Steen Pedersen

From Calculus to Analysis

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To Katherine, without your support this book would not have been completed.

Preface

The text is intended to form a bridge between calculus and analysis. It is based on the author's lecture notes, notes used and revised nearly every year over the last decade or so. The students typically were a mixture of students intending to go on to graduate work in mathematics and students intending to teach mathematics in grades 9–12. The later students typically only take the first semester of the two semester sequence. In order to cover the material, the prospective teachers might need to teach one variable calculus in high school; the text is designed to comfortably allow coverage of the standard calculus topics including the Fundamental Theorem of Calculus in the first semester. The material needed to prepare the remaining students for a graduate level Real Analysis course can comfortably be covered in the second semester.

To learn something new, for example, some mathematical topic or a foreign language, requires exposure to many examples, practice, and the passage of time. To quote one of the famous mathematicians of the twentieth century:

> "... in mathematics you don't understand things. You just get used to them." – John Von Neumann

Hence, to help the reader learn from the text, the text includes many examples and a variety of interesting applications are included, several dealing with surprising properties of irrational numbers. The applications include some of the "jewels" of analysis, for example, e is transcendental, π is irrational, a space filling curve, and examples of nowhere differentiable continuous functions. Many of the applications are not standard, but of interest to the students. There are exercises embedded in the text and problem sections at the end of each chapter. The exercises in the text are intended to slow the reader down and allow immediate engagement with some aspect of the material being discussed. Essential ideas related to limits and continuity are treated early and revisited in greater depth later. Topics are broken into small easily digestible modules containing detailed explanations. Brief biographical material is included for many of the mathematicians who contributed to the development of the subject.

The text is basically a one variable treatment, in many places this variable is a complex variable. Using a complex variables (i) places emphasis on the triangle inequality, hence aids in the transition to more advanced analysis courses, (ii) keeps the level of abstraction at the same level as the standard one variable treatment, and (iii) some two (real) variable topics are easily accessible.

Introduction

We define the set of real numbers to be the set of infinite decimals. Apart from leaving the arithmetic of infinite decimals as a mystery, we attempt an axiomatic approach to the subject.

The chapters are intended to be read in the order in which they are presented. Chap. 13 on topology is special, and could perhaps be regarded as an appendix. Most of Chap. 13 can be covered at the same time as Chap. 5. It could also be covered instead of parts of that chapter. The subsection on sequential compactness in Sect. 13.3, requires Sect. 9.1. The appendices contain various kinds of background material. The author has found it useful for students to begin with Appendix D. In addition to reinforcing the rules for working with inequalities, this gets the students used to proving things they already "know." Since most students have taken calculus prior to taking this course that is in itself useful, this appendix also introduces the students to do work based on a set of axioms.

Some sections are marked with a star. This does not indicate they are less important or more difficult than other section. It only signifies that they are not used elsewhere in the text. Some of these sections contain the most "interesting" results in this text. Several of the starred sections involve investigating properties of irrational and transcendental numbers. Among these are proofs that e and π are irrational as well as the explicit construction of some transcendental numbers.

Except in a few clearly labeled places, we do not use functions whose existence we have not established. We establish the existence of roots, logarithms, exponentials, and the trigonometric functions using the methods of analysis.

The ideas in some proofs reoccur multiple times. In fact, some proofs have been chosen specifically for this reason. Examples include the proof that a number is a root of a polynomial if and only if there is a corresponding linear factor and the proof that the square root of two is irrational. Consequently, some of the proofs in the text are deliberately not the most elegant proof of the statement under consideration.

The early parts of the text uses formal notation. Writing a statement in this manner indicates what must be done to prove this statement and in which order this must be done. Using formal notation also makes it clear how a statement can be used in a proof. Hence, using formal notation aids the beginning analysis student understand and construct proofs. In places, the formality is relaxed to help the student transition to the standard writing style used in many mathematics books.

Many results in the text are credited to one or two persons, in many (most?) cases this is misleading. Most likely the result was know to earlier mathematicians in some form or other. Dates in the text may only be approximate. At least the author found contradictory information in several instances. Despite these shortcomings, mentioning names and dates serves to illustrate that analysis was developed by many people over a long period of time.

Note to the Student Reader

Working the exercises embedded in the text is intended to help you assimilate the material. Working these exercises without assistance of any kind, for example, from friends, other books, or the internet is necessary in order to gain a basic understanding of the subject. Each chapter contains a section of solutions and hints that can be consulted after a significant amount of time has been spent attempting to find a solution. Once you have written down a complete solution to an exercise, it is useful to compare this solution to other solutions. This is where friends, other books, and the Internet are useful.

Each chapter also contains a selection of problems. Working these problems will give the reader additional practice with the concepts and tools developed in the text. The results established in these problems are not used elsewhere.

When solving exercises and problems try only to rely on the definition and the major theorems: the named theorems. Try not to use the various minor results contained in un-named theorems, lemmas, and exercises. When solving exercises and problems do not refer to other problems in the text. An exception to rule is that some problems are part of a sequence of problems leading to up to a major conclusion.

Some easy exercises are stated because they are useful when proving some of the more interesting statements. Claims stated without proofs as well as all claims made in exercises/problems are meant to be proven by the reader.

When reading a proof (including your own proofs/solutions) you need to be able to: (a) explain why the proof proves the theorem/problem, (b) identify the main steps in the proof, (often they take the form of smaller claims made in the proof) (c) fill in all missing details (go through the proof line by line to check the details), (d) identify where each hypothesis in the theorem is used.

Acknowledgements

Preliminary versions of this book were class room tested over several years. I thank the students' suffering thought preliminary versions of this book for helping me refine the content of this book. I would also like to thank the reviewers and the staff at Springer for their help improving this book in so many ways.

Contents

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Part I

Calculus

The theory of differential and integral calculus is the main focus of this part. This includes discussions of number, sets of numbers, limits, continuity, derivative, and Riemann integral. Applications includes a proof of the Steinhaus three distance conjecture, Liouville's theorem on transcendental numbers, convex function theory, a construction of the exponential and logarithmic functions, an example of a compactly supported smooth function, and a proof that the number e is transcendental.

Chapter 1
Limits

The set real numbers is defined as the set of infinite decimals. Density of the set of rational numbers and of the set of irrational numbers in the set of real numbers is established. This naturally leads to a discussion of accumulation points that serves as a precursor for the main part of this chapter: the theory of limits of functions. Convergence of sequences and series of numbers are discussed briefly. A bounded function, the Dirichlet function, that does not have a limit at any real number is presented. Section 1.8 contains a proof of Steinhaus' three distance conjecture.

1.1 Infinite Decimals

The basis set underlying all of our considerations is the set of all real numbers, that is, the set of all infinite decimals. In this section we investigate a few basic properties of infinite decimals. We begin by introducing some terminology.

An *infinite decimal* is an expression of the form

$$\pm d_0.d_1d_2\cdots,$$

where $d_0 \in \mathbb{N}_{0,}$ and for $k \geq 1$, d_k is a decimal digit meaning

$$d_k \in \{0,1,2,3,4,5,6,7,8,9\}.$$

For example, if our infinite decimal is $13.23674\cdots$, then $d_0 = 13$, $d_1 = 2$, $d_2 = 3$, $d_3 = 6$, $d_4 = 7$, etc.

