ubi

LIB. I. ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur, ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producat. Conditio autem diversitatis omittetur, si Factores illos in denominatorem transponamus. Sit igitur proposita hæc expressio

$$\frac{1}{(1 - x^a z)(1 - x^b z)(1 - x^c z)(1 - x^d z)(1 - x^e z) \&c.,}$$

quæ per divisionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c..$$

Atque manifestum est fore P aggregatum Potestatum ipsius x , quarum Exponentes contineantur in hac Serie

$$a, b, c, d, e, f, g, \&c.,$$

Deinde Q erit aggregatum Potestatum ipsius x , quarum Exponentes sint summæ duorum terminorum hujus Seriei, five eorundem five diversorum. Tum erit R summa Potestatum ipsius x , quarum Exponentes ex additione trium terminorum illius Seriei oriantur; & S summa Potestatum, quarum Exponentes ex additione quatuor terminorum in illa Serie contentorum formantur, & ita porro.

303. Si igitur tota expressio per singulos terminos explicetur, & termini similes conjunctim exprimantur, intelligetur quot variis modis propositus numerus n per additionem m terminorum, five diversorum five non diversorum, Seriei $a, b, c, d, e, f, g, \&c.$, produci queat. Quærat scilicet in expressione evoluta terminus $x^n z^m$, ejusque coëfficiens, qui sit

N , ita ut totus terminus sit $= Nx^n z^m$, atque coëfficiens N indicabit quot variis modis numerus n per additionem m terminorum

minorum in Serie $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$, contentorum produci queat. Hoc igitur pacto quæstio priori, quam ante sumus contemplati, similis resolvetur. C A P.
XVI.

304. Accommodemus hæc ad casum inprimis notatu dignum, sitque proposita hæc expressio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\&c.},$$

quæ per divisionem evoluta dabit

$$\begin{aligned} &1+z(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.) \\ &+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+5x^{10}+\&c.) \\ &+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+10x^{11}+\&c.) \\ &+z^4(x^4+x^5+2x^6+3x^7+5x^8+6x^9+9x^{10}+11x^{11}+15x^{12}+\&c.) \\ &+z^5(x^5+x^6+2x^7+3x^8+5x^9+7x^{10}+10x^{11}+13x^{12}+18x^{13}+\&c.) \\ &+z^6(x^6+x^7+2x^8+3x^9+5x^{10}+7x^{11}+11x^{12}+14x^{13}+20x^{14}+\&c.) \\ &+z^7(x^7+x^8+2x^9+3x^{10}+5x^{11}+7x^{12}+11x^{13}+15x^{14}+21x^{15}+\&c.) \\ &+z^8(x^8+x^9+2x^{10}+3x^{11}+5x^{12}+7x^{13}+11x^{14}+15x^{15}+22x^{16}+\&c.) \\ &\&c., \end{aligned}$$

Ex his ergo Seriebus statim definire licet quot variis modis propositus numerus per additionem ex dato terminorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, &c., numero produci queat. Sic, si quæratür quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debet terminus $x^{13}z^5$, cujus coëfficiens 18 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z=1$, atque fimiles Potestates ipsius x conjunctim exprimantur, hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\&c.},$$

evolvetur in hanc Seriem

LIB. I. $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 \&c.$;

in qua quilibet coëfficiens indicat, quot variis modis Exponens Potestatis adjunctæ per additionem produci queat ex numeris integris, five æqualibus five inæqualibus. Scilicet ex termino $11x^6$ cognoscitur numerum 6 undecim modis per additionem numerorum integrorum produci posse, qui sunt

$6 = 6$	$6 = 3 + 1 + 1 + 1$
$6 = 5 + 1$	$6 = 2 + 2 + 2$
$6 = 4 + 2$	$6 = 2 + 2 + 1 + 1$
$6 = 4 + 1 + 1$	$6 = 2 + 1 + 1 + 1 + 1$
$6 = 3 + 3$	$6 = 1 + 1 + 1 + 1 + 1 + 1$
$6 = 3 + 2 + 1$	

ubi quoque notari debet, ipsum numerum propositum, cum in Serie numerorum 1, 2, 3, 4, 5, 6, &c., proposita contineatur, unum modum præbere.

306. His in genere expositis, diligentius inquiramus in modum hanc compositionum multitudinem inveniendi. Ac primo quidem consideremus eam ex numeris integris compositionem, in qua numeri tantum diversi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem proposita hæc expressio

$$Z = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z) \&c.,$$

quæ evoluta & secundum Potestates ipsius z digesta præbeat

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c.,$$

ubi methodus desideratur has ipsius x Functiones $P, Q, R, S, T, \&c.$, expedite inveniendi, hoc enim pacto quæstioni propositæ convenientissime satisfaciet.

307. Patet autem, si loco z ponatur xz , prodire

$$(1 +$$

$$(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z) \&c. = \frac{Z}{1+xz} \quad \text{CAP. XVI.}$$

ergo, posito xz loco z , valor producti, qui erat Z , abibit in

$\frac{Z}{1+xz}$; sicque, cum sit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

erit

$$\frac{Z}{1+xz} = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.,$$

multiplicetur ergo actu per $1+xz$, atque prodibit

$$Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c. \\ + xz + Pxz^2 + Qx^3z^3 + Rx^4z^4 + \&c.,$$

qui valor ipsius Z cum superiori comparatus dabit

$$P = \frac{x}{1-x}; Q = \frac{Px^2}{1-x^2}; R = \frac{Qx^3}{1-x^3}; S = \frac{Rx^4}{1-x^4} \&c.,$$

Sequentes ergo pro P , Q , R , S , $\&c.$, obtinentur valores

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^3}{(1-x)(1-x^2)}$$

$$R = \frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

$$T = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} \\ \&c.$$

308. Sic igitur seorsim unamquamque Seriem Potestatum ipsius x exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem porro est has singulas Series esse recurrentes, quia ex evolutione Functionis fractæ ipsius x nascuntur. Prima scilicet expressio

$$K \quad k \quad 2 \quad P =$$

LIB. I. $P = \frac{x}{1-x}$, dat Seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

ex qua quidem manifestum est quemvis numerum semel in Serie numerorum integrorum contineri.

309. Expressio secunda $\frac{x^3}{(1-x)(1-xx)}$, dat hanc Seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \&c.,$$

in qua cujusvis termini coëfficiens indicat quot modis Exponens ipsius x in duas partes inæquales dispertiri possit. Sic terminus $4x^9$ indicat, numerum 9 quatuor modis in duas partes inæquales secari posse. Quod si hanc Seriem per x^3 dividamus, prodibit Series, quam præbet ista fractio $\frac{1}{(1-x)(1-x^2)}$, quæ erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \&c.,$$

cujus terminus generalis fit $= Nx^n$; atque ex genesi hujus Seriei intelligitur coëfficientem N indicare, quot variis modis Exponens n ex numeris 1 & 2 per additionem nasci queat. Cum igitur prioris Seriei terminus generalis fit $= Nx^{n+3}$, deducitur hinc istud theorema.

Quot variis modis numerus n per additionem ex numeris 1 & 2 produci potest, totidem variis modis numerus $n+3$ in duas partes inæquales secari poterit.

310. Expressio tertia $\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$ in Seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \&c.,$$

in qua cujusvis termini coëfficiens indicat quot variis modis Exponens Potestatis x adjunctæ in tres partes inæquales disper-
tiri

tiri possit. Quod si autem hæc fractio $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ CAP.
XVI.
evolvatur, prodibit hæc Series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \&c.,$$

cujus terminus generalis si ponatur $= Nx^n$, coëfficiens N indicabit quot variis modis numerus n ex numeris 1, 2, 3, per additionem produci possit. Cum igitur prioris Seriei terminus generalis sit Nx^{n+6} , sequetur hinc istud theorema.

Quot variis modis numerus n per additionem ex numeris 1, 2, 3, produci potest, totidem variis modis numerus n + 6 in tres partes inæquales secari poterit.

311. Expressio quarta $\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ in Seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + \&c.,$$

in qua cujusvis termini coëfficiens indicabit quot variis modis Exponens Potestatis x adjunctæ in quatuor partes inæquales dispertiri possit. Quod si autem hæc expressio

$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ evolvatur, prodibit superior Series per x^{10} divisa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \&c.,$$

cujus terminum generalem ponamus $= Nx^n$; atque hinc patebit coëfficientem N indicare, quot variis modis numerus n per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris Seriei terminus generalis futurus sit $= Nx^{n+10}$, deducitur hoc theorema.

LIB. I. *Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, totidem variis modis numerus $n + 10$ in quatuor partes inaequales secari poterit.*

312. Generaliter ergo, si hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots\dots(1-x^m)}$$

in Seriem evolvatur, ejusque terminus generalis fuerit $= Nx^n$, coëfficiens N indicabit, quot variis modis numerus n per additionem produci possit ex his numeris 1, 2, 3, 4 $\dots\dots m$. Quod si autem hæc expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\dots\dots(1-x^m)}$$

in Seriem evolvatur, erit ejus terminus generalis $= Nx^{n+\frac{m(m+1)}{2}}$: atque hic coëfficiens N indicat quot variis modis numerus $n + \frac{m(m+1)}{2}$ in m partes inaequales secari possit, unde hoc habetur theorema.

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4 $\dots\dots m$, totidem modis numerus $n + \frac{m(m+1)}{2}$ in m partes inaequales secari poterit.

313. Ex posita partitione numerorum in partes inaequales, perpendamus quoque partitionem in partes, ubi æqualitas partium non excluditur; quæ partitio ex hac expressione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\&c..}$$

Ponamus evolutione per divisionem instituta prodire

$$Z =$$

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c.. \quad \text{CAP. XVI.}$$

Perfpicuum autem eft, fi loco z ponatur xz , prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\&c.} = (1-xz)Z.$$

Facta ergo in Serie evoluta eadem mutatione, fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c..$$

Multiplisetur ergo superior Series pariter per $(1-xz)$, eritque

$$(1-xz)Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c. \\ -xz - Pxz^2 - Qxz^3 - Rxz^4 - \&c..$$

Comparatione ergo instituta orietur

$$P = \frac{x}{1-x}; \quad Q = \frac{Px}{1-x^2}; \quad R = \frac{Qx}{1-x^3}; \quad S = \frac{Rx}{1-x^4} \quad \&c.,$$

unde pro $P, Q, R, S, \&c.$, fequentes valores proveniunt.

$$P = \frac{x}{1-x} \\ Q = \frac{x^2}{(1-x)(1-x^2)} \\ R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)} \\ S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \\ \&c.$$

314. Exprefiones iftæ a fuperioribus aliter non difcrepant, nifi quod numeratores hic minores habeant Exponentes quam cafu præcedente. Atque hanc ob rem Series, quæ per evolutionem nascuntur, ratione coëfficientium omnino convenient, quæ convenientia jam ex comparatione (§. §. 300. & 304.) perfpici-

LIB. I. perspicitur, nunc vero demum ejus ratio intelligitur. Hinc
 — ergo omnino similia theoremata consequentur, quæ sunt.

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, totidem modis numerus $n + 2$ in duas partes dispertiri poterit.

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, totidem modis numerus $n + 3$ in tres partes dispertiri poterit.

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, totidem modis numerus $n + 4$ in quatuor partes dispertiri poterit.

Atque generaliter habebitur hoc theorema :

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, m , totidem modis numerus $n + m$ in m partes dispertiri poterit.

315. Sive ergo quærat^{ur} quot modis datus numerus in m partes inæquales, sive in m partes, æqualibus non exclusis, dispertiri possit, utraque quæstio resolvetur si cognoscatur quot modis quisque numerus per additionem produci possit ex numeris 1, 2, 3, 4 m , quemadmodum hoc patebit ex sequentibus theorematis, quæ ex superioribus sunt derivata.

Numerus n tot modis in m partes inæquales dispertiri potest, quot modis numerus $n - \frac{m(m+1)}{2}$ per additionem produci potest ex numeris 1, 2, 3, 4, m .

Numerus n , tot modis in m partes sive æquales sive inæquales dispertiri potest quot modis numerus $n - m$ per additionem produci potest ex numeris 1, 2, 3, m .

Hinc porro sequuntur hæc theoremata.

Numerus n totidem modis in m partes inæquales secari potest, quot modis numerus $n - \frac{m(m-1)}{2}$ in m partes, sive æquales sive inæquales, dispertitur.

Numerus n totidem modis in m partes, sive inæquales sive æquales, secari

secari potest, quot modis numerus $n + \frac{m(m-1)}{2}$ in m partes
inaequales dispertiri potest.

316. Per formationem autem Serierum recurrentium inveniri poterit, quot variis modis datus numerus n per additionem produci possit ex numeris $1, 2, 3, \dots, m$. Ad hoc enim inveniendum evolvi debet fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^m)}$$

atque Series recurrens continuari debet usque ad terminum Nx^n , cujus coëfficiens N indicabit, quot modis numerus n per additionem produci possit ex numeris $1, 2, 3, 4, \dots, m$. At vero hic solvendi modus non parum habebit difficultatis, si numeri m & n sint modice magni; scala enim relationis, quam præbet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit Seriem ad plures terminos continuare.