An infinite decimal

$$\pm d_0.d_1d_2\cdots$$

is *repeating*, if there are $k,m \in \mathbb{N}$, such that $d_{j+m} = d_j$ for all $j \geq k$. The digits $d_kd_{k+1}\cdots d_{k+m-1}$ are the *repeating part* of the infinite decimal and the repeating

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part has *length m*. We will write

$$d_0.d_1d_2\cdots d_{k-1}\overline{d_kd_{k+1}\cdots d_{k+m-1}} := d_0.d_1d_2\cdots.$$

For example, $1.67234523452345\cdots = 1.67\overline{2345}$ is an infinite decimal with repeating part 2345 of length 4. Another example of a repeating decimal is $0 = 0.\overline{0}$. More generally, we will say that decimals of the form $2.453 := 2.453\overline{0}$ are *finite decimals*.

We assume the reader is familiar with the usual properties of addition and multiplication, including the associative, commutative, and distributive properties. These properties are, for example, encoded in the field axioms. Inequality between two infinite decimals has the usual meaning, in particular, inequality satisfies the Axioms of Trichotomy and Positive Closure. Consequently, we can freely use the consequences of these axioms derived in Sects. D.1 and D.2.

Finite Decimals

Let $x := 0.\overline{9}$, then $10x = 9.\overline{9} = 9 + x$, hence $9x = 9$, so $x = 1$. That is

$$0.999\cdots = 0.\overline{9} = 1.$$

Similarly, any infinite decimal terminating in $\overline{9}$ equals a finite decimal. Conversely, any finite decimal equals an infinite decimal terminating in $\overline{9}$. For example, $1.23\overline{9} = 1.24$. We have shown:

Theorem 1.1.1. *The set of finite decimals equals the set of infinite decimals terminating in repeating nines.*

Repeating Decimals

The set of rational numbers is

$$\mathbb{Q} := \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

We provide a characterization of the rationals in terms of decimals.

The number $0.\overline{234}$ is rational. In fact, if $x := 0.\overline{234}$, then $1000x = 234 + x$, hence $x = 234/999$. Similarly, any repeating decimal is a rational number.

Calculating the decimal form of $1/7$ by long division, there are at most 7 different remainders, namely $0, 1, 2, 3, 4, 5$, and 6. Hence, after at most 7 divisions, the division problem repeats itself. Thus the infinite decimal form of $1/7$ is a repeating decimal whose repeating part has length at most 6 (If the remainder is zero, the process stops). Similarly, the repeating decimal form of any rational p/q, with $1 < q$, is a repeating decimal whose repeating part has length at most $q - 1$.

We have shown:

Theorem 1.1.2. *An infinite decimal is a rational number iff it is repeating.*

Example 1.1.3. By long division $\frac{1}{3} = 0.\overline{3}$, and $\frac{2}{3} = 0.\overline{6}$. Since $\frac{1}{3} + \frac{2}{3} = \frac{1}{1}$ and $\frac{3}{1}\frac{1}{3} = \frac{1}{1}$ we have

$$0.\overline{3} + 0.\overline{6} = 1 \text{ and}$$
$$3 \cdot 0.\overline{3} = 1.$$

Leading to two further verifications that $0.\overline{9} = 1$.

A real number that is not rational is called *irrational*. Irrational numbers exist because non-repeating decimals exist. For example, the infinite decimal

$$0.1010010001\cdots \tag{1.1}$$

(keep increasing the number of zeros) is non-repeating, hence an irrational number.

We will prove that $\sqrt{2}$, e, and π exist and are irrational. For $\sqrt{2}$ existence is contained in Theorem 3.5.1 and irrationality is established by Theorem 3.5.4. We construct e in Sect. 8.2 and show e is irrational (even transcendental) in Sect. 8.3. In Sect. 11.2 we construct π and in Sect. 11.5 we show π is irrational.

Density

Let A and B be two sets. We say A is *dense* in B, if any ball centered at a point in B must contain at least one point from A. In symbols,

$$\forall b \in B, \forall r > 0, A \cap B_r(b) \neq \emptyset.$$

Alternatively, using the definition of an open ball, we can rewrite the definition of A being dense in B as

$$\forall b \in B, \forall r > 0, \exists a \in A, |a - b| < r.$$

Usually, when we say A is dense in B, the set A is a subset of the set B, but this is not required by the definition.

Lemma 1.1.4. *If $r > 0$, then there is an integer $N \geq 1$, such that $r > 1/10^N$.*

Proof. Let $r = d_0.d_1d_2\cdots$ be the infinite decimal representation of r. Since $r \neq 0$, for at least one n, d_n is non-zero. Consequently, $r > 1/10^{n+1}$. ☺

A direct consequence of this result is that there is no infinitely small positive real number:

Corollary 1.1.5. *If $0 \leq r$ and $r < 1/10^n$ for all integers n, then $r = 0$.*

The following theorems establish the density of the set of rational numbers and of the set of irrational numbers in the set of all real numbers.

Theorem 1.1.6 (Density of Rationals). *The set of all rational numbers is dense in the set of all real numbers.*

Proof. Let real number x and $r > 0$ be given. We must show that $B_r(x)$ contains at least one rational number. Suppose $x > 0$. Let $x = d_0.d_1d_2\cdots$ be an infinite decimal representation of x. Let $N \geq 1$ be an integer such that $r > 1/10^N$. Let $y := d_0.d_1\cdots d_N$ and $z := d_0.d_1\cdots d_N\overline{9} = y + 1/10^N$. Then y and z are rationals.

Since y is obtained from $x = d_0.d_1d_2\cdots$ by replacing the d_k with $k > N$ by 0's and z is obtained from x by replacing the d_k with $k > N$ by 9 's we have

$$y \leq x \leq z, \text{ and } z = y + \frac{1}{10^N}. \tag{1.2}$$

Hence,

$$x \leq z = y + \frac{1}{10^N} \leq x + \frac{1}{10^N} < x + r.$$

Where the last inequality follows from the choice of N. Consequently, z is a rational in $B_r(x)$.

The case where $x < 0$ is similar. If $x = 0$, then x is already a rational in $B_r(x)$. ☺

Exercise 1.1.7. If $r \neq 0$ is rational and x is irrational, then $1/x$, x/r, rx and $r + x$ are irrational.

Theorem 1.1.8 (Density of Irrationals). *Any open interval contains an irrational number.*

Proof. Let real numbers x and $r > 0$ be given. We will show that $B_r(x)$ contains at least one irrational number. Let t be the irrational in (1.1). Then $0 < t < 1$. Suppose $x > 0$. Let $x = d_0.d_1d_2\cdots$ be an infinite decimal representation of x. Let $N \geq 1$ be an integer such that $r > 1/10^N$. Then

$$y := d_0.d_1\cdots d_N + \frac{t}{10^N}$$

is an irrational such that $|x - y| \leq \frac{1}{10^N} < r$.

The case $x < 0$ is similar. If $x = 0$, then $t/10^N$ is an irrational in $B_r(x)$. ☺

1.2 Accumulation Points

Let D be a subset of $\mathbb{C}$ and let a be some complex number. We say a is an *accumulation point* of D, if there are points in $D \setminus \{a\}$ arbitrarily close to a. Hence, a point a is an accumulation point of D, if given any distance $\varepsilon > 0$, there is at least one point in $D \setminus \{a\}$ whose distance to a is less than ε. We restate this as:

Definition 1.2.1. Let D be a subset of $\mathbb{C}$ and let a be some complex number. We say a is an *accumulation point* of D, if for all $\varepsilon > 0$, there is an $x \in D$, such that $0 < |x-a| < \varepsilon$. In symbols,

$$\forall \varepsilon > 0, \exists x \in D, 0 < |x-a| < \varepsilon.$$

Accumulation points are also called limits points.