317. Hæc autem disquisitio minus erit molesta, si casus simplices primum expediantur, ex his enim facile erit ad casus magis compositos progredi. Sit Seriei, quæ ex hac fractione oritur,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^m)}$$

terminus generalis $\equiv Nx^n$; at Seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\dots(1-x^m)}$$

ortæ terminus generalis sit Mx^n , ubi coëfficiens M indicabit quot variis modis numerus $n - m$ per additionem produci

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L 1

possit

LIB. I. possit ex numeris $1, 2, 3, \dots, m$. Subtrahatur posterior expressio a priori, ac remanebit

I

$$(1-x)(1-x^2)(1-x^3)\dots(1-x^{m-1})$$

atque manifestum est Seriei hinc ortæ terminum generalem futurum esse $(N-M)x^n$; quare coëfficiens $N-M$ indicabit quot variis modis numerus n per additionem produci possit ex numeris $1, 2, 3, \dots, (m-1)$.

318. Hinc ergo sequentem regulam nanciscimur.

Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris $1, 2, 3, \dots, (m-1)$.

Sit M numerus modorum, quibus numerus $n-m$ per additionem produci potest ex numeris $1, 2, 3, \dots, m$.

Sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris $1, 2, 3, \dots, m$.

His positis, erit, ut vidimus, $L=N-M$; ideoque $N=L+M$. Quod si ergo jam invenerimus quot variis modis numeri n & $n-m$ per additionem produci queant, ille ex numeris $1, 2, 3, \dots, (m-1)$ hic vero ex numeris $1, 2, 3, \dots, m$; hinc addendo cognoscemus, quot variis modis numerus n per additionem produci queat ex numeris $1, 2, 3, \dots, m$. Ope hujus theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit, hocque modo tabula hic annexa est computata, * cujus usus ita se habet.

Si quæratür quot variis modis numerus 50 in 7 partes inæquales dispertiri possit; sumatur in prima columna verticali numerus $50 - \frac{7 \cdot 8}{2} = 22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quæsitum.

Sin autem quæratür, quot variis modis numerus 50 in 7 partes, five æquales five inæquales, dispertiri possit, in prima columna

* Vide Tab. pag. 275.

columna verticali sumatur numerus $50 - 7 = 43$, cui in columna 7^{ma} respondebit numerus quæsitus 8946. CAP.
XVI.

319. Series hujus tabulæ verticales, etsi sunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus, & sequentibus, quam paucis exponere operæ pretium erit. Quoniam enim ex fractione

$\frac{1}{(1-x)(1-xx)}$ oritur Series $1+x+2x^2+2x^3+3x^4+3x^5+\&c.$, ac proinde ex fractione $\frac{x}{(1-x)(1-xx)}$ hæc $x+x^2+2x^3+2x^4+3x^5+3x^6+\&c.$. Si duæ hæ Series addantur, nascitur ista

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \&c.,$$

quæ per divisionem oritur ex fractione $\frac{1+x}{(1-x)(1-xx)} = \frac{1}{(1-x)^2}$; unde patet Seriei postremæ terminos numericos Seriem numerorum naturalium constituere. Hinc ex Serie tabulæ secunda addendo binos terminos proveniet Series numerorum naturalium, posito $x = 1$.

$$\begin{array}{cccccccccccccccc} 1 & + & 1 & + & 2 & + & 2 & + & 3 & + & 3 & + & 4 & + & 4 & + & 5 & + & 5 & + & 6 & + & 6 & + & \&c. \\ 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & 6 & + & 7 & + & 8 & + & 9 & + & 10 & + & 11 & + & 12 & + & \&c. \end{array}$$

Vicissim ergo ex Serie numerorum naturalium superior invenitur, subtrahendo quemque terminum Seriei superioris a termino inferioris sequente.

320. Series verticalis tertia oritur ex fractione

$\frac{1}{(1-x)(1-xx)(1-xx^2)}$. Cum autem sit $\frac{1}{(1-x)^3} = \frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-xx^2)}$, manifestum est, si primo Seriei illius terni termini addantur, tum bini hujus novæ Seriei denuo addantur, prodire debere numeros trigonales, id quod ex schemate sequente apparebit

L 1 2

1 +

LIB. I. 1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + 14 + 16 + 19 &c.
 ——— 1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + 36 + 42 + 49 &c.
 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 + 91 &c.

Vicissim autem apparet quomodo ex Serie trigonalium erui debeat Series superior.

321. Simili modo, quia Series quarta oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)}, \text{ erit } \frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}$$

$$= \frac{1}{(1-x)^4}.$$

Si in Serie quarta primum quaterni termini addantur, tum in Serie resultante terni, denique in hac bini, prodibit Series numerorum pyramidalium uti ex sequenti calculo videre licet.

1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + 23 + 27 + &c.
 1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + 67 + 83 + &c.
 1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + 161 + 203 + &c.
 1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + 286 + 364 + &c.

Simili autem modo Series quinta deducet ad numeros pyramidales secundi ordinis, sexta ad tertii ordinis, & ita porro.

322. Vicissim igitur ex numeris figuratis illæ ipsæ Series, quæ in tabulis occurrunt, formari poterunt, per operationes, quæ ex inspectione calculi sequentis sponte elucebunt.

N U M E R O R U M.

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C A P.

I + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + &c.

I + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + &c.

II

XVI.

I + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + &c.

I + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + &c.

I + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + &c.

III

I + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + &c.

I + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + &c.

I + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + &c.

I + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + &c.

IV

I + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + &c.

I + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + &c.

I + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + &c.

I + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + &c.

I + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + &c.

V

&c.

In his ordinibus primæ Series sunt numeri figurati, unde subtrahendo quemvis terminum Seriei secundæ a termino primæ sequente formatur Series secunda. Tum Seriei tertiæ bini termini conjunctim subtrahantur a termino sequente Seriei secundæ, sicque oritur Series tertia; hocque modo subtrahendo ulterius summam trium, quatuor, & ita porro terminorum a termino superioris Seriei sequente, formabuntur reliquæ Series donec perveniatur ad Seriem, quæ incipit ab $1 + 1 + 2$ &c., hæcque erit Series in tabula exhibita.

323. Series verticales tabulæ omnes similiter incipiunt, continuoque plures habent terminos communes; ex quo intelligitur in infinitum has Series inter se fore congruentes. Prohibet autem Series, quæ oritur ex hac fractione

I

$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)&c.,$

quæ cum sit recurrens, primum denominator spectari debet, ut

L 1 3

hinc

LIB. I. hinc scala relationis habeatur. Quod si autem Factores denominatoris continuo in se multiplicentur, prodibit

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \&c.,$$

quæ Series si attentius consideretur, aliæ Potestates ipsius x adesse non deprehenduntur, nisi quarum Exponentes contineantur in hac formula $\frac{3nn \pm n}{2}$; atque, si n sit numerus impar, Potestates erunt negativæ; affirmativæ autem si n fuerit numerus par.

324. Cum igitur scala relationis sit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \&c.,$$

Series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\&c.,}$$

oriunda erit hæc

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20} + 792x^{21} + 1002x^{22} + 1250x^{23} + 1570x^{24} \&c..$$

In hac ergo Serie coëfficiens quisque indicat, quot variis modis Exponens ipsius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri potest.

$7=7$	$7=4+2+1$	$7=3+1+1+1+1$
$7=6+1$	$7=4+1+1+1$	$7=2+2+2+1$
$7=5+2$	$7=3+3+1$	$7=2+2+1+1+1$
$7=5+1+1$	$7=3+2+2$	$7=2+1+1+1+1+1$
$7=4+3$	$7=3+2+1+1$	$7=1+1+1+1+1+1+1$

325. Quod

325. Quod si autem hoc productum

C A P.
X V I.

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \&c.,$$

evolvatur, sequens prodibit Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+ \&c.,$$

in qua quisque coëfficiens indicat, quot variis modis Exponens ipsius x per additionem numerorum inæqualium oriri possit, Sic numerus 9 octo variis modis per additionem ex numeris inæqualibus formari potest.

$$\begin{array}{l|l} 9 = 9 & 9 = 6 + 2 + 1 \\ 9 = 8 + 1 & 9 = 5 + 4 \\ 9 = 7 + 2 & 9 = 5 + 3 + 1 \\ 9 = 6 + 3 & 9 = 4 + 3 + 2 \end{array}$$

326. Ut comparisonem inter has formas instituamus, sit
 $P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6) \&c.,$
 $\&$

$$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \&c.,$$

erit

$$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12}) \&c.,$$

qui Factores cum omnes in P contineantur, dividatur P per PQ , erit $\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9) \&c.,$
 ideoque

$$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9) \&c.},$$

quæ fractio si evolvatur, prodibit Series, in qua quisque coëfficiens indicabit, quot variis modis Exponens ipsius x , per additionem ex numeris imparibus produci possit. Cum igitur hæc expressio æqualis sit illi, quam in §. præcedente contemplati sumus, sequitur hinc istud theorema.

Quot

LIB. I.

Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inæqualibus; totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, siue æqualibus siue inæqualibus.

327. Cum igitur, ut ante vidimus, fit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \&c.,$$

erit, scribendo xx loco x ,

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \&c.,$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \&c.}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \&c.}.$$

Erit ergo Series Q pariter recurrens, atque ex Serie $\frac{1}{P}$ oritur, hanc per $1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} \&c.$, multiplicando. Nempe, cum sit ex (324), $\frac{1}{P} = 1 + x + 2x^2 +$

$$3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \&c.,$$

si is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \&c.,$$

fiet

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \&c.$$

$$- x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \&c.$$

$$- x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \&c.$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \&c.$$

$= Q$ Hinc ergo, si formatio numerorum per additionem numerorum, siue æqualium siue inæqualium constet, deducetur formatio numerorum per additionem numerorum inæqualium, hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe hæc expressio

$$(1 + x)$$

$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32}) \&c.$, CAP. XVI.

in qua Exponentes ipsius x in ratione dupla progrediuntur. Hæc expressio si evolvatur, reperietur quidem hæc Series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \&c.;$$

quoniam vero dubium esse potest, utrum hæc Series in infinitum hac lege geometrica progrediatur, hanc ipsam Seriem investigemus. Sit igitur

$$P = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16}) \&c.,$$

ac ponatur Series per evolutionem oriunda

$$P = 1 + ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7 + hx^8 + \&c.,$$

Patet autem si loco x scribatur xx , tum prodire productum

$$(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32}) \&c. = \frac{P}{1+x} :$$

facta ergo in Serie eadem substitutione erit

$$\frac{P}{1+x} = 1 + ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \&c.,$$

multiplicetur ergo per $1+x$, eritque

$$P = 1 + x + ax^2 + ax^3 + bx^4 + bx^5 + cx^6 + cx^7 + dx^8 + dx^9 + \&c.,$$

qui valor ipsius P si cum superiori comparetur, habebitur

$$a=1; b=a; c=a; d=b; e=b; f=c; g=c; \&c.,$$

erunt ergo omnes coefficientes $=1$, ideoque productum propositum P evolutum dabit Seriem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

329. Cum igitur hic omnes ipsius x Potestates, singulaeque semel occurrant, ex forma producti $(1+x)(1+x^2)(1+x^4) \&c.$, sequitur, omnem numerum integrum ex terminis progressionis

Euleri *Introduct. in Anal. infin. parv.*

M m

geo-

LIB. I. geometricæ duplæ 1, 2, 4, 8, 16, 32, &c., diversis per additionem formari posse, hocque unico modo. Nota est hæc proprietas in praxi ponderandi, si enim habeantur pondera 1, 2, 4, 8, 16, 32, &c., librarum; his solis ponderibus omnia onera ponderari poterunt, nisi partes libræ requirant. Sic his decem ponderibus, nempe 1 lb, 2 lb, 4 lb, 8 lb, 16 lb, 32 lb, 64 lb, 128 lb, 256 lb, 512 lb, omnia pondera usque ad 1024 lb, librari possunt, & si unum pondus 1024 lb, addatur omnibus oneribus usque ad 2048 lb, ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quæ scilicet in ratione geometrica tripla progrediantur, nempe 1, 3, 9, 27, 81, &c., librarum pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent. Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricæ triplæ 1, 3, 9, 27, 81, &c., diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet.

$$\begin{array}{ccc|ccc}
 1 = 1 & & & 5 = 9 - 3 - 1 & & 9 = 9 \\
 2 = 3 - 1 & & & 6 = 9 - 3 & & 10 = 9 + 1 \\
 3 = 3 & & & 7 = 9 - 3 + 1 & & 11 = 9 + 3 - 1 \\
 4 = 3 + 1 & & & 8 = 9 - 1 & & 12 = 9 + 3 \\
 & & & \text{\&c.} & &
 \end{array}$$

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

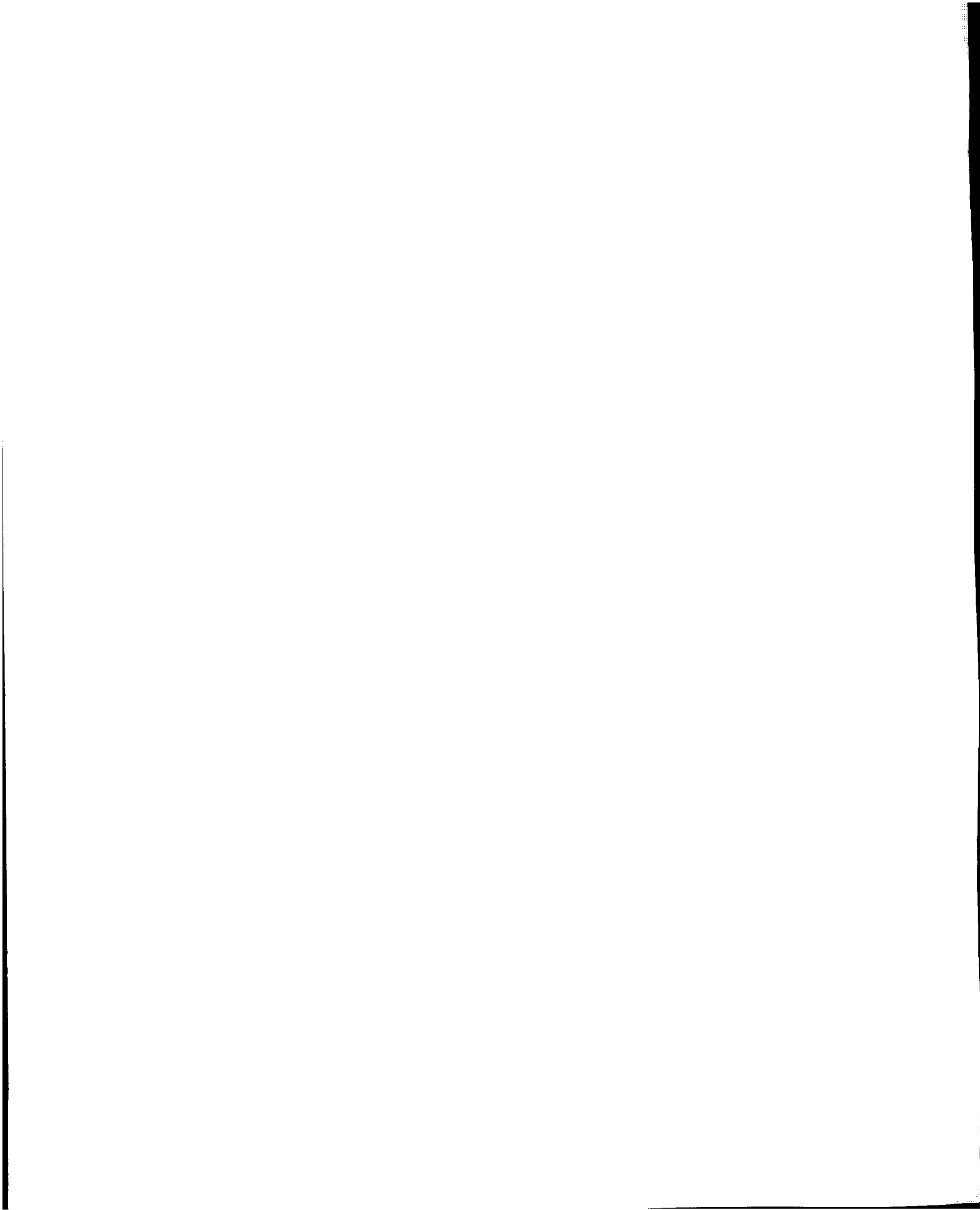
$$(x^{-1} + 1 + x^1)(x^{-3} + 1 + x^3)(x^{-9} + 1 + x^9)(x^{-27} + 1 + x^{27}) \&c. \\
 = P,$$

quod evolutum alias non dabit Potestates ipsius x , nisi quarum Exponentes formari possint ex numeris 1, 3, 9, 27, 81, &c.,
five

T A B U L A

ad paginam 275 Tom. I.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56
12	1	7	19	34	47	58	65	70	73	75	76
13	1	7	21	39	57	71	82	89	94	97	99
14	1	8	24	47	70	90	105	116	123	128	131
15	1	8	27	54	84	110	131	146	157	164	169
16	1	9	30	64	101	136	164	186	201	212	219
17	1	9	33	72	119	163	201	230	252	267	278
18	1	10	37	84	141	199	248	288	318	340	355
19	1	10	40	94	164	235	300	352	393	423	445
20	1	11	44	108	192	282	364	434	488	530	560
21	1	11	48	120	221	331	436	525	598	653	695
22	1	12	52	136	255	391	522	638	732	807	863
23	1	12	56	150	291	454	618	764	887	984	1060
24	1	13	61	169	333	532	733	919	1076	1204	1303
25	1	13	65	185	377	612	860	1090	1291	1455	1586
26	1	14	70	206	427	709	1009	1297	1549	1761	1930
27	1	14	75	225	480	811	1175	1527	1845	2112	2331
28	1	15	80	249	540	931	1367	1801	2194	2534	2812
29	1	15	85	270	603	1057	1579	2104	2592	3015	3370
30	1	16	91	297	674	1206	1824	2462	3060	3590	4035
31	1	16	96	321	748	1360	2093	2857	3589	4242	4802
32	1	17	102	351	831	1540	2400	3319	4206	5013	5788
33	1	17	108	378	918	1729	2738	3828	4904	5888	6751
34	1	18	114	411	1014	1945	3120	4417	5708	6912	7972
35	1	18	120	441	1115	2172	3539	5066	6615	8070	9373
36	1	19	127	478	1226	2432	4011	5812	7657	9418	11004
37	1	19	133	511	1342	2702	4526	6630	8824	10936	12866
38	1	20	140	551	1469	3009	5102	7564	10156	12690	15021
39	1	20	147	588	1602	3331	5731	8588	11648	14663	17475
40	1	21	154	632	1747	3692	6430	9749	13338	16928	20298
41	1	21	161	672	1898	4070	7190	11018	15224	19466	23501
42	1	22	169	720	2062	4494	8033	12450	17354	22367	27169
43	1	22	176	764	2233	4935	8946	14012	19720	25608	31316
44	1	23	184	816	2418	5427	9953	15765	22380	29292	36043
45	1	23	192	864	2611	5942	11044	17674	25331	33401	41373
46	1	24	200	920	2818	6510	12241	19805	28629	38047	47420
47	1	24	208	972	3034	7104	13534	22122	32278	43214	54218
48	1	25	217	1033	3266	7760	14950	24699	36347	49037	61903
49	1	25	225	1089	3507	8442	16475	27493	40831	55494	70515
50	1	26	234	1154	3765	9192	18138	30588	45812	62740	80215
51	1	26	243	1215	4033	9975	19928	33940	51294	70760	91058
52	1	27	252	1285	4319	10829	21873	37638	57358	79725	103226
53	1	27	261	1350	4616	11720	23961	41635	64015	89623	116792
54	1	28	271	1425	4932	12692	26226	46031	71362	100654	131970
55	1	28	280	1495	5260	13702	28652	50774	79403	112804	148847
56	1	29	290	1575	5608	14800	31275	55974	88252	126299	167672
57	1	29	300	1650	5969	15944	34082	61575	97922	141136	188556
58	1	30	310	1735	6351	17180	37108	67696	108527	157564	211782
59	1	30	320	1815	6747	18467	40340	74280	120092	175586	237429
60	1	31	331	1906	7166	19858	43819	81457	132751	195491	266006
61	1	31	341	1991	7599	21301	47527	89162	146520	217280	297495
62	1	32	352	2087	8056	22856	51508	97539	161554	241279	332337
63	1	32	363	2178	8529	24473	55748	106522	177884	267507	370733
64	1	33	374	2280	9027	26207	60289	116263	195666	296320	413112
65	1	33	385	2376	9542	28009	65117	126692	214944	327748	459718
66	1	34	397	2484	10083	29941	70281	137977	235899	362198	511045
67	1	34	408	2586	10642	31943	75762	150042	258569	399705	567377
68	1	35	420	2700	11229	34085	81612	163069	283161	440725	629281
69	1	35	432	2808	11835	36308	87816	176978	309729	485315	697097



five addendo five subtrahendo : num vero omnes Potestates prodeant , singulæque semel, sic exploro. Sit

C A P.
X V I.

$$P = \&c. + cx^{-3} + bx^{-2} + ax^{-1} + 1 + ax^1 + \zeta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.,$$

manifestum vero est, si x^3 loco x scribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^1} = bx^{-6} + ax^{-3} + 1 + ax^3 + \zeta x^6 + \gamma x^9 + \&c..$$

Hinc igitur reperitur $P = \&c.$

$$+ ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + ax^2 + ax^3 + ax^4 + \zeta x^5 + \zeta x^6 + \zeta x^7 + \&c.,$$

quæ expressio cum assumpta comparata dabit

$$\begin{aligned} a &= 1; \zeta = a; \gamma = a; \delta = a; \varepsilon = \zeta; \xi = \zeta; \&c., \& \\ a &= 1, b = a, c = a, d = a, e = b, \&c.. \end{aligned}$$

Hinc itaque erit

$$P = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c. \\ + x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \&c.,$$

unde patet omnes ipsius x Potestates, tam affirmativas quam negativas, hic occurrere, atque adeo omnes numeros ex terminis progressionis geometricæ triplæ, vel addendo vel subtrahendo, formari posse; & unumquemque numerum unico tantum modo.

De usu Serierum recurrentium in radicibus æquationum indagandis.

332. **I**ndicavit *Vir Celeb. Daniel* BERNOULLI insignem usum Serierum recurrentium in investigandis radicibus æquationum cujusvis gradus, in *Comment. Acad. Petropol. Tomo III.*, ubi ostendit, quemadmodum cujusque æquationis algebraicæ, quotcunque fuerit dimensionum, valores radicum veris proximi ope Serierum recurrentium assignari queant. Quæ inventio, cum sæpenumero maximam afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim præter expectationem evenit, ut nulla æquationis radix ope hujus methodi cognosci queat. Quocirca, ut vis hujus methodi clarius perspiciatur, ex proprietatibus Serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis Series recurrens ex evolutione cujusdam fractionis rationalis oritur, sit ista fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.},$$

unde oriatur sequens Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes *A, B, C, D, &c.*, ita determinantur ut sit

$$A = a$$

$$\begin{aligned} A &= a \\ B &= \alpha A + b \\ C &= \alpha B + \zeta A + c \\ D &= \alpha C + \zeta B + \gamma A + d \\ E &= \alpha D + \zeta C + \gamma B + \delta A + e \\ &\quad \&c. \end{aligned}$$

Terminus autem generalis, seu coëfficiens Potestatis z^n , invenitur ex resolutione fractionis propositæ in fractiones simplices, quarum denominatores sint Factores denominatoris $1 - \alpha z - \zeta z^2 - \gamma z^3 - \&c.$, uti (Cap. XIII.) est ostensum.

334. Forma autem termini generalis potissimum pendet ab indole Factorum simplicium denominatoris, utrum sint reales an imaginarii, & utrum sint inter se inæquales & eorum bini pluresve æquales. Quos varios casus ut ordine percurramus, ponamus primum omnes denominatoris Factores simplices cum reales esse tum inter se inæquales. Sint ergo Factores simplices denominatoris omnes $(1 - pz)(1 - qz)(1 - rz)(1 - sz) \&c.$, ex quibus fractio proposita in sequentes fractiones simplices resolvatur $\frac{A}{1 - pz} + \frac{B}{1 - qz} + \frac{C}{1 - rz} + \frac{D}{1 - sz} + \&c.$. Quibus cognitis erit Seriei recurrentis terminus generalis $= z^n (Ap^n + Bq^n + Cr^n + Ds^n + \&c.)$, quem statuamus $= Pz^n$; sit scilicet P coëfficiens Potestatis z^n , sequentiumque $Q, R, \&c.$, ita ut Series recurrens fiat $A + Bz + Cz^2 + Dz^3 + \dots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.$

335. Ponamus jam n esse numerum maximum, seu Seriem recurrentem ad plurimos terminos esse continuatam; quoniam numerorum inæqualium Potestates eo magis fiunt inæquales, quo fuerint altiores; tanta erit diversitas in Potestatibus

LIB. I. $A p^n, B q^n, C r^n, \&c.$, ut ea, quæ oritur ex maximo numerorum $p, q, r, \&c.$, reliquas magnitudine longe superet, præ eaque reliquæ penitus evanescant, si n fuerit numerus plane infinite magnus. Cum igitur numeri $p, q, r, \&c.$, sint inter se inæquales, ponamus inter eos p esse maximum; ac propterea, si n sit numerus infinitus, fiet $P = A p^n$; sin autem n sit numerus vehementer magnus erit tantum proxime $P = A p^n$.

Simili vero modo erit $Q = A p^{n+1}$, ideoque $\frac{Q}{P} = p$. Unde patet, si Series recurrens jam longe fuerit producta, coëfficientem cujusque termini per præcedentem divisum proxime esse exhibiturum valorem maximæ litteræ p .