A ball without its center is called a *punctured ball*; hence a punctured ball is a set of the form

$$B'_r(c) := B_r(c) \setminus \{c\} = \{z \in \mathbb{C} \mid 0 < |z-c| < r\}.$$

A punctured ball is also called a *punctured neighborhood* of c. In the same manner the open ball $B_r(a)$ is sometimes called a *neighborhood* of a.

In terms of balls, we can rewrite the definition of an accumulation point as

$$\forall \varepsilon > 0, B'_\varepsilon(a) \cap D \neq \emptyset. \tag{1.3}$$

The following exercise shows that it is sufficient to consider small values of $\varepsilon > 0$.

Exercise 1.2.2. Let $m > 0$ be given. If for all $0 < r \leq m$, $D \cap B'_r(c) \neq \emptyset$, then c is an accumulation point of D.

Example 1.2.3. The number 0 is an accumulation point of $B_1(0)$.

Proof. To see this consider an arbitrary $\varepsilon > 0$. We may assume $\varepsilon \leq 1$. Let $x := \varepsilon/2$. Then $0 < x = |x-0| < 1$ (so $x \in B_1(0)$,) and $0 < |x-0| < \varepsilon$. See Fig. 1.1. ☺

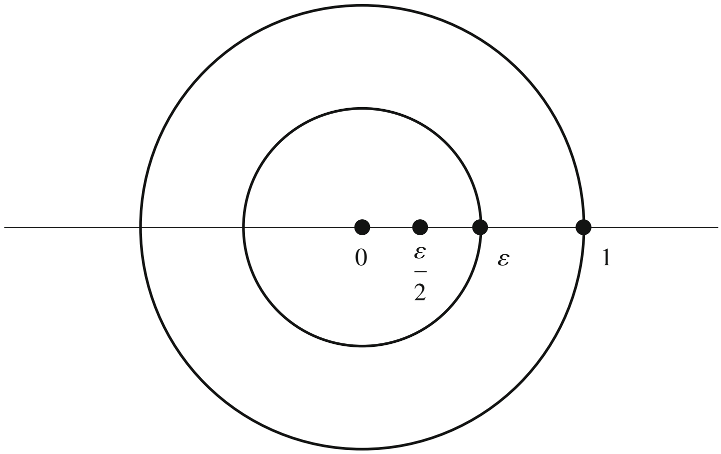

Fig. 1.1 Illustrating Example 1.2.3 using balls as in (1.3). In the notation of (1.3) the *large disk* is D and the *small disk* is $B_\varepsilon(a)$, with $a = 0$

Example 1.2.4. The number 1 is an accumulation point of $B_1(0)$.

Proof. To see this, consider $\varepsilon > 0$. We may assume $\varepsilon \leq 1$. Then $0 < \varepsilon/2 \leq 1/2$. Hence, $0 < \frac{1}{2} \leq 1 - \frac{\varepsilon}{2} < 1$. Consequently, if $x := 1 - \frac{\varepsilon}{2}$, then $0 < x < 1$. Thus $x \in B_1(0)$ and $0 < |x-1| < \varepsilon$, as we needed to show. See Fig. 1.2. ☺

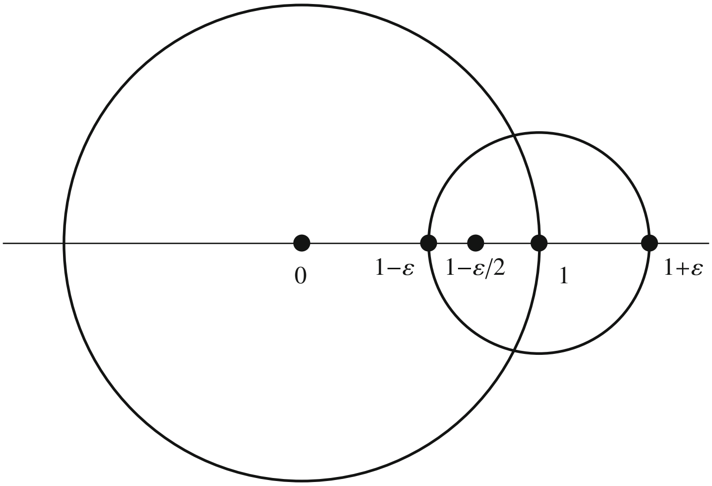

Fig. 1.2 Illustrating Example 1.2.4 using balls as in (1.3). In the notation of (1.3) the *large disk* is D and the *small disk* is $B_\varepsilon(a)$, with $a = 1$

Example 1.2.5. The number 2 is not an accumulation point of $B_1(0)$.

Proof. Let $\varepsilon := \frac{1}{2}$. Let $x \in B_1(0)$. Then $|x-2| \nless \varepsilon$, because $|x-2| \geq ||x|-|2||$ $= 2-|x| \geq 2-1 > 1/2 = \varepsilon$. See Fig. 1.3. ☺

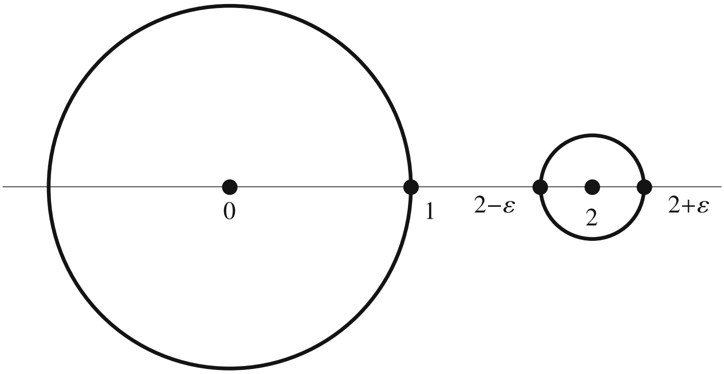

Fig. 1.3 Illustrating Example 1.2.5 using balls as in (1.3). In the notation of (1.3) the *large disk* is D and the *small disk* is $B_\varepsilon(a)$, with $a = 2$

Remark 1.2.6. Using the methods from Examples 1.2.3 to 1.2.5 it follows that the set of accumulation points of an open ball in the corresponding closed ball.

1.3 Limits of Functions

Understanding the definition of limits is central to understanding a modern development of calculus. The modern concept of limits is highly nontrivial: It took mathematicians several hundred years, after the discovery of calculus, to organize the logical development of calculus around the concept of the limits and another hundred years or so to write down the modern (precise) definition of the limit.

Definition of Limits

If A and B are sets, then the notation $f : A \to B$ is used to indicate that f is defined on all of A and has values in B. If every element of B is a value of f, then f is said to be *onto* or *surjective*. See Sect. A.4 for more background information regarding functions.

Let D be a subset of $\mathbb{C}$. Let function $f : D \to \mathbb{C}$ and suppose $a \in \mathbb{C}$ is an accumulation point of D. Let $L \in \mathbb{C}$. We would like to turn the vague statement

$$f(x) \text{ is close to } L, \text{ when } x \text{ is close to } a$$

into a precise mathematical statement, that is, into a statement that can be described using set theory and logic. As a step in this direction we reformulate the previous vague statement as:

$$\text{we can arrange that } f(x) \text{ is as close to } L, \text{ as we wish,}$$

$$\text{for all } x \text{ sufficiently close to } a$$

The first part "that $f(x)$ is as close to L, as we wish" can be made precise as follows:

$$\text{for any (small) } \varepsilon > 0, |f(x) - L| < \varepsilon.$$

The second part "for all x sufficiently close to a" can be made precise by saying that

$$0 < |x-a| < \delta \text{ for some } \delta > 0 \text{ depending on } \varepsilon,$$

where $0 < |x-a| < \delta$ says x is close to a and "sufficiently" is encoded in the dependence of δ on ε. We want a to be an accumulation point of D, because that guarantees that $D \cap B'_\delta(a)$ is nonempty, that is, $0 < |x-a| < \delta$ for at least one $x \in D$. Hence we have arrived at:

Definition 1.3.1. Let $D \subseteq \mathbb{C}$, $a, L \in \mathbb{C}$, and $f : D \to \mathbb{C}$. If a is an accumulation point of D, we will say that $f(x)$ *converges* to L as x goes to a provided: Given any $\varepsilon > 0$, there is a $\delta > 0$, such that for any $x \in D$ with $0 < |x-a| < \delta$, we have $|f(x) - L| < \varepsilon$.