336. Si igitur in fractione proposita

$$\frac{a + b z + c z^2 + d z^3 + \&c.}{1 - a z - b z^2 - c z^3 - d z^4 - \&c.}$$

denominator habeat omnes Factores simplices reales & inter se inæquales, ex Serie recurrente inde orta cognosci poterit unus Factor simplex, is scilicet $1 - p z$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coëfficientes numeratoris $a, b, c, d, \&c.$, in computum ingrediuntur, sed quicunque ii statuuntur, tamen denique idem verus valor litteræ maximæ p invenitur. Verus quidem valor ipsius p tum demum innotescit, quando Series in infinitum fuerit continuata; interim tamen si jam plures ejus termini fuerint formati, eo propius valor ipsius p cognoscetur, quo major fuerit terminorum numerus, & quo magis littera ista p excedat reliquas $q, r, s, \&c.$: perinde vero est utrum hæc maxima littera p fuerit signo $+$ an signo $-$ affecta, quoniam ejus Potestates æque crescunt.

337. Quemadmodum nunc hæc investigatio ad inventionem radicum æquationis cujusvis algebraicæ accommodari possit,

fit, satis est perspicuum. Ex Factoribus enim denominatoris ^{CAP.}
 $1 - az - bz^2 - \gamma z^3 - \delta z^4 - \&c.$, cognitis facile XVII.
 assignantur radices æquationis hujus

$$1 - az - bz^2 - \gamma z^3 - \delta z^4 - \&c. = 0,$$

ita ut, si Factor fuerit $1 - pz$, hujus æquationis radix una futura sit $z = \frac{1}{p}$. Cum igitur ex Serie recurrente reperia-
 tur maximus numerus p , indidem obtinebitur minima radix
 æquationis $1 - az - bz^2 - \gamma z^3 - \&c. = 0$. Vel, si
 ponatur $z = \frac{1}{x}$ ut prodeat hæc æquatio

$$x^m - ax^{m-1} - bx^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

ejusdem methodi ope eruitur maxima hujus æquationis radix
 $x = p$.

338. Si igitur proponatur æquatio hæc

$$x^m - ax^{m-1} - bx^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

quæ omnes radices habeat reales & inter se inæquales, harum
 radicum maxima sequenti modo reperietur. Formetur ex coëf-
 ficientibus hujus æquationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - az - bz^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Hincque formetur Series recurrens, assumendo pro arbitrio
 numeratorem, seu, quod eodem redit, assumendo pro libitu
 terminos initiales; quæ sit

$$A + Bz + Cz^2 + Dz^3 + \dots + Pz^n + Qz^{n+1}$$

dabitque fractio $\frac{Q}{P}$ valorem radice maximæ x pro æquatione
 proposita, eo propius, quo major fuerit numerus n .

EXEM-

EXEMPLUM I.

Sit proposita ista æquatio $xx - 3x - 1 = 0$, cujus maximam radicem inveniri oporteat.

Formetur fractio $\frac{a + bz}{1 - 3z - zz}$, unde positis duobus primis terminis 1, 2, orietur ista Series recurrens

1, 2, 7, 23, 76, 251, 829, 2738, &c.,

erit ergo $\frac{2738}{829}$ proxime æqualis radici æquationis propositæ maximæ. Valor autem hujus fractionis in partibus decimalibus expressus est

3, 3027744

æquationis vero radix maxima est $= \frac{3 + \sqrt{13}}{2} =$

3, 3027756,

quæ inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse majores & minores.

EXEMPLUM II.

Proposita sit ista æquatio $3x - 4x^3 = \frac{1}{2}$ cujus radices exhibent Sinus trium Arcuum, quorum triplorum Sinus est $= \frac{1}{2}$.

Æquatione perducta ad hanc formam $0 = 1 - 6x + 8x^3$, quærat hujus, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{2}$. Formetur ergo hæc fractio

$$\frac{a + bx + cxx}{1 - 6x + 8x^3}$$

ex qua sumendis pro lubitu tribus terminis initialibus 0, 0, 1, quia

quia hoc modo calculus facillime expeditur, orietur hæc Series recurrens, omittendis potestatibus ipsius x quia tantum CAP. XVII.
coëfficientibus opus est,

0; 0; 1; 6; 36; 208; 1200; 6912; 39808; 229248.

Erit ergo proxime æquationis radix minima $= \frac{39808}{229248} =$

$\frac{311}{1791} = 0,1736515$, quæ propterea esse deberet Sinus anguli 10° ; hic autem ex tabulis est $0,1736482$, quem superat radix inventa parte $\frac{33}{10000000}$. Facilius autem hæc eadem radix in-

veniri potest ponendo $x = \frac{1}{2}y$, ut prodeat æquatio $1 - 3y * + y^3 = 0$, ex qua simili modo tractata oritur Series

0, 0, 1, 3, 9, 26, 75, 216, 622, 1791, 5157 &c.,

erit ergo proxime æquationis radix minima $y = \frac{1791}{5157} =$

$\frac{199}{573} = 0,3472949$, unde fit $x = \frac{1}{2}y = 0,1736479$, qui valor decies propius accedit quam præcedens.

EXEMPLUM III.

*Si desideretur ejusdem æquationis proposita $0 = 1 - 6x * + 8x^3$, radix maxima.*

Ponatur $x = \frac{y}{2}$, eritque $y^3 * - 3y + 1 = 0$. Cujus æquationis radix maxima reperietur per Seriem recurrentem cujus scala relationis est 0, 3, — 1, unde ergo oritur, sumtis tribus terminis initialibus pro arbitrio,

1, 1, 1, 2, 2, 5, 4, 13, 7, 35, 8, 98, — 11, &c.,

in qua Serie cum ad terminos negativos perveniatur, id indicio est maximam radicem esse negativam, est enim $x = -$

Euleri *Introduct. in Anal. infin. parv.*

N n

fin.

LIB. I. $\sin. 70^\circ = -0,9396926$. Quare hujus ratio in terminis
 ——— initialibus est habenda, hoc modo

$$1 - 2 + 4 - 7 + 14 - 25 + 49 - 89 + 172 - 316 + 605 - \&c.,$$

ex qua erit $y = \frac{-605}{316}$ & $x = \frac{-605}{632} = -0,957$, quæ a
 veritate vehementer abludit.

339. Ratio hujus dissensus potissimum est, quod æquationis propositæ radices sint $\sin. 10^\circ$, $\sin. 50^\circ$, & $-\sin. 70^\circ$, quarum binæ maximæ tam parum a se invicem discrepant, ut in Potestatibus, ad quas Seriem continuavimus, secunda radix $\sin. 50^\circ$ adhuc notabilem teneat rationem ad radicem maximam, ideoque præ ea non evanescant. Hincque etiam saltus pendet, quod alternatim valores inventi fiant nimis magni & nimis parvi: Sic, sumendo

$$y = \frac{-316}{172}, \text{ fit } x = \frac{-158}{172} = \frac{-79}{86} = -0,918.$$

Nam, quoniam Potestates radices maximæ alternatim fiunt affirmativæ & negativæ, alternatim quoque Potestates secundæ radices adduntur & tolluntur: quamobrem, quo hæc discrepantia fiat insensibilis, Series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri potest, transmutando æquationem ope idoneæ substitutionis in aliam formam, cujus radices sibi non amplius sint tam vicinæ. Sic, si in æquatione $0 = 1 - 6x + 8x^3$ cujus radices sunt $-\sin. 70^\circ$, $+\sin. 50^\circ$, $+\sin. 10^\circ$, ponatur $x = y - 1$, æquationis $0 = 8y^3 - 24yy + 18y - 1$ radices erunt $1 - \sin. 70^\circ$; $1 + \sin. 50^\circ$; $1 + \sin. 10^\circ$; ideoque ejus radix minima erit $1 - \sin. 70^\circ$, cum tamen hæc $\sin. 70^\circ$ esset radix maxima æquationis præcedentis; atque $1 + \sin. 50^\circ$ nunc est radix maxima, cum $\sin. 50^\circ$ ante esset media. Atque hoc modo quævis radix per substitutionem in maximam minimamve radicem novæ æquationis transmutari, ideoque per methodum hic traditam inveniri poterit.

poterit. Quia præterea in hoc exemplo radix $1 - \sin. 70^\circ$ CAP. XVII.
multo minor est, quam binæ reliquæ, etiam facile per Seriem
recurrentem proxime cognoscetur.

EXEMPLUM IV.

Invenire radicem minimam æquationis $0 = 8y^3 - 24yy + 8y - 1$, *quæ ab unitate subtracta relinquet Sinum anguli* 70° .

Ponatur $y = \frac{1}{2}z$, ut sit $0 = z^3 - 6zz + 9z - 1$, cujus
radix minima invenietur per Seriem recurrentem, cujus scala
relationis est $9, -6, +1$, pro radice autem maxima inve-
nienda scala relationis sumi deberet $6, -9, +1$. Pro mi-
nima ergo formetur hæc Series

1, 1, 1, 4, 31, 256, 2122, 17593; 145861; &c.,

erit ergo proxime $z = \frac{17593}{145861} = 0,12061483$ & $y =$
 $0,06030741$, atque $\sin. 70^\circ = 1 - y = 0,93969258$, quæ
a veritate ne in ultima quidem figura discrepat. Ex hoc ergo
exemplo intelligitur quantam utilitatem idonea transformatio
æquationis ope substitutionis ad inventionem radicum afferat,
& quod hoc pacto methodus tradita non solum ad maximas
minimasve radices adstringatur, sed etiam omnes radices exhi-
bere queat.

341. Cognita ergo jam quacunque æquationis propositæ
radice proxime, ita ut, verbi gratia, numerus k quam mini-
me a quapiam radice differat, ponatur $x - k = y$ seu $x =$
 $y + k$, hocque modo prodibit æquatio, cujus radix minima
erit $= x - k$, quæ igitur si per Series recurrentes indage-
tur, quod facillime fiet, quia hæc radix multo minor erit,
quam ceteræ, si ea ad k addatur habebitur radix vera ipsius
 x , pro æquatione proposita. Hoc vero artificium tam late
patet, ut etiam si æquatio contineat radices imaginarias, usum
suum retineat.

342. Imprimis autem sine hoc artificio radix cognosci ne-
quit.

LIB. I. quit, cui datur alia æqualis sed signo contrario affecta. Scilicet, si æquatio cuius maxima radix p , eadem radicem habeat $-p$, tum, etiam si Series recurrens in infinitum continetur, tamen radix hæc p nunquam obtinebitur. Sit, ut hoc exemplo illustremus, proposita æquatio $x^3 - x^2 - 5x + 5 = 0$, cuius maxima radix est $\sqrt{5}$, præter quam vero inest quoque $-\sqrt{5}$. Si igitur modo ante præscripto, pro radice maxima invenienda, utamur, atque Seriem recurrentem formemus ex scala relationis $1, +5, -5$, quæ erit

$1, 2, 3, 8, 13, 38, 63, 188, 313, 938, 1563, \&c.,$

ubi ad nullam rationem constantem pervenitur. Termini vero alterni rationem æquabilem induunt, quorum si quisque per præcedentem dividatur, reperietur quadratum maximæ radice, sic enim est proxime $5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}$. Quoties ergo termini tantum alterni sese ad rationem constantem componunt, toties quadratum radice quæsitæ proxime obtinetur. Ipsa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$ unde fit $1 - 3y - 5y^2 - y^3 = 0$, cuius radix minima cognoscetur ex Serie

$1, 1, 1, 9, 33, 145, 609, 2585, 10945, \&c.,$

erit enim proxime $= \frac{2585}{10945} = 0,2361$, at $2,2361$ est proxime $= \sqrt{5}$, quæ est radix maxima æquationis.

343. Quanquam numerator fractionis, ex qua Series recurrens formatur, a nostro arbitrio pendet, tamen idonea ejus constitutio plurimum confert, ut valor radice cito vero proxime exhibeatur. Cum enim assumtis, ut supra, Factoribus denominatoris (334.), fit terminus generalis Seriei recurrentis $= z^n (Ap^n + Bq^n + Cr^n + \&c.)$, isti coëfficientes $A, B, C, \&c.$, per numeratorem fractionis determinantur; unde fieri potest, ut A five magnum five parvum valorem obtineat: priori casu radix maxima p cito reperitur, posteriore vero tarde. Quin etiam numerator ita accipi potest ut A prorsus evanescat, quo

quo casu, etiamsi Series in infinitum continetur, tamen nunquam radicem maximam p præbebit. Hoc autem evenit si numerator ita accipiatur, ut ipse eundem habeat Factorem $1 - pz$, sic enim ex computo penitus tolletur. Sic, si proponatur æquatio $x^3 - 6xx + 10x - 3 = 0$, cujus maxima radix est $= 3$, indeque formetur fractio

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3}$$

ut Seriei recurrentis sit scala relationis $6, - 10, + 3$

$$1 + 3, 8, 21, 55, 144, 377, \&c.,$$

cujus termini prorsus non convergunt ad rationem, $1 : 3$. Eadem enim Series oritur ex fractione $\frac{1}{1 - 3z + z^2}$, ac propterea maximam radicem æquationis $x^2 - 3x + 1 = 0$ exhibet.