In symbols,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, 0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon. \tag{1.4}$$

This definition of limit is due to Karl Theodor Wilhelm Weierstrass (31 October 1815 Ostenfeldeto 19 February 1897 Berlin). Using neighborhoods, as in Sect. 1.2, Eq. (1.4) can be written as

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x \in B'_\delta(a) \implies f(x) \in B_\varepsilon(L). \tag{1.5}$$

If $g : A \to B$ and $C \subseteq A$, then $g(C) := \{g(x) \mid x \in C\}$ is the *image* of C. Clearly, $g(A) \subseteq B$. Note that g is onto, if $g(A) = B$. Using this image notation we can write (1.5) and therefore (1.4) as

$$\forall \varepsilon > 0, \exists \delta > 0, f(D \cap B'_\delta(a)) \subseteq B_\varepsilon(L). \tag{1.6}$$

Restating (1.6) in words: Given a ball B centered at L, there is a punctured ball B' centered at a, such that f maps B' into B.

We will abbreviate Eq. (1.4) as

$$\lim_{x \to a} f(x) = L$$

and say that L is the *limit* of f as x goes to a. Using notation that is similar to saying that $f(x)$ goes to L as x goes to a, we will also abbreviate (1.4), and therefore also the equivalent formulations (1.5) and (1.6), as

$$f(x) \to L, \text{ as } x \to a$$

and as

$$f(x) \underset{x \to a}{\longrightarrow} L.$$

Model Nothing moves in the definition of convergence. Hence, notation using arrows to describe convergence is somewhat deceptive. Here we discuss a more appropriate model of $\lim_{x \to a} f(x) = L$.

<u>Preamble</u> The model below is chosen because traditionally, at least in Europe during times of war, mathematicians were employed calculating ballistic orbits, sometimes even on the battlefield.

<u>Set-up</u> Suppose we have a canon. When this canon is fired at an angle x to the horizon the projectile lands a distance $f(x)$ from the canon. The intended target is a distance L from the canon and a is an angle such that $f(a) = L$. In practice, it is not possible to set up the canon in such a way that the angle x is determined with absolute precision, i.e., such that $x = a$. However, we only need the projectile to land so close to the target that the target is destroyed. We call this the target tolerance.

<u>Discussion</u> The function $f(x)$ encodes a lot of information, e.g., how far the target is above (or below) the canon (or more complicated information about the terrain), wind speed, atmospheric pressure, etc. How close to the target

we need the projectile to land depends on how well the target is fortified. Considering both the angle to the horizon x and the direction y in which the canon is fired, leads to a problem where $f(x+iy)$ is a function mapping the plane into the plane—into the plane because the projectile can land left or right of the target as well as in front of or behind the target.

Problem Given a target, set up the canon such that the target is destroyed.

Solution For any target tolerance, there is an angle tolerance, such that, if the angle adjustment of the canon falls within the angle tolerance, then the projectile will land within the target tolerance. Using the symbols a, x, L, and $f(x)$ we restate this as: Given any target tolerance $\varepsilon > 0$, there is an angle tolerance $\delta > 0$, such that, if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$. See Table 1.1.

Ballistic model	Definition (1.4) of limit
For any target tolerance	$\forall \varepsilon > 0$
there is an angle tolerance	$\exists \delta > 0$
such that	such that
if ..., then ...	$\Longrightarrow$
angle adjustment falls within angle tolerance	$0 < |x-a| < \delta$
projectile lands within the target tolerance	$|f(x)-L| < \varepsilon$

Table 1.1 Comparison of the ballistic model to the definition (1.4) of the limit of a function

Remark The condition $0 < |x-a|$ in Table 1.1 corresponds to the fact that it is not possible to adjust the angle of the canon with absolute precision.

End-of-Model

It is a consequence of Exercise 1.2.2 that when working with limits only "small" values of $\varepsilon > 0$ and $\delta > 0$ need to be considered. This is illustrated in the following two remarks. We give the arguments both in terms of (1.4) and in terms of (1.6).

Remark 1.3.2. If we can find a δ corresponding to $\varepsilon = 1/2$, then that δ also works for $\varepsilon = 1$. Simply

$$0 < |x-a| < \delta \implies |f(x)-b| < \frac{1}{2} \implies |f(x)-b| < 1$$

by transitivity of inequality. In terms of ball, if $f(B'_\delta(a)) \subseteq B_{1/2}(b)$, then $f(B'_\delta(a)) \subseteq B_1(b)$, because $B_{1/2}(b) \subseteq B_1(b)$.

Thus, when trying to verify (1.4), it is sufficient to consider small values of ε. That is, if it is convenient to assume, for example, $\varepsilon < 1/3$, you can safely do so.

Remark 1.3.3. Suppose $\delta = 1$ works for some value of ε, then $\delta = 1/2$ also works for that value of ε. Again, this is by transitivity of inequalities:

$$0 < |x-a| < \frac{1}{2} \implies 0 < |x-a| < 1 \implies |f(x)-b| < \varepsilon.$$

In terms of ball, if $f(B_1'(a)) \subseteq B_\varepsilon(b)$, then $f\left(B_{1/2}'(a)\right) \subseteq B_\varepsilon(b)$, because $B_{1/2}'(a) \subseteq B_1'(a)$.

Since we only need to find one value of δ, this means that, if it is convenient to assume, for example, $\delta < 1/3$, we can safely do so. We are not looking for the "best" or largest δ, or trying to classify all the possible δ's that might work.

The arguments in Remarks 1.3.2 and 1.3.3 show that we can replace $0 < |x-a| < \delta$ by $0 < |x-a| \leq \delta$ and/or $|f(x)-L| < \varepsilon$ by $|f(x)-L| \leq \varepsilon$, if convenient. That is, we can use closed balls in place of open balls, or a mixture of open and closed balls, whenever it is convenient.

Simple, but important, limits are established in the following two examples.

Example 1.3.4. Let $a, k \in \mathbb{C}$. If $f(x) := k$ for all $x \in \mathbb{C}$, then $f(x) \to k$ as $x \to a$.

Proof. Let $\varepsilon > 0$ be given. Let $\delta := 1$. Then $0 < |x-a| < \delta = 1$, implies $|f(x)-k| = |k-k| = 0 < \varepsilon$. Hence, $\delta = 1$ works for any ε. ☺

Example 1.3.5. Let $a \in \mathbb{C}$. If $f(x) := x$ for all $x \in \mathbb{C}$, then $f(x) \to a$ as $x \to a$.

Proof. Let $\varepsilon > 0$ be given. Let $\delta := \varepsilon$, then $\delta > 0$ and

$$0 \leq |x-a| < \delta \implies |f(x)-a| = |x-a| < \delta = \varepsilon.$$

Hence, $\delta = \varepsilon$ works. ☺

In order to analyze more interesting examples we need some techniques. Using (1.6) is often intuitive, since it allows us a pictorial argument. (The reader may want to illustrate Remarks 1.3.2 and 1.3.3 using drawings of appropriate balls.) However, in concrete cases (1.4) is often more useful because it allows us to "solve" $|f(x)-L| < \varepsilon$ for δ. This is illustrated in the examples below.

Example 1.3.6. Let $f : \mathbb{C} \to \mathbb{C}$ be determined by $f(x) := 3x-5$. Then $f(x) \to 7$ as $x \to 4$.

Proof. To prove this, let $\varepsilon > 0$ be given. We need to find a $\delta > 0$, such that $|f(x) - 7| < \varepsilon$, when $0 < |x-4| < \delta$. A useful algebraic manipulation is

$$|f(x)-7| = |3x-12| = 3\,|x-4| \tag{1.7}$$

since this displays $|x-4|$ as a factor of $|f(x)-7|$. The choice of δ will control the size of $|x-4|$. Our goal

$$|f(x)-7| < \varepsilon$$

takes the form

$$3|x-4| < \varepsilon.$$

Dividing by 3, gives

$$|x-4| < \frac{\varepsilon}{3}.$$

So, let $\delta := \varepsilon/3$. Then $\delta > 0$ since $\varepsilon > 0$. Having constructed a δ, we must verify that it works. Using (1.7) it follows that

$$|x-4| < \delta \implies 3|x-4| < 3\delta \implies |f(x)-7| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Thus $\delta = \varepsilon/3$ works. ☺

Having constructed δ, one should in principle verify that it works, as we did above. This step is often left for the reader.

Example 1.3.7. Let $f : \mathbb{C} \to \mathbb{C}$ be determined by $f(x) := x^2$. Then $f(x) \to 9$ as $x \to 3$.

Proof. To prove this, let $\varepsilon > 0$ be given. We need to find a $\delta > 0$, such that $|f(x) - 9| < \varepsilon$, when $0 < |x-3| < \delta$. A useful algebraic manipulation is

$$|f(x)-9| = |x+3|\,|x-3| \tag{1.8}$$

since this displays $|x-3|$ as a factor of $|f(x)-9|$. The choice of δ will control the size of $|x-3|$, but before we can choose δ, we need to control the size of $|x+3|$. Since we want x to be close to 3, we can restrict our deliberations to $|x-3| < 1$ and see what that says about $|x+3|$. By the triangle inequality

$$|x+3| = |x-3+6| \le |x-3| + |6| < 1+6 = 7$$

when $|x-3| < 1$. Hence, using (1.8)

$$|x-3| < 1 \implies |f(x)-9| \le 7|x-3|.$$

The last inequality suggests we want $7|x-3| < \varepsilon$. In fact, if $|x-3| < 1$ and $|x-3| < \varepsilon/7$, then

$$|f(x)-9| \le 7|x-3| < 7\frac{\varepsilon}{4} = \varepsilon.$$

Thus $\delta := \min\{1, \varepsilon/7\}$ works. ☺

The choice of the 1 in $|x-3| < 1$ above, was arbitrary. Replacing that 1 by any other positive number would work in this example.

The proof above could be written using neighborhood notation, to summarize:

$$\begin{aligned} x \in B_1(3) &\implies |x+3| < 7 \text{ and} \\ x \in B_{\varepsilon/7}(3) &\implies |x-3| < \varepsilon/7 \text{ hence} \\ x \in B_1(3) \cap B_{\varepsilon/7}(3) &\implies |x+3|\,|x-3| < 7(\varepsilon/7). \end{aligned}$$

The assignment $\delta := \min\{1, \varepsilon/7\}$ then results from

$$B_1(3) \cap B_{\varepsilon/7}(3) = B_{\min\{1,\varepsilon/7\}}(3).$$

Example 1.3.8. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be determined by $f(x) := \frac{1}{x}$. Then $f(x) \to 2$ as $x \to \frac{1}{2}$.

Proof. Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$, such that $|f(x) - 2| < \varepsilon$, when $0 < \left|x - \frac{1}{2}\right| < \delta$. A useful, since it displays $|x - 1/2|$ as a factor of $|f(x) - 2|$, algebraic manipulation is

$$\begin{aligned}
|f(x) - 2| &= \left|\frac{1}{x} - 2\right| \\
&= \left|\frac{1-2x}{x}\right| \\
&= \left|\frac{2}{x}\right| \left|x - \frac{1}{2}\right|.
\end{aligned}$$

Since we want x to be close to $\frac{1}{2}$, we can restrict our deliberations to $\left|x - \frac{1}{2}\right| < 1$ and see what that says about $\left|\frac{2}{x}\right|$.We would like to find a number $M > 0$ ($M = 7$ worked in the previous example), such that $\left|\frac{2}{x}\right| < M$, that is, such that $2 < M|x|$, when $\left|x - \frac{1}{2}\right| < 1$. Unfortunately, this does not work. In fact, $\left|x - \frac{1}{2}\right| < 1$ is satisfied by values of x that are arbitrarily close to 0, and $2 < M|x|$ fails, when $|x| < 2/M$. ☺

Exercise 1.3.9. Repair the proof by considering $\left|x - \frac{1}{2}\right| < \frac{1}{4}$ in place of $\left|x - \frac{1}{2}\right|$<1.

Example 1.3.10 (Pseudo-sine Function). We have not yet constructed the trigonometric functions. This example introduces a function that roughly behaves like the sine function. Let

$$f(x) := \begin{cases} 4x(1-x) & \text{when } 0 \leq x \leq 1 \\ 4x(1+x) & \text{when } -1 \leq x \leq 0 \end{cases}$$

Note f is an odd function: $f(-x) = -f(x)$. Extend f to all of $\mathbb{R}$ as a periodic function σ with period 2 :

$$\sigma(x + 2n) := f(x), \text{ when } x \in [-1,1] \text{ and } n \in \mathbb{Z}.$$

σ is well-defined because $f(-1) = f(1)$.

The graph of σ roughly looks like the sine function, more precisely, roughly like the graph of $\sin(\pi x)$, see Fig. 1.4. We call σ the *pseudo-sine function*. The sine function is constructed in Sect. 11.2.

Exercise 1.3.11. Let $g(x) := \sigma(1/x)$ when $x \neq 0$. Suppose every real number $y \geq 0$ has a square root (this is established in Theorem 3.5.1). Sketch proofs of:

(*i*) For all $\delta > 0$, $\sigma(\{x \in \mathbb{R} \mid 0 < |x - 0| < \delta\}) \supseteq \{-1, 1\}$. See Fig. 1.5.
(*ii*) The limit $\lim_{x \to 0} g(x)$ does not exists.