344. Quin etiam numerator ita assumi potest, ut per Seriem recurrentem quavis radix æquationis reperiatur, quod fiet si numerator fuerit productum ex omnibus Factoribus denominatoris præter eum, cui responderet radix quam velimus. Sic, si in priori exemplo sumatur numerator $1 - 3z + 2z^2$, fractio

$$\frac{1 - 3z + 2z^2}{1 - 6z + 10z^2 - 3z^3},$$

dabit hanc Seriem recurrentem $1, 3, 9, 27, 81, 243, \&c.$, quæ, cum sit geometrica, statim monstrat radicem $x = 3$. Fractio enim illa æqualis est huic simplici

$\frac{1}{1 - 3z}$. Hinc apparet, si termini initiales, quos pro lubitu assumere licet, ita accipiantur, ut progressionem geometricam constituent, cujus Exponens æquetur uni radici æquationis, tum totam Seriem recurrentem fore geometricam, ideoque eam ipsam radicem esse exhibituram, etiamsi neque sit maxima neque minima.

345. Ne igitur, dum quærimus radicem vel maximam vel minimam, præter expectationem nobis alia radix per Seriem recurrentem exhibeatur, ejusmodi numerator debet eligi, qui

N n 3 cum

LIB. I. cum denominatore nullum Factorem habeat communem, quod fiet si pro numeratore unitas accipiat, unde terminus primus Seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix æquationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic, proposita æquatione $y^3 * - 3y + 1 = 0$, cujus radix maxima desideratur, ex scala relationis $0, + 3, - 1$ incipiendo ab unitate sequens oritur Series recurrens

$$1 - 0 + 3 - 1 + 9 - 6 + 28 - 27 + 90 - 109 + 297 \\ - 517 + 1000 - 1848 + 3517 - 6544 + \&c.,$$

quæ manifesto ad rationem constantem convergit, ostenditque radicem maximam esse negativam, atque proxime $y = \frac{-6544}{3517} = -1,860676$, quæ esse debebat $= -1,86793852$. Ratio autem supra est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima, simulque sit affirmativa.

346. His probe perpensis, quæ cum in genere tum ad exempla allata monuimus, summa utilitas hujus methodi ad investigandas æquationum radices luculenter perspicietur; artificia vero, quibus operatio contrahi, eoque promptior reddi queat, satis quoque sunt indicata; ita ut nihil insuper addendum esset, nisi casus, quibus æquatio vel radices habet æquales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - az - bz^2 - \gamma z^3 - d^2 z^4 - \&c.}$$

habere Factorem $(1 - pz)^2$, reliquis Factoribus existentibus $1 - qz$, $1 - rz$, $\&c.$, Seriei ergo recurrentis hinc natae terminus generalis erit $= z^n (n+1)Ap^n + Bp^n + Cq^n + \&c.)$, quæ cujusmodi valorem sit adeptura, si n fuerit numerus vehemen-

vehementer magnus, duo casus sunt distinguendi, alter quo p CAP. est numerus major reliquis $q, r, \&c.$, alter quo p non præbet XVII. radicem maximam. Casu priori, quo p simul est radix maxima, — ob coëfficientem $(n+1)$ reliqui termini $Bp^n + Cq^n \&c.$, non tam cito præ eo evanescunt, quam ante: fin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)Ap^n$ præ Bq^n evanescet, ideoque investigatio radice maximæ admodum evadet molesta.

EXEMPLUM I.

Sit proposita æquatio $x - 3xx + 4 = 0$, cujus maxima radix 2 bis occurrit.

Quærat ergo maxima radix hæc modo ante exposito per evolutionem fractionis

$$\frac{1}{1 - 3z + 4z^2}$$

quæ dabit hanc Seriem recurrentem

1, 3, 9, 23, 57, 135, 313, 711, 1593, &c.,

ubi quidem quivis terminus per præcedentem divisus dat quotientum binario majorem. Cujus ratio ex termino generali facillime patet, rejectis enim in eo terminis $Bp^n, Cq^n \&c.$, erit terminus potestati z^n respondens $= (n+1)Ap^n + Bp^n$, sequens $= (n+2)Ap^{n+1} + Bp^{n+1}$, qui per illum divisus dat $\frac{(n+2)A+B}{(n+1)A+B}p > p$, nisi n jam in infinitum excreverit.

EXEM-

E X E M P L U M II.

Sit jam proposita æquatio $x^3 - xx - 5x - 3 = 0$, cujus maxima radix $= 3$, reliquæ duæ æquales $= -1$, & quaeratur maxima radix ope Seriei recurrentis, cujus scala relationis est $1, +5, +3$; unde oritur

$$1, 1, 6, 14, 47, 135, 412, 1228, \&c.,$$

quæ ideo satis cito valorem 3 exhibet, quod Potestates minoris radices -1 , etiam si multiplicentur per $n+1$, tamen mox præ Potestatibus ipsius 3 evanescant.

E X E M P L U M III.

Sin autem proponeretur æquatio $x^3 + xx - 8x - 12 = 0$, cujus radices sunt $3, -2, -2$, multo tardius maxima sese prodet. Orietur enim hæc Series

$$1, -1, 9, -5, 65, 3, 457, 347, 3345, 4915, \&c.,$$

quæ adhuc longissime continuari deberet, antequam pateret, radicem inde oriundam esse $= 3$.

347. Simili modo si tres Factores essent æquales, ita ut denominatoris Factor unus esset $(1 - pz)^3$, reliqui $1 - qz$, $1 - rz$, &c., Seriei recurrentis terminus generalis erit $= z^n \left(\frac{(n+1)(n+2)}{1 \cdot 2} Ap^n + (n+1)Bp^n + Cp^n + Dq^n + Cr^n \&c. \right)$ Si ergo p fuerit maxima radix, atque n fuerit numerus tantus, ut Potestates q^n, r^n &c. præ p^n evanescant, tum ex Serie recurrente orietur radix $=$

$$\frac{\frac{1}{2}(n+2)(n+3)A + (n+2)B + C}{\frac{1}{2}(n+1)(n+2)A + (n+1)B + C} p,$$

quæ, nisi sit n numerus maximus & quasi infinitus, verum ipsius

fius p valorem indicabit. Erit autem iste radicis valor $= p + \frac{C A P.}{XVII.}$

$$\frac{(n+2)A+B}{\frac{1}{2}(n+1)(n+2)A+(n+1)B+C} p.$$

Quod si autem p non fuerit radix maxima, tum inventio maxime multo magis adhuc impiedetur; unde sequitur æquationes, quæ contineant radices æquales, hac methodo per Series recurrentes multo difficilius resolvi, quam si omnes radices essent inter se inæquales.

348. Videamus nunc quomodo Series recurrens in infinitum continuata debeat esse comparata, quando denominator fractionis habet Factores imaginarios. Sint igitur fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - az - bz^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Factores denominatoris reales $1 - qz$, $1 - rz$, &c., insuperque Factor trinomialis $1 - 2pz \cos \phi + ppzz$ continens duos Factores simplices imaginarios. Quod si ergo Series recurrens ex illa fractione orta fuerit

$$A + Bz + Cz^2 + Dz^3 + \dots + Pz^n + Qz^{n+1},$$

erit, per ea quæ supra exposuimus, coëfficiens $P =$

$$\frac{A \sin(n+1)\phi + B \sin n\phi}{\sin \phi} p^n + C q^n + D r^n + \&c..$$

Si igitur numerus p minor fuerit, quam unus ceterorum q , r , &c., ita ut maxima radix æquationis

$$x^m - ax^{m-1} - bx^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

fit realis, tum ea per Series recurrentes æque reperietur, ac si nullæ radices inessent imaginariæ.

349. Inventio ergo maximæ radicis realis per radices imaginarias non perturbabitur, si hæ ita fuerint comparatæ, ut binarum, quæ Factorem realem componunt, productum non sit

Euleri *Introduct. in Anal. infin. parv.* O o magis

LIB. I. majus quadrato radice maximæ. Sin autem binæ ejusmodi insint radices imaginariæ, ut earum productum adæquet vel adeo superet quadratum maximæ radice realis, tum investigatio ante exposita nihil declarabit, propterea quod Potestas p^n , præ simili Potestate radice maximæ nunquam evanescit, etiamsi Series in infinitum continetur. Cujus exempla illustrationis causa hic adjicere visum est.

E X E M P L U M I.

Sit proposita æquatio $x^3 - 2x - 4 = 0$, cujus radicem maximam investigari oporteat.

Resolvitur hæc æquatio in duos Factores $(x - 2)(xx + 2x + 2)$; unde unam habet radicem realem 2 & duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radice realis. Quam ob rem ea per modum hætenus traditum cognosci poterit. Formetur ergo Series recurrens ex scala relationis 0, + 2, + 4, quæ erit

1, 0, 2, 4, 4, 16, 24, 48, 112, 192, 416, 832, &c.,

unde satis luculenter radix realis 2 cognosci potest.

E X E M P L U M II.

Proposita sit æquatio $x^3 - 4xx + 8x - 8 = 0$, cujus radix una realis est 2, binarum imaginariarum productum vero = 4, ideoque æquale quadrato radice realis 2.

Quæramus ergo radicem per Seriem recurrentem, quod quo facilius fieri queat, ponamus $x = 2y$, ut habeatur $y^3 - 2yy + 2y - 1 = 0$, unde formetur Series recurrens

1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, 1, 2, 2, 1, &c.,

in qua cum iidem termini perpetuo revertantur, nihil inde aliud

aliud colligi potest, nisi radicem maximam vel non esse rea- CAP.
lem, vel dari imaginarias, quarum productum æquale sit aut su- XVII.
peret quadratum radicis realis.

E X E M P L U M I I I.

Sit jam proposita æquatio $x^3 - 3xx + 4x - 2 = 0$, cujus
radix realis est 1, imaginariarum vero productum = 2.

Formetur ergo ex scala relationis 3, — 4, + 2, Series

1, 3, 5, 5, 1, — 7, — 15, — 15, —, + 1, 33, 65, 65, 1, &c.,

in qua cum termini modo fiant affirmativi, modo negativi,
radix realis 1 inde nullo modo cognosci poterit. Hujusmodi
vero revolutiones semper ostendunt radicem, quam Series præ-
bere debebat, esse imaginariam; hic enim radices imaginariæ
potestate sunt majores quam realis 1.

350. Sit igitur in fractione generali productum binarum ra-
dicum imaginariarum pp majus quam ullius radicis realis qua-
dratum, ita ut præ p^n reliquæ potestates q^n, r^n , &c., eva-
nescant si n sit numerus infinitus. Hoc ergo casu fiet $P =$

$$\frac{A.\sin.(n+1)\phi + B.\sin.n\phi}{\sin.\phi} p^n, \text{ \& } Q = \frac{A.\sin.(n+2)\phi + B.\sin.(n+1)\phi}{\sin.\phi} p^{n+1}$$

ideoque $\frac{Q}{P} = \frac{A.\sin.(n+2)\phi + B.\sin.(n+1)\phi}{A.\sin.(n+1)\phi + B.\sin.n\phi} p$. Quæ ex-

expressio nunquam valorem constantem induet, etiam si n sit
numerus infinitus. Sinus enim Angulorum perpetuo maxime
manent mutabiles, ita ut mox sint affirmativi mox negativi.

351. Interim tamen si fractiones sequentes $\frac{R}{Q}, \frac{S}{R}$ simili
modo sumantur, indeque litteræ A & B eliminantur, simul nu-
merus n ex calculo egredietur; reperietur enim $Ppp + R =$

$$2Qp.\cos.\phi, \text{ unde fit } \cos.\phi = \frac{Ppp + R}{2Qp}; \text{ similiter vero erit}$$

$$002 \qquad \cos.\phi =$$

LIB. I. $\cos. \phi = \frac{Qpp + S}{2Rp}$, ex quorum duorum valorum compa-

ratione fit $p = \sqrt{\frac{RR - QS}{QQ - PR}}$, atque $\cos. \phi =$

$\frac{QR - PS}{2\sqrt{(Q^2 - PR)(R^2 - QS)}}$. Quam ob rem si Series recurrens

jam eo usque fuerit continuata, ut præ p^n reliquarum radicum Potestates evanescant, tum hoc modo Factor trinomialis $1 - 2pz.\cos.\phi + ppz^2$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ in-

vento oritur $AP.p.\sin.(n+2)\phi + BPp.\sin.(n+1)\phi = AQ.\sin.(n+1)\phi + BQ.\sin.n\phi$, unde fit $\frac{A}{B} =$

$\frac{Q.\sin.n\phi - Pp.\sin.(n+1)\phi}{Pp.\sin.(n+2)\phi - Q.\sin.(n+1)\phi}$. Pari ratione erit $\frac{A}{B} =$

$\frac{R.\sin.(n+1)\phi - Qp.\sin.(n+2)\phi}{Qp.\sin.(n+3)\phi - R.\sin.(n+2)\phi}$; æquatis his duobus valoribus fiet

$0 = QQp.\sin.n\phi.\sin.(n+3)\phi - QR.\sin.n\phi.\sin.(n+2)\phi - PQpp.\sin.(n+1)\phi.\sin.(n+3)\phi - QQp.\sin.(n+1)\phi.\sin.(n+2)\phi + QR.\sin.(n+1)\phi.\sin.(n+1)\phi + PQpp.\sin.(n+1)\phi.\sin.(n+2)\phi.$

Cum autem sit $\sin.a.\sin.b = \frac{1}{2}.\cos.(a-b) - \frac{1}{2}.\cos.(a+b)$

fiet $0 = \frac{1}{2} QQp.(\cos.3\phi - \cos.\phi) + \frac{1}{2} QR.(1 - \cos.2\phi) +$

$\frac{1}{2} PQpp.(1 - \cos.2\phi)$ quæ per $\frac{1}{2} Q$ divisa dat

$(Ppp + R)(1 - \cos.2\phi) = Qp.(\cos.\phi - \cos.3\phi)$. At est

$\cos.\phi = \cos.2\phi.\cos.\phi + \sin.2\phi.\sin.\phi$ & $\cos.3\phi = \cos.2\phi.\cos.\phi - \sin.2\phi.\sin.\phi$ unde $\cos.\phi - \cos.3\phi = 2\sin.2\phi.\sin.\phi = 4\sin.\phi^2 \times \cos.\phi$ & $1 - \cos.\phi = 2\sin.\phi^2$, ex quo erit $Ppp + R =$

$2Qp.\cos.\phi$, & $\cos.\phi = \frac{Ppp + R}{2Qp}$, atque $\cos.\phi = \frac{Qpp + S}{2Rp}$: unde
superiores

superiores valores prodeunt, scilicet $p = \sqrt{\frac{RR - QS}{QQ - PR}}$ & $\cos. \phi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(RR - QS)}}$ CAP. XVII.