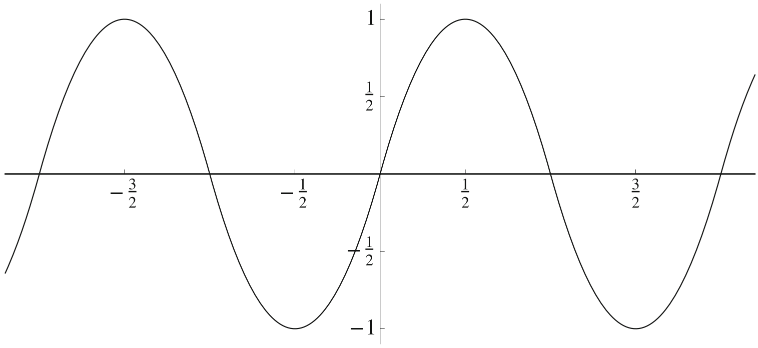

Fig. 1.4 Pseudo-sine function

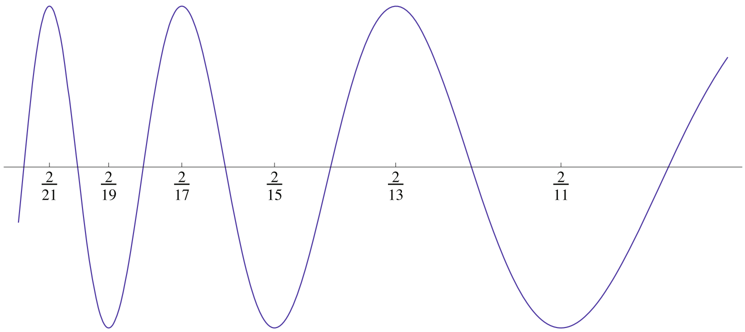

Fig. 1.5 A part of the graph of $\sigma(1/x)$ with some important values of x indicated

Some Consequences of the Existence of a Limit

The results in this subsection may appear simple. But they are very useful when we want to use the existence of some limit(s) to establish the existence of related limits. They also serve to give us some experience using the assumption $f(x) \to L$ as $x \to a$ in proofs.

In the following D is some subset of $\mathbb{C}$ and a is an accumulation point of D.

We say $f : D \to \mathbb{C}$ has a property P *near* a, if there is a $\delta > 0$, such that $f(x)$ satisfies P for all x in $D \cap B'_\delta(a)$. If f has a limit at a, then f is bounded near a. More precisely:

Theorem 1.3.12 (Local Boundedness). *Let* $f : D \to \mathbb{C}$. *If* $\lim_{x\to a} f(x)$ *exists, then there is a* $\delta > 0$ *and an* $M > 0$, *such that for all* $x \in D$, $0 < |x-a| < \delta \implies |f(x)| \leq M$.

Proof. Suppose $f(x) \to L$ as $x \to a$. Let $\varepsilon := 1$. Since $f(x) \to L$ as $x \to a$, there is a $\delta > 0$, such that for all $x \in D$, $0 < |x-a| < \delta \implies |f(x)-L| < 1$. Hence, for $x \in D$ with $0 < |x-a| < \delta$, we have

$$|f(x)| \leq |f(x)-L| + |L| < 1 + |L|.$$

Thus $M = 1 + |L|$ works (Fig. 1.6). ☺

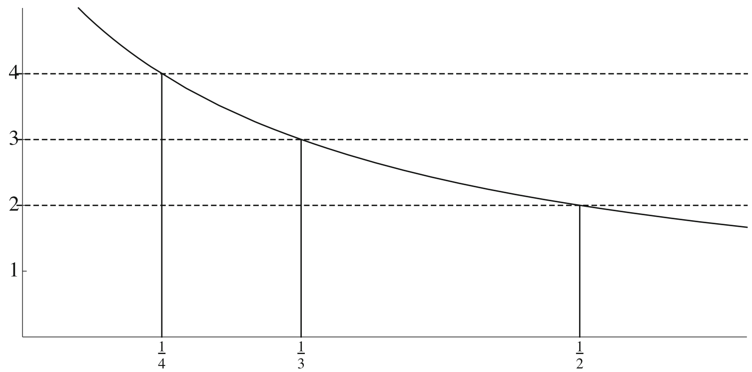

Fig. 1.6 Illustrating the proof of local boundedness (Theorem 1.3.12) in the case where $f(x) = 1/x$ and $a = 1/3$. In this case $L = 3$, hence $M = 1 + |L| = 4$ and the proof gives $\delta \le \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

Example 1.3.13. Suppose $f(x) := \frac{1}{x}$ and $a := \frac{1}{2}$. If $0 < \delta < \frac{1}{2}$, then any $M \ge \frac{2}{1-2\delta}$ satisfies the conclusions of the previous theorem. Note if $\delta = \frac{1}{2}$, then f is defined at all points with $\left|x - \frac{1}{2}\right| < \frac{1}{2}$, but there is no corresponding value for M. In fact, let $M > 0$ be given. Then

$$f\left(\frac{1}{M+2}\right) = M + 2 > M$$

and $1/(M+2) < 1/2$. In particular, we must choose M, i.e., ε, before we can determine δ. This is what we did in the proof of Theorem 1.3.12, and it corresponds to the structure of our definition (1.4) of limits.

If a real valued function has a positive limit as $x \to a$, then the function is positive near a. More precisely:

Theorem 1.3.14 (Local Positivity). *If $f : D \to \mathbb{R}$ is a function such that $f(x) \to L$ as $x \to a$ and $L > 0$, then there is a $\delta > 0$, such that for all $x \in D$, $0 < |x - a| < \delta \implies \frac{L}{2} < f(x)$.*

Proof. Let $\varepsilon := L/2$. Then $\varepsilon > 0$ since $L > 0$. Since $f(x) \to L$ as $x \to a$, there is a $\delta > 0$, such that for all $x \in D$, $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon = L/2$. For $x \in D$, with $0 < |x - a| < \delta$ we have

$$\begin{aligned} f(x) &= L - (L - f(x)) \\ &\ge L - |L - f(x)| \\ &> L - \varepsilon \\ &= \frac{L}{2}. \end{aligned}$$

This calculation completes the proof (Fig. 1.7). ☺

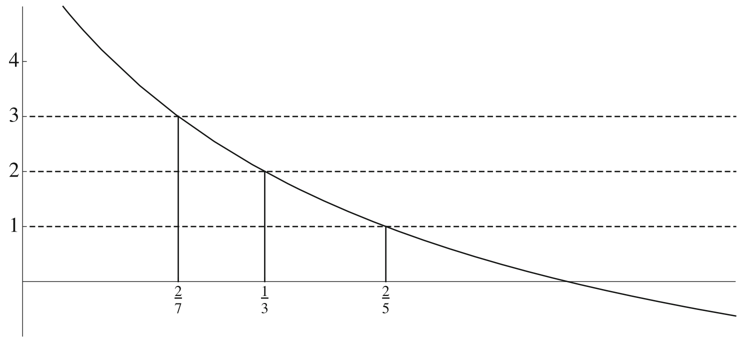

Fig. 1.7 Illustrating the proof of local positivity (Theorem 1.3.14) in the case where $f(x) = \frac{2}{x} - 4$ and $a = 1/3$. In this case $L = 2$, hence $\varepsilon = L/2 = 1$ and the proof gives $\delta \leq \frac{1}{3} - \frac{2}{7} = \frac{1}{21}$

Exercise 1.3.15. If $f : D \to \mathbb{C}$, $f(x) \to L$ as $x \to a$, and $L \neq 0$, then there is a $\delta > 0$, such that for all $x \in D$, $0 < |x - a| < \delta \implies \frac{|L|}{2} < |f(x)|$.

1.4 Calculating with Limits

The following allows us to perform basic algebra with limits. All the ideas needed for the proofs are present in the examples above.

Theorem 1.4.1 (Linearity). *Let D be a subset of $\mathbb{C}$, $f, g : D \to \mathbb{C}$, and $x_0, a, b, L, M \in \mathbb{C}$. Suppose x_0 is an accumulation point of D. If*

$$f(x) \to L \text{ and } g(x) \to M \text{ as } x \to x_0,$$

then

$$(af + bg)(x) \to aL + bM \text{ as } x \to x_0.$$

Here $(af + bg)(x) := af(x) + bg(x)$ for $x \in D$.

Linearity is a direct consequence of the constant multiple and sum rules below.