353. Si denominator fractionis, ex qua Series recurrens formatur, plures habeat Factores trinomiales inter se æquales, tum, spectata forma termini generalis supra data, patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quæcunque radix realis jam proxime fuerit detecta, tum æquationis transformatione semper valor ejusdem radice multo propior eruetur. Ponatur enim x æqualis valori illi jam detecto $+y$, atque novæ æquationis quærat minima radix pro y , quæ addita ad illum valorem præbebit verum ipsius x valorem.

EXEMPLUM.

Sit proposita ista æquatio $x^3 - 3xx + 5x - 4 = 0$, cujus unam radicem fere esse $= 1$ inde constat, quod, posito $x = 1$, prodit $x^3 - 3xx + 5x - 4 = -1$.

Ponatur ergo $x = 1 + y$, fietque $1 - 2y - y^3 = 0$, unde pro radice minima invenienda formetur Series recurrens, cujus scala relationis 2, 0, +1, quæ erit

1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296, &c.,

unde radix minima ipsius y erit proxime $\frac{1041}{2296} = 0,453397$, ita ut sit $x = 1,453397$, qui valor tam prope vix alia methodo æque facile obtineri poterit.

354. Quod si autem Series quæcunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit, cujusnam æquationis radix sit futura quotus qui ex divisione unius termini per præcedentem oritur. Sint

O O 3

P, Q,

termini Seriei recurrentis a principio jam longissime remoti, ita ut cum progressionem geometricam confundantur; sitque $T = \alpha S + \epsilon R + \gamma Q + \delta P$, seu scala relationis $\alpha, +\epsilon, +\gamma, +\delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit $\frac{R}{P} = xx$; $\frac{S}{P} = x^3$ & $\frac{T}{P} = x^4$, qui in superiori æquatione substituti dabunt

$$x^4 = \alpha x^3 + \epsilon x^2 + \gamma x + \delta.$$

unde patet quotum $\frac{Q}{P}$ tandem præbere radicem unam æquationis inventæ. Hoc vero & præcedens methodus indicat, præterea vero docet fractionem $\frac{Q}{P}$ dare maximam æquationis radicem.

355. Potest quoque hæc methodus investigandarum radicum sæpenumero utiliter adhiberi, si æquatio sit infinita. Ad quod ostendendum proposita sit æquatio $\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \&c.$, cujus radix minima z exhibet Arcum 30° , seu Semiperipheriæ Circuli sextantem. Perducatur ergo æquatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \&c. = 0.$$

Hinc ergo formetur Series recurrens, cujus scala relationis est infinita, scilicet

$$2, 0, -\frac{1}{3}, 0, +\frac{1}{60}, 0, -\frac{1}{2520}, 0 \&c.,$$

eritque Series recurrens

$$1, 2, 4, \frac{23}{3}, \frac{44}{3}, \frac{1681}{60}, \frac{2408}{45}, \&c.,$$

erit

erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1681.3}{2408.4} = \frac{5043}{9632} = 0,52356$: CAP. XVII

At ex proportione Peripheriæ ad Diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$

a vero discrepet. Hoc autem in hac æquatione commode usu venit, quod ejus omnes radices sint reales, atque a minima reliquæ satis notabiliter discrepent. Quæ conditio cum rarissime in æquationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

CAPUT XVIII.

De fractionibus continuis.

356. **Q**Uoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis differui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est ex-cultum, tamen non dubitamus, quin ex eo amplissimus usus in analyfin infinitorum aliquando sit redundaturus. Exhibui enim jam aliquoties ejusmodi specimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipsam Arithmetica & Algebram communem non contemnenda subsidia affert ista speculatio, quæ hoc Capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco ejusmodi fractionem, cujus denominator constat ex numero integro cum fractione, cujus denominator denuo est aggregatum ex integro & fractione, quæ porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Hujusmodi ergo fractio continua erit sequens expressio

$a +$

LIB. I.

$$\text{---} a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f + \dots}}}}} \text{ vel } a + \frac{a}{b + \frac{c}{c + \frac{d}{d + \frac{e}{e + \frac{f}{f + \dots}}}}} \&c.,$$

in quarum forma priori omnes fractionum numeratores sunt unitates, quam potissimum hic contemplabor, in altera vero forma sunt numeratores numeri quicunque.

358. Expofita ergo fractionum harum continuarum forma, primum videndum est, quemadmodum earum significatio confueto more expreffa inveniri queat. Quæ ut facilius inveniri poffit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in fecunda, poft in tertia & ita porro fractione; quo factò patebit fore

$$\begin{aligned} a &= a \\ a + \frac{1}{b} &= \frac{ab + 1}{b} \\ a + \frac{1}{b + \frac{1}{c}} &= \frac{abc + a + c}{bc + 1} \\ a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}} &= \frac{abcd + ab + ad + cd + 1}{bcd + b + d} \\ a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e}}}} &= \frac{abcde + abe + ade + cde + abc + a + c + e}{bcde + be + de + bc + 1} \\ &\&c. \end{aligned}$$

359. Etſi in his fractionibus ordinariis non facile lex, ſecundum quam numerator ac denominator ex litteris $a, b, c, d, \&c.$, componantur, perſpicitur, tamen attendenti ſtatim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator eſt aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex numeratore

meratore penultimo simplici: eademque lex in denominatoribus observatur. Scriptis ergo ordine litteris $a, b, c, d, \&c.$, CAP. XVIII.
ex iis fractiones inventæ facile formabuntur hoc modo

$$\frac{a}{1}; \frac{b}{1}; \frac{c}{b+1}; \frac{d}{bc+1}; \frac{e}{bcd+b+d+1}$$

ubi quilibet numerator invenitur, si præcedentium ultimus per indicem supra scriptum multiplicetur atque ad productum antepenultimus addatur; quæ eadem lex pro denominatoribus valet. Quo autem hac lege ab ipso initio uti liceat, præfixi fractionem $\frac{1}{0}$ quæ, etiamsi e fractione continua non oriatur, tamen progressionis legem clariorem efficit. Quælibet autem fractio exhibet valorem fractionis continuæ usque ad eam litteram, quæ antecedenti imminet, inclusive continuata.

360. Simili modo altera fractionum continuarum forma

$$a + \frac{\alpha}{b + \frac{\epsilon}{c + \frac{\gamma}{d + \frac{\delta}{e + \frac{f}{\&c.}}}}}$$

dabit, prout aliis aliisque locis abrumpitur, sequentes valores

$$\begin{aligned} a &= a \\ a + \frac{\alpha}{b} &= \frac{ab + \alpha}{b} \\ a + \frac{\alpha}{b + \frac{\epsilon}{c}} &= \frac{abc + \epsilon a + \alpha c}{bc + \epsilon} \\ a + \frac{\alpha}{b + \frac{\epsilon}{c + \frac{\gamma}{d}}} &= \frac{abcd + \epsilon ad + \alpha cd + \gamma ab + \alpha \gamma}{bcd + \epsilon d + \gamma b} \\ &\&c., \end{aligned}$$

LIB. I. quarum fractionum quæque ex binis præcedentibus sequentem in modum invenietur

$$\frac{a}{\alpha} ; \frac{b}{\epsilon} ; \frac{c}{\gamma} ; \frac{d}{\delta} ; \frac{e}{\epsilon} ; \frac{abc + \epsilon a + \alpha c}{b c + \epsilon} ; \frac{abcd + \epsilon ad + \alpha cd + \gamma ab + \alpha \gamma}{b c d + \epsilon d + \gamma b}$$

361. Fractionibus scilicet formandis supra inscribantur indices $a, b, c, d, \&c.$, infra autem subscribantur indices $\alpha, \epsilon, \gamma, \delta, \&c.$. Prima fractio iterum constituatur $\frac{1}{0}$. secunda

$\frac{a}{1}$, tum sequentium quævis formabitur si antecedentium ultimæ numerator per indicem supra scriptum, penultimæ vero numerator per indicem infra scriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis sequentis: simili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum, & ex penultimo denominatore per indicem infra scriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatæ inclusive.

362. Quod si ergo hæ fractiones eousque continuentur quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractionis vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuæ

$$a + \frac{\alpha}{b + \frac{\epsilon}{c + \frac{\gamma}{d + \frac{\epsilon}{e + \&c.}}}} \quad \text{esse} = x$$

atque manifestum est fractionem primam $\frac{1}{0}$ esse majorem quam

quam x ; secunda vero $\frac{a}{1}$ minor erit quam x ; tertia $a + \frac{a}{b}$ CAP. XVIII.
 iterum vero valore erit major; quarta denuo minor, atque
 ita porro hæ fractiones alternatim erunt majores & minores
 quam x . Porro autem perspicuum est quamlibet fractionem
 propius accedere ad verum valorem x quam ulla præceden-
 tium; unde hoc pacto citissime & commodissime valor ipsius x
 proxime obtinetur; etiam si fractio continua in infinitum pro-
 grediarur, dummodo numeratores $a, c, e, g, \&c.$, non nimis
 crescant; sin autem omnes isti numeratores fuerint unitates,
 tum appropinquatio nulli incommodo est obnoxia.

363. Quo ratio hujus appropinquationis ad verum fractio-
 nis continuæ valorem melius percipiatur, consideremus fractio-
 num inventarum differentias. Ac, prima quidem $\frac{1}{0}$ præter-
 missa, differentia inter secundam ac tertiam est $= \frac{a}{b}$; quar-
 ta a tertia subtracta relinquit $\frac{ac}{b(bc+c)}$; quarta a quinta sub-
 tracta relinquit $\frac{ace}{(bc+c)(bcd+cd+g)}$, &c.. Hinc ex-
 primetur valor fractionis continuæ per Seriem terminorum con-
 suetam hoc modo, ut sit

$$x = a + \frac{a}{b} - \frac{ac}{b(bc+c)} + \frac{ace}{(bc+c)(bcd+cd+gb)} - \&c.,$$

quæ Series toties abrumpitur quoties fractio continua non in
 infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quam-
 cunque in Seriem terminorum, quorum signa alternantur,
 convertendi, si quidem prima littera a evanescat. Si enim
 fuerit

LIB. I.

$$x = \frac{a}{b + \frac{c}{c + \frac{d}{d + \frac{e}{e + \frac{f}{f + \dots}}}}}$$

erit per ea, quæ modo invenimus,

$$x = \frac{a}{b} - \frac{a c}{b(b c + c)} + \frac{a c \gamma}{(b c + c)(b c d + c d + \gamma b)} - \frac{a c \gamma d}{(b c d + c d + \gamma b)(b c d e + c d e + \gamma b e + d b c + c d)} + \dots$$

Unde, si $a, c, \gamma, d, \&c.$ fuerint numeri non crescentes, uti omnes unitates, denominatores vero $a, b, c, d, \&c.$ numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe consideratis, poterit vicissim Series quæcunque terminorum alternantium in fractionem continuam converti, seu fractio continua inveniri cujus valor æqualis sit summæ Seriei propositæ. Sit enim proposita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, singulis terminis cum Serie ex fractione continua orta comparandis

$$\begin{aligned} A &= \frac{a}{b}; & \text{hincque } a &= A b, \\ \frac{B}{A} &= \frac{c}{b c + c}; & \text{unde fit } c &= \frac{B b c}{A - B} \\ \frac{C}{B} &= \frac{\gamma b}{b c d + c d + \gamma b}; & \gamma &= \frac{C d (b c + c)}{b (B - C)} \\ \frac{D}{C} &= \frac{d (b c + c)}{b c d e + c d e + \gamma b e + d b c + c d}; & d &= \frac{D e (b c d + c d + \gamma b)}{(b c + c) (C - D)} \\ & & & \&c.. \end{aligned}$$

$$\text{At, cum fit } c = \frac{B b c}{A - B}, \text{ erit } b c + c = \frac{A b c}{A - B}; \text{ unde } \gamma =$$

$$\gamma = \frac{ACcd}{(A-B)(B-C)}. \text{ Porro fit } bcd + \zeta d + \gamma b = \frac{CAP.}{XVIII.}$$

$$(bc + \zeta)d + \gamma b = \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)},$$

$$\text{unde erit } \frac{bcd + \zeta d + \gamma b}{bc + \zeta} = \frac{Bd}{B-C} \text{ \& } \delta = \frac{BDde}{(B-C)(C-D)};$$

$$\text{simili modo reperietur } \varepsilon = \frac{CEef}{(C-D)(D-E)} \text{ \& ita porro.}$$

366. Quo ista lex clarius appareat, ponamus esse

$$\begin{aligned} P &= b \\ Q &= bc + \zeta \\ R &= bcd + \zeta d + \gamma b \\ S &= bcde + \zeta de + \gamma be + \delta bc + \zeta \delta \\ T &= bcdef + \&c. \\ V &= bcdefg + \&c., \end{aligned}$$

erit ex lege harum expressionum

$$\begin{aligned} Q &= P\zeta + \zeta \\ R &= Qd + \gamma P \\ S &= Re + \delta Q \\ T &= Sf + \varepsilon R \\ V &= Tg + \xi S \\ &\&c.. \end{aligned}$$

Cum igitur his adhibendis litteris fit

$$x = \frac{a}{P} - \frac{a\zeta}{PQ} + \frac{a\zeta\gamma}{QR} - \frac{a\zeta\gamma\delta}{RS} + \frac{a\zeta\gamma\delta\varepsilon}{ST} - \&c.,$$

367. Quoniam ergo ponimus esse

$$x = A - B + C - D + E - F + \&c.,$$

erit

$$A = \frac{a}{P}; a = AP$$

P p 3

$\frac{B}{A}$

$$\begin{aligned}\frac{B}{A} &= \frac{\zeta}{Q}; & \zeta &= \frac{BQ}{A} \\ \frac{C}{B} &= \frac{\gamma P}{R}; & \gamma &= \frac{CR}{BP} \\ \frac{D}{C} &= \frac{\delta Q}{S}; & \delta &= \frac{DS}{CQ} \\ \frac{E}{D} &= \frac{\varepsilon R}{T}; & \varepsilon &= \frac{ET}{DR} \\ && & \&c. \qquad \&c.\end{aligned}$$

Porro vero differentiis sumendis habebitur

$$\begin{aligned}A - B &= \frac{\alpha(Q - \zeta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q} \\ B - C &= \frac{\alpha\zeta(R - \gamma P)}{PQR} = \frac{\alpha\zeta d}{PR} = \frac{BQd}{R} \\ C - D &= \frac{\alpha\zeta\gamma(S - \delta Q)}{QRS} = \frac{\alpha\zeta\gamma e}{QS} = \frac{CR\varepsilon}{S} \\ D - E &= \frac{\alpha\zeta\gamma\delta(T - \varepsilon R)}{RST} = \frac{\alpha\zeta\gamma\delta f}{RT} = \frac{DSf}{T}, \\ && \&c. \qquad \&c. \qquad \&c.\end{aligned}$$

Si bini igitur in se invicem ducantur, fiet

$$\begin{aligned}(A - B)(B - C) &= ABcd \cdot \frac{P}{R}; & \& \frac{R}{P} &= \frac{ABcd}{(A - B)(B - C)} \\ (B - C)(C - D) &= BCde \cdot \frac{Q}{S}; & \& \frac{S}{Q} &= \frac{BCed}{(B - C)(C - D)} \\ (C - D)(D - E) &= CDef \cdot \frac{R}{T}; & \& \frac{T}{R} &= \frac{CDef}{(C - D)(D - E)} \\ && \&c.\end{aligned}$$

Unde, cum fit $P = b$; $Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\begin{aligned}\alpha &= Ab \\ \zeta &= \frac{Bbc}{A - B} \\ \gamma &= \frac{ACcd}{(A - B)(B - C)} \\ && \delta =\end{aligned}$$

$$\begin{aligned} d &= \frac{B D d e}{(B - C)(C - D)} \\ e &= \frac{C E e f}{(C - D)(D - E)} \\ &\quad \&c.. \end{aligned}$$

368. Inventis ergo valoribus numerorum $a, b, c, d, e, \&c.$, denominatores $b, c, d, e, \&c.$, arbitrio nostro relinquuntur: ita autem eos assumi convenit, ut, cum ipsi sint numeri integri, tum valores integros pro $a, b, c, d, e, \&c.$, exhibeant. Hoc vero pendet quoque a natura numerorum $A, B, C, \&c.$, utrum sint integri an fracti. Ponamus esse numeros integros, atque quæsito satisfiet statuendo

$$\begin{aligned} b &= 1 & a &= A \\ c &= A - B & b &= B \\ d &= B - C & c &= AC \\ e &= C - D & d &= BD \\ f &= D - E & e &= CE \\ &\&c. & &\&c. \end{aligned} \quad \text{unde fit}$$

Quocirca, si fuerit,

$$x = A - B + C - D + E - F + \&c.,$$

idem ipsius x valor per fractionem continuam ita exprimi poterit, ut sit

$$x = \frac{A}{1 + \frac{B}{A - B + \frac{AC}{B - C + \frac{BD}{C - D + \frac{CE}{D - E + \&c.}}}}}$$

369. Sin autem omnes termini Seriei sint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \&c.,$$

habebuntur pro $a, b, c, d, e, \&c.$, sequentes valores

$$a =$$

$$\text{LIB. I. } \alpha = \frac{b}{A}; \zeta = \frac{A b c}{B - A}; \gamma = \frac{B^2 c d}{(B - A)(C - B)}; \\ \delta = \frac{C^2 d e}{(C - B)(D - C)}; \epsilon = \frac{D^2 e f}{(D - C)(E - D)}; \&c..$$

Ponatur ergo ut sequitur

$$\begin{array}{ll} b = A; & \alpha = 1 \\ c = B - A; & \zeta = AA \\ d = C - B; & \gamma = BB \\ e = D - C; & \delta = CC \\ & \&c., \end{array}$$

eritque per fractionem continuam

$$x = \frac{1}{A + \frac{AA}{B - A + \frac{BB}{C - B + \frac{CC}{D - C + \&c.}}}}$$

EXEMPLUM I.

Transformetur hac Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.,$$

in fractionem continuam.

Erit ergo $A=1$, $B=2$, $C=3$, $D=4$, $\&c.$, atque, cum Series propositæ valor sit $=l_2$, erit

$$l_2 = \frac{1}{1 + \frac{1}{1 + \frac{4}{1 + \frac{9}{1 + \frac{16}{1 + \frac{25}{1 + \&c.}}}}}}$$

EXEMPLUM II.

Transformetur hac Series infinita

$$\frac{\pi}{4}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.,$$

ubi π denotat peripheriam circuli, cujus diameter $= 1$, in fractionem continuam.

Substitutis loco $A, B, C, D, \&c.$, numeris $1, 3, 5, 7, \&c.$, orietur

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \&c.}}}}}$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \&c.}}}}$$

quæ est expressio, quam BROUNCKERUS primum pro quadratura circuli protulit.

EXEMPLUM III.

Sit proposita ista Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.,$$

quæ, ob $A = m; B = m+n; C = m+2n; \&c.$, in hanc fractionem continuam mutatur

$$x = \frac{1}{m + \frac{mn}{n + \frac{(m+n)^2}{n + \frac{(m+2n)^2}{n + \frac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

Euleri *Introduct. in Anal. infin. parv.*

Q q

$\frac{1}{x}$

$$\text{LIB. I. } \frac{1}{x} - m = \frac{m}{n} + \frac{(m+n)^2}{n} + \frac{(m+2n)^2}{n} + \frac{(m+3n)^2}{n} + \&c.$$

EXEMPLUM IV.

Quoniam, supra §. 178., invenimus esse

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \&c.,$$

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; $\&c.$, unde fiet

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} + \frac{m}{n-2m} + \frac{(n-m)^2}{2m} + \frac{(n+m)^2}{n-2m} + \frac{(2n-m)^2}{2m} + \frac{(2n+m)^2}{n-2m} + \&c.$$

370. Si Series proposita per continuos Factores progrediat, ut sit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.,$$

tum prodibunt sequentes determinationes

$$a = \frac{b}{A}; \quad c = \frac{bc}{B-1}; \quad d = \frac{bcd}{(B-1)(C-1)};$$

$$e = \frac{cde}{(C-1)(D-1)}; \quad f = \frac{def}{(D-1)(E-1)}; \quad \&c.,$$

fiat ergo ut sequitur,

$$b = A;$$

$$\begin{array}{ll}
 b = A; & \alpha = 1 \\
 c = B - 1; & \zeta = A \\
 d = C - 1; & \gamma = B \\
 e = D - 1; & \delta = C \\
 f = E - 1; & \varepsilon = D
 \end{array}$$

&c.,

unde consequenter fiet

$$x = \frac{1}{A + \frac{A}{B-1 + \frac{B}{C-1 + \frac{C}{D-1 + \frac{D}{E-1 + \dots}}}}}$$

EXEMPLUM I.

Quoniam, posito e numero cujus Logarithmus est $= 1$, supra invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \dots$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \dots$$

hæc Series in fractionem continuam convertetur ponendo $A=1$, $B=2$, $C=3$, $D=4$, &c.: quo ergo facto habebitur

$$1 - \frac{1}{e} = \frac{1}{1 + \frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \frac{5}{5 + \dots}}}}}}}$$

unde, asymmetria initio rejecta, erit

$$\frac{1}{e-1} = \frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \frac{5}{5 + \dots}}}}}$$

Q q 2

EXEM-

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis sumitur, cosinum esse $= 1 - \frac{1}{2} + \frac{1}{2 \cdot 12} - \frac{1}{2 \cdot 12 \cdot 30} + \frac{1}{2 \cdot 12 \cdot 30 \cdot 56} - \dots$. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$, $E = 56$, &c., atque Cofinus arcus qui radio æquatur, ponatur $= x$; erit

$$x = \frac{1}{1 + \frac{1}{1 + \frac{2}{11 + \frac{12}{29 + \frac{30}{55 + \dots}}}}} \text{ &c. ,}$$

feu

$$\frac{1}{x} - 1 = \frac{1}{1 + \frac{2}{11 + \frac{12}{29 + \frac{30}{55 + \dots}}}} \text{ &c.}$$

371. Sit Series insuper cum geometrica conjuncta, scilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \dots,$$

erit

$$a = Ab; \quad c = \frac{Bbcz}{A - Bz}; \quad y = \frac{ACcdz}{(A - Bz)(B - Cz)};$$

$$d = \frac{BDdez}{(B - Cz)(C - Dz)}; \quad e = \frac{CEefz}{(C - Dz)(D - Ez)}; \quad \text{&c.}$$

Ponatur nunc

$b = 1;$	$a = A$
$c = A - Bz;$	$c = Bz$
$d = B - Cz;$	$y = ACz$
$e = C - Dz;$	$d = BDz,$

unde fiet

$$x =$$

$$x = \frac{A}{1} + \frac{Bz}{A - Bz} + \frac{ACz}{B - Cz} + \frac{BDz}{C - Dz} + \&c.$$

372. Quo autem hoc negotium generalius absolvamus, ponamus esse

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione instituta,

$$\begin{aligned} \delta &= \frac{Ab}{L}; \epsilon = \frac{BLbcy}{AMz - BLy}; \gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)}; \\ \delta &= \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)}; \&c., \end{aligned}$$

statuantur valores $b, c, d, \&c.$, sequenti modo

$\begin{aligned} b &= L; \\ c &= AMz - BLy; \\ d &= BNz - CMy; \\ e &= COz - DNy; \\ f &= DPz - EOy; \\ &\&c. \end{aligned}$	erit	$\begin{aligned} a &= A \\ \epsilon &= BLLy \\ \gamma &= ACM^2yz \\ \delta &= BDN^2yz \\ \epsilon &= CEO^2yz \\ &\&c. \end{aligned}$
--	------	--

unde Series proposita per sequentem fractionem continuam exprimetur

$$x = \frac{A}{L} + \frac{BLLy}{AMz - BLy} + \frac{ACMMyz}{BNz - CMy} + \frac{BDNNyz}{COz - DNy} + \&c.$$

373. Habeat denique Series proposita hujusmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque sequentes valores prodibunt

Q q 3

$x =$

$$\text{LIB. I. } \alpha = \frac{Ab}{L}; \xi = \frac{Bbcy}{Mz - By}; \gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)};$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)}; \varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)};$$

$$\&c.,$$

ad valores ergo integros inveniendos fiat

$$\begin{array}{ll} b = Lz; & \text{erit } \alpha = Az \\ c = Mz - By; & \xi = BLyz \\ d = Nz - Cy; & \gamma = CMyz \\ e = Oz - Dy; & \delta = DMyz \\ f = Pz - Ey; & \varepsilon = EMyz \\ & \&c. \end{array}$$

Unde valor Seriei propositæ ita exprimetur, ut fit

$$x = \frac{Az}{Lz +} \frac{BLyz}{Mz - By +} \frac{CMyz}{Nz - Cy +} \frac{DMyz}{Oz - Dy +} \&c..$$

Vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \frac{BLyz}{Mz - By +} \frac{CMyz}{Nz - Cy +} \frac{DMyz}{Oz - Dy +} \&c..$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex supra traditis, infinitæ Series, quarum summæ constant, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor summæ illius Seriei est æqualis. Exempla, quæ jam hic sunt allata, sufficiunt ad hunc usum ostendendum: verumtamen optandum esset, ut methodus detegeretur, cujus beneficio, si proposita fuerit fractio continua quæcunque, ejus valor immediate inveniri posset. Quanquam enim fractio continua

tinua transmutari potest in Seriem infinitam, cujus summa per CAP.
XVIII. methodos cognitæ investigari queat, tamen plerumque istæ Series tantopere fiunt intricatæ, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit.