Lemma 1.4.2 (Constant Multiple Rule). *Let D be a subset of $\mathbb{C}$, $f : D \to \mathbb{C}$, and $a, k, L \in \mathbb{C}$. If a is an accumulation point of D and $f(x) \to L$ as $x \to a$, then $(kf)(x) \to kL$ as $x \to a$.*

Proof. Let $\varepsilon > 0$ be given. The proof is based on the equality

$$|(kf)(x) - kL| = |k|\,|f(x) - L|.$$

Suppose $k \neq 0$. Since $\varepsilon/|k| > 0$ and $f(x) \underset{x\to a}{\longrightarrow} L$, there is a $\delta > 0$, such that, if $x \in D$, then $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon/|k|$. Multiplying the last inequality by $|k|$ yields, $0 < |x-a| < \delta \implies |kf(x) - kL| < \varepsilon$. Thus $(kf)(x) \underset{x\to a}{\longrightarrow} kL$. ☺

Example 1.4.3. If $a := 5$, $k := 3$ and $f(x) := x$, we get $3x \to 3\cdot 5 = 15$ as $x \to 5$.

Lemma 1.4.4 (Sum Rule). *Let D be a subset of $\mathbb{C}$ and $a, L, M \in \mathbb{C}$. Suppose $f, g : D \to \mathbb{C}$, a is an accumulation point of D, $f(x) \to L$ as $x \to a$, and $g(x) \to M$ as $x \to a$, then $(f+g)(x) \to L + M$ as $x \to a$.*

Proof. Let $\varepsilon > 0$ be given. The proof is based on the triangle inequality

$$|(f+g)(x) - (L+M)| \leq |f(x) - L| + |g(x) - M|.$$

Since $f(x) \underset{x\to a}{\longrightarrow} L$, there is a $\delta_1 > 0$, such that, if $x \in D$, then $x \in B'_{\delta_1}(a) \implies |f(x) - L| < \varepsilon/2$. Similarly, since $g(x) \underset{x\to a}{\longrightarrow} M$, there is a $\delta_2 > 0$, such that, if $x \in D$, then $x \in B'_{\delta_2}(a) \implies |g(x) - M| < \varepsilon/2$. Hence, $x \in B'_{\delta_1}(a) \cap B'_{\delta_2}(a)$ implies $|(f+g)(x) - (L+M)| < \varepsilon/2 + \varepsilon/2$. Thus $\delta := \min\{\delta_1, \delta_2\}$ works. ☺

Example 1.4.5. If $a = 5$, $f(x) := 7$ and $g(x) := x$, we get $7 + x \to 7 + 5 = 12$ as $x \to 5$.

Theorem 1.4.6 (Product Rule). *Let D be a subset of $\mathbb{C}$ and let $a, L, M \in \mathbb{C}$. Suppose $f, g : D \to \mathbb{C}$ and a is an accumulation point of D. If*

$$f(x) \to L \text{ and } g(x) \to M \text{ as } x \to a,$$

then

$$(fg)(x) \to LM \text{ as } x \to a.$$

Here $(fg)(x) := f(x)g(x)$.

Proof. Let $\varepsilon > 0$ be given. The proof is based on the triangle inequality

$$\begin{aligned}|(fg)(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM|\\ &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|.\end{aligned}$$

By Local Boundedness (Theorem 1.3.12) there is a constant $K > 0$ and a $\delta_1 > 0$, such that $|g(x| \leq K$, when $0 < |x-a| < \delta_1$. Hence

$$|(fg)(x) - LM| \leq K\,|f(x) - L| + |L|\,|g(x) - M|$$

when $0 < |x-a| < \delta_1$.

Using $f(x) \to L$, and $g(x) \to M$ as $x \to a$, we conclude there are $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2K} \text{ when } 0 < |x-a| < \delta_2$$
$$|g(x) - M| < \frac{\varepsilon}{2|L|+1} \text{ when } 0 < |x-a| < \delta_3$$

The $+1$ is there in case $L = 0$. Consequently, if $0 < |x-a| < \min\{\delta_1, \delta_2, \delta_3\}$, then the last three inequalities displayed above all hold. Plugging the last two of these into the first gives us

$$|(fg)(x) - LM| < K\frac{\varepsilon}{2K} + |L|\frac{\varepsilon}{2|L|+1} < \varepsilon.$$

Thus, $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ works. ☺

Exercise 1.4.7. Let $n \in \mathbb{N}$ and $a \in \mathbb{C}$. Show that $x^n \to a^n$ as $x \to a$.

A *polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

for some $n \in \mathbb{N}_0$ and some complex numbers a_k, $k = 0, 1, \ldots, n$. The *degree* of p is the largest subscript k such that $a_k \neq 0$.

Exercise 1.4.8. If $a \in \mathbb{C}$ and p is a polynomial, then $p(x) \to p(a)$ as $x \to a$.

Theorem 1.4.9 (Quotient Rule). *Let D be a subset of $\mathbb{C}$. Suppose $f, g : D \to \mathbb{C}$, and $a, L, M \in \mathbb{C}$. Let a be an accumulation point of D. If $M \neq 0$ and*

$$f(x) \to L \text{ and } g(x) \to M \text{ as } x \to a,$$

then

$$\frac{f}{g}(x) \to \frac{L}{M} \text{ as } x \to a.$$

Here $\frac{f}{g}(x) := \frac{f(x)}{g(x)}$ for $x \in D$.

Proof. We showed in Exercise 1.3.15 above that $|g(x)| > |M|/2$ for all $x \in D$ near a. In particular, $g(x) \neq 0$ for all $x \in D$ near a. Consequently, $f(x)/g(x)$ makes sense near a.

Let $\varepsilon > 0$ be given. The proof is based on the triangle inequality

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \left|\frac{f(x)}{g(x)} - \frac{L}{g(x)} + \frac{L}{g(x)} - \frac{L}{M}\right| \\
&= \left|\frac{f(x)}{g(x)} - \frac{L}{g(x)} + L\frac{M - g(x)}{Mg(x)}\right| \\
&\leq \left|\frac{1}{g(x)}\right| |f(x) - L| + \left|\frac{L}{M}\right| \left|\frac{1}{g(x)}\right| |M - g(x)|.
\end{aligned}$$

Exercise 1.4.10. Complete the proof of the Quotient Rule.

☺

The derivation of the inequalities that form the basis for the proofs of the rules above are all based on the desire to find expressions we know something about, for example, in the proof of the quotient rule we began by looking for the factor $|f(x)-L|$ and then we "found" the factor $|g(x)-M|$.

A *rational function* is the ratio of two polynomials,

$$f(x) := \frac{p(x)}{q(x)}$$

where p and q are polynomial functions. If $q(a) \neq 0$, then the quotient rule shows that

$$\frac{p(x)}{q(x)} \to \frac{p(a)}{q(a)} \text{ as } x \to a.$$

Actually, this requires that a is an accumulation point of $\{x \in \mathbb{C} \mid q(x) \neq 0\}$. But, this is true because, the set of roots $\{x \in \mathbb{C} \mid q(x) = 0\}$ is finite:

Proposition 1.4.11. *If p is a polynomial of degree n, then p has at most n roots.*

This is part of the Fundamental Theorem of Algebra. The full version of the Fundamental Theorem of Algebra is established in Sect. 9.4. We precede the proof by two lemmas. The idea of the proof of the first lemma is used at several points in the text.