375. Quo autem clarius perspiciatur, dari ejusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex Seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

cujus omnes denominatores sunt inter se æquales; si enim hinc modo supra exposito, fractiones formemus

$$0, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2 \\ \frac{1}{0}, \quad \frac{0}{1}, \quad \frac{1}{2}, \quad \frac{2}{5}, \quad \frac{5}{12}, \quad \frac{12}{29}, \quad \frac{29}{70}, \quad \&c. :$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 12} - \frac{1}{12 \cdot 29} + \frac{1}{29 \cdot 70} - \&c.,$$

vel, si bini termini conjungantur, erit

$$x = \frac{2}{1 \cdot 5} + \frac{2}{5 \cdot 29} + \frac{2}{29 \cdot 169} + \&c.,$$

vel

$$x = \frac{1}{2} - \frac{2}{2 \cdot 12} - \frac{2}{12 \cdot 70} - \&c..$$

Quin etiam, cum sit

$$x = \frac{1}{4} - \frac{1}{2 \cdot 2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 12} - \frac{1}{2 \cdot 12 \cdot 29} + \&c.$$

+

LIB. I.

$$+ \frac{1}{4} - \frac{1}{2 \cdot 2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 12} - \frac{1}{2 \cdot 12 \cdot 29} + \&c.,$$

erit

$$x = \frac{1}{4} + \frac{1}{1 \cdot 5} - \frac{1}{2 \cdot 12} + \frac{1}{5 \cdot 29} - \frac{1}{12 \cdot 70} \&c.,$$

quæ Series etiamfi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro hujusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt æquales, vel iidem revertuntur; ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit æqualis, facilis habetur modus earum summas explorandi. In exemplo enim proposito, cum sit

$$x = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \&c.}}}}$$

erit $x = \frac{1}{2 + x}$, ideoque $xx + 2x = 1$ & $x + 1 = \sqrt{2}$; ita ut valor hujus fractionis continuæ sit $= \sqrt{2} - 1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Est enim $\sqrt{2} - 1$ tam prope $= \frac{29}{70}$, ut error sit insensibilis: namque, radicem extrahendo, erit

$$\sqrt{2} - 1 = 0,41421356236,$$

atque

$$\frac{29}{70} = 0,41428571428,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuæ commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum C A P.
proxime investigandas. Ponamus hunc in finem XVIII.

$$x = \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \dots}}}}}$$

erit $x = \frac{1}{a + x}$ & $xx + ax = 1$, unde fit $x = -\frac{1}{2}a + \sqrt{(1 + \frac{1}{4}aa)} = \frac{\sqrt{(aa + 4)} - a}{2}$. Hæc ergo fractio continua inserviet valori radice quadratæ ex numero $aa + 4$ inveniendi. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4, &c., reperientur $\sqrt{5}$; $\sqrt{2}$; $\sqrt{13}$; $\sqrt{5}$; $\sqrt{29}$; $\sqrt{10}$; $\sqrt{53}$; &c., perductis scilicet his radicibus ad formam simplicissimam: erit ergo

$$\begin{array}{l} 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \\ \frac{0}{1}, \quad \frac{1}{1}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{5}, \quad \frac{5}{8}, \quad \&c. = \frac{\sqrt{5}-1}{2} \\ 2, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2 \\ \frac{0}{1}, \quad \frac{1}{2}, \quad \frac{2}{5}, \quad \frac{5}{12}, \quad \frac{12}{29}, \quad \frac{29}{70}, \quad \&c. = \sqrt{2}-1 \\ 3, \quad 3, \quad 3, \quad 3, \quad 3, \quad 3 \\ \frac{0}{1}, \quad \frac{1}{3}, \quad \frac{3}{10}, \quad \frac{10}{33}, \quad \frac{33}{109}, \quad \frac{109}{360}, \quad \&c. = \frac{\sqrt{13}-3}{2} \\ 4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 4 \\ \frac{0}{1}, \quad \frac{1}{4}, \quad \frac{4}{17}, \quad \frac{17}{72}, \quad \frac{72}{305}, \quad \frac{305}{1292}, \quad \&c. = \sqrt{5}-2 \\ \&c., \end{array}$$

notandum autem eo promptiorem esse approximationem, quo major fuerit numerus a : sic in ultimo exemplo erit $\sqrt{5} = 2 \frac{305}{1292}$, ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

LIB. I. 378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus esse

$$x = \frac{1}{a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \dots}}}}}} \text{ \&c.,}$$

erit

$$x = \frac{1}{a + \frac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}; \text{ ideoque } axx + abx = b.$$

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}bb + \frac{b}{a}\right)} = \frac{-ab \pm \sqrt{(aabb + 4ab)}}{2a}.$$

Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi gratia, $a = 2, b = 7$; erit $x = \frac{-14 \pm \sqrt{14 \cdot 18}}{4} = \frac{-7 \pm 3\sqrt{7}}{2}$; at valorem ipsius x proxime exhibebunt sequentes fractiones

$$\frac{2}{1}, \frac{7}{2}, \frac{2}{15}, \frac{7}{32}, \frac{2}{239}, \frac{7}{510}, \text{ \&c.,}$$

Erit ergo proxime $\frac{-7 + 3\sqrt{7}}{2} = \frac{239}{510}$ & $\sqrt{7} = \frac{2024}{765} = 2,6457516$; at revera est $\sqrt{7} = 2,64575131$; ita ut error minor sit quam $\frac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{a + \dots}}}}}}}} \text{ \&c.,}$$

erit

$$x = \frac{1}{a + \frac{1}{b + \frac{1}{c + x}}} = \frac{1}{a + \frac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c}$$

unde $(ab + 1)xx + (abc + a - b + c)x = bc + 1$ atque

$x =$

$$x = \frac{abc - a + b - c + \sqrt{((abc + a + b + c)^2 + 4)}}{2(ab + 1)}; \quad \text{CAP. XVIII.}$$

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum, neque ergo hæc forma radicibus ex aliis numeris extrahendis inservit, nisi ad quos prima forma jam suffecerat. Simili modo si quatuor litteræ a, b, c, d , continuo repetitæ denominatores fractionis continuæ constituent, tum ea plus non inserviet quam secunda, quæ duas tantum litteras continebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extractionem radicis quadratæ adhiberi queant, simul inservient æquationibus quadratis resolvendis; quod quidem ex ipso calculo est manifestum, dum x per æquationem quadraticam affectam determinatur. Potest autem vicissim facile cujusque æquationis quadratæ radix per fractionem continuam hoc modo exprimi. Sit proposita ista æquatio

$$xx = ax + b,$$

ex qua, cum sit $x = a + \frac{b}{x}$, substituatur in ultimo termino loco x valor idem jam inventus, eritque

$$x = a + \frac{b}{a + \frac{b}{x}},$$

simili ergo modo procedendo, erit per fractionem continuam infinitam

$$x = a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \frac{b}{\ddots}}}}$$

quæ autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem con-

LIB. I.

tinuam converti posse. Sit enim proposita fractio $x = \frac{A}{B}$; in qua sit $A > B$; dividatur A per B , sitque quotus $= a$ & residuum C ; tum per hoc residuum C dividatur præcedens divisor B , prodeatque quotus b & relinquatur residuum D , per quod denuo præcedens divisor C dividatur; sicque hæc operatio, quæ vulgo ad maximum communem divisorem numerorum A & B investigandum usurpari solet, continuetur, donec ipsa finiatur; sequenti modo

$$\begin{array}{r} B \overline{) A} (a \\ \quad C \overline{) B} (b \\ \quad \quad D \overline{) C} (c \\ \quad \quad \quad E \overline{) D} (d \\ \quad \quad \quad \quad F \text{ \&c.}, \end{array}$$

eritque per naturam divisionis

$$A = aB + C; \text{ unde } \frac{A}{B} = a + \frac{C}{B};$$

$$B = bC + D; \quad \frac{B}{C} = b + \frac{D}{C}; \quad \frac{C}{B} = \frac{1}{b + \frac{D}{C}}$$

$$C = cD + E; \quad \frac{C}{D} = c + \frac{E}{D}; \quad \frac{D}{C} = \frac{1}{c + \frac{E}{D}}$$

$$D = dE + F; \quad \frac{D}{E} = d + \frac{F}{E}; \quad \frac{E}{D} = \frac{1}{d + \frac{F}{E}}$$

&c.

&c.

&c.

hinc, sequentes valores in præcedentibus substituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b + \frac{D}{C}} = a + \frac{1}{b + \frac{1}{c + \frac{E}{D}}},$$

unde tandem x per meros quotos inventos $a, b, c, d, \text{ \&c.}$, sequentem in modum exprimetur, ut sit

$$x =$$

$$x = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f + \dots}}}}}$$

EXEMPLUM I.

Sit proposita ista fractio $\frac{1461}{59}$, quæ sequenti modo in fractionem continuam transmutabitur, cujus omnes numeratores erunt unitates. Instituatur scilicet eadem operatio, qua maximus communis divisor numerorum 59 & 1461 quæri solet.

$$\begin{array}{r} 59 \overline{) 1461} (24 \\ \underline{118} \\ 281 \\ \underline{236} \\ 45 \overline{) 59} (1 \\ \underline{45} \\ 14 \overline{) 45} (3 \\ \underline{42} \\ 3 \overline{) 14} (4 \\ \underline{12} \\ 2 \overline{) 3} (1 \\ \underline{2} \\ 1 \overline{) 2} (2 \\ \underline{2} \\ 0 \end{array}$$

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \frac{1}{1 + \frac{1}{3 + \frac{1}{4 + \frac{1}{1 + \frac{1}{2}}}}}$$

EXEMPLUM II.

Fractiones quoque decimales eodem modo transmutari poterunt; sit enim proposita

$$\sqrt{2} = 1,41421356 = \frac{141421356}{100000000},$$

unde hæc operatio instituatur

R r 3

100000000

adeo esse $\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$,
cujus expressionis ratio jam ex superioribus patet.

Imprimis vero etiam hic attentione dignus est numerus e , cujus logarithmus est $= 1$, qui est $e = 2,718281828459$, unde oritur $\frac{e-1}{2} = 0,8591409142295$, quæ fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	1000000000000000	1
8451545146224	8591409142295	6
<u>139863996071</u>	<u>1408590857704</u>	10
<u>139312557916</u>	<u>1398639960710</u>	14
551438155	9950896994	18
550224488	9925886790	22
<u>1213667</u>	<u>25010204</u>	
	&c.	

1, 6, 10, 14, 18, 22, 26, 30, 34, &c.,
qui, demto primo, progressionem arithmeticam constituunt, unde
patet fore e — 1

$$\frac{e-1}{2} = \frac{1}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \frac{1}{22 + \frac{1}{\&c.}}}}}}}$$

cujus fractionis ratio ex calculo infinitesimali dari potest.

382. Cum igitur ex hujusmodi expressionibus fractiones erui queant, quæ quam citissime ad verum valorem expressionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita cujus numerator & denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt quæ, etiam si propositæ non sint penitus æquales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO olim tractatum facile resolvi potest, quo quærentur fractiones minoribus numeris expressæ, quæ tam prope exhauriant valorem fractionis cujuscumque in numeris majoribus propositæ, quantum fieri poterit numeris non majoribus. Fractiones autem nostra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus constantes dentur quæ propius accedant.

EXEMPLUM I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri majores adhibeantur. Si fractio decimalis cognita

3, 1415926535 &c.,

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

3, 7, 15, 1, 292, 1, 1, &c.,

ex quibus sequentes fractiones formabuntur,

$\frac{1}{0}, \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \&c.,$

secunda fractio jam ostendit esse diametrum ad peripheriam utt

1 : 3,

LIB. I. 1: 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7: 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor sit parte $\frac{1}{113.33102}$. Ceterum hæ fractiones alternatim vero sunt majores minoresque.

EXEMPLUM II.

Exprimatur ratio diei ad annum solare medium in numeris minimis proxime. Cum annus iste sit $365^d, 5^b, 48', 55''$, continebit in fractione annus unus $365 \frac{22935}{86400}$ dies. Tantum ergo opus est ut hæ fractio evolvatur, quæ dabit sequentes quotos

4, 7, 1, 6, 1, 2, 2, 4
unde istæ eliciuntur fractiones

$$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{260}, \frac{181}{747}, \&c..$$

Horæ ergo cum minutis primis & secundis, quæ supra 365 dies adsunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde sequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inserat 100 dies, *Gregorianus* quaternis seculis tres annos bissextiles in communes convertit.

FINIS TOMI PRIMII.