Lemma 1.4.12. *If p is a polynomial of degree n and z_0 is a constant, then $q(z) := p(z+z_0)$ is a polynomial of degree n.*

Proof. Write $p(z) = \sum_{k=0}^{n} a_k z^k$. Then

$$q(z) := p(z+z_0) = \sum_{k=0}^{n} a_k (z+z_0)^k,$$

where $a_n \neq 0$. Expanding each $(z+z_0)^k$ completes the proof. We include one version of the details below. Simply expanding everything using the Binomial Theorem and collection the coefficients to the powers of z is an alternative way to finish the argument

Expanding $(z+z_0)^k$, we see $r_k(z) := (z+z_0)^k$ is a polynomial of degree k. Since the sum of two polynomials of degree $\leq m$ is a polynomial of degree $\leq m$, we conclude

$$q(z) = \sum_{k=0}^{n} a_k r_k(z) = a_n r_n(z) + \sum_{k=0}^{n-1} a_k r_k(z)$$

is a polynomial of degree $\leq n$. The polynomial $a_n r_n(z)$ has degree n since r_n has degree n and $a_n \neq 0$. And the polynomial $\sum_{k=0}^{n-1} a_k r_k(z)$ has degree $\leq n-1$, since it is a sum of polynomials whose degrees all are $\leq n-1$. Consequently, the only term in $q(z)$ containing z^n is the term in $a_n r_n(z)$.Thus q has degree n. ☺

Lemma 1.4.13. *If p is a polynomial of degree $n+1$ and $p(z_0)=0$, then there is a polynomial q of degree n, such that $p(z)=(z-z_0)q(z)$.*

Proof. Suppose p is a polynomial of degree $n+1$ and $p(z_0)=0$. Let $q(z):=p(z+z_0)$. By the first lemma q is a polynomial of degree $n+1$. Write $q(z)=\sum_{k=0}^{n+1} b_k z^k$. Now

$$b_0 = q(0) = p(0+z_0) = 0.$$

Hence, z is a common factor of the terms in $q(z)$:

$$q(z) = \sum_{k=0}^{n+1} b_k z^k = \sum_{k=1}^{n+1} b_k z^k = z\sum_{k=1}^{n+1} b_k z^{k-1}.$$

So, $r(z) := \sum_{k=1}^{n+1} b_k z^{k-1}$ is a polynomial of degree n and $q(z)=zr(z)$. Finally,

$$p(z) = q(z-z_0) = (z-z_0)r(z-z_0) = (z-z_0)s(z),$$

where, $s(z) := r(z-z_0) = r(z+(-z_0))$ is a polynomial of degree n, by the first lemma. ☺

We are now ready for the proof of Proposition 1.4.11.

Proof. Let R_p be the set of roots of the polynomial p. That is $R_p = \{z \mid p(z)=0\}$. We must show that if p has degree n, then R_p has at most n elements.

The proof is by induction on the degree of the polynomial. If p has degree one, then $p(x)=a+bx$, where $b\neq 0$. Hence, $R_p=\{-a/b\}$. This set has one element.

Let $n\in\mathbb{N}$. Suppose any polynomial of degree n has at most n roots. Let p be some polynomial of degree $n+1$. If $R_p=\emptyset$, then we are done, since $0\le n+1$. If $R_p\neq\emptyset$, let $z_0\in R_p$. Then $p(z_0)=0$. Hence, by the second lemma, there is a polynomial q of degree n, such that $p(z)=(z-z_0)q(z)$. Since $p(z)=(z-z_0)q(z)$ we have $R_p=\{z_0\}\cup R_q$. By the inductive hypothesis R_q has at most n elements, hence R_p has at most $n+1$ elements. ☺

Theorem 1.4.14 (Composition Rule). *Let A, B, and C be subsets of $\mathbb{C}$. Suppose $f:A\to B$, $g:B\to C$, a is an accumulation point of A, b is an accumulation point of B, and $f(x)\neq b$ when $x\neq a$ is close to a. If*

$$f(x)\to b \text{ as } x\to a \text{ and } g(x)\to c \text{ as } x\to b,$$

then

$$g\circ f(x)\to c \text{ as } x\to a.$$

Here, $g\circ f:A\to C$ is determined by $g\circ f(x):=g(f(x))$. And $f(x)\neq b$ when $x\neq a$ is close to a, means that there is a $\gamma>0$, such that $\forall x\in A$, $0<|x-a|<\gamma \implies f(x)\neq b$.

Proof. Let $\varepsilon>0$ be given. Since $g(y)\to c$ as $y\to b$, there is a $\delta_1>0$ be such that

$$0<|y-b|<\delta_1 \implies |g(y)-c|<\varepsilon.$$

Since $f(x) \to b$ as $x \to a$, there is a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |f(x)-b| < \delta_1.$$

Let $\gamma > 0$ be such that $0 < |x-a| < \gamma \implies f(x) \neq b$. Let $\delta := \min\{\gamma, \delta_2\}$, then

$$\begin{aligned} 0 < |x-a| < \delta &\implies 0 < |f(x)-b| < \delta_1 \\ &\implies |g(f(x))-c| < \varepsilon. \end{aligned}$$

As we needed to show. ☺

Exercise 1.4.15. Let $D := \mathbb{R}$. Supposing $\sqrt{x} \to 2$ as $x \to 4$, explain why the Composition Rule can be used to show $\sqrt{1+x} \to 2$ as $x \to 3$.

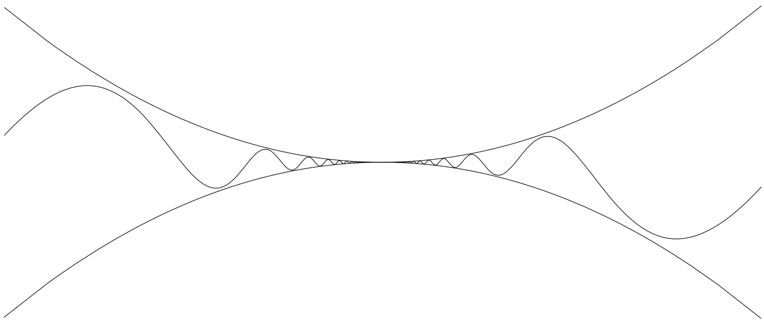

Fig. 1.8 Illustrating the Squeeze Rule (Theorem 1.4.16)

Theorem 1.4.16 (Squeeze Rule). *Let $f, g, h : D \to \mathbb{R}$. Suppose $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ near x_0. If*

$$f(x) \to L \text{ and } h(x) \to L \text{ as } x \to x_0,$$

then

$$g(x) \to L \text{ as } x \to x_0.$$

See Fig. 1.8

Proof. Let $\varepsilon > 0$ be given. Let $\delta_f > 0$ and $\delta_h > 0$ be such that $x \in B'_{\delta_f}(a)$ implies $|f(x)-L| < \varepsilon$ and $x \in B'_{\delta_h}(a)$ implies $|h(x)-L| < \varepsilon$. Let $\delta := \min\{\delta_f, \delta_h\}$. Then $x \in B'_{\delta}(a)$ implies $x \in B'_{\delta_f}(a)$ and $x \in B'_{\delta_h}(a)$ hence $-\varepsilon < f(x)-L < \varepsilon$ and $-\varepsilon < h(x)-L < \varepsilon$. Hence, for $x \in B'_{\delta}(a)$ we have

$$-\varepsilon < f(x)-L \leq g(x)-L \leq h(x)-L < \varepsilon.$$

Consequently, $0 < |x-a| < \delta$ implies $-\varepsilon < g(x)-L < \varepsilon$. As we needed to show. ☺

Example 1.4.17. Let $f(x) := \frac{x}{1+x^2}$ be defined on $D := \mathbb{R}$. Then $-|x| \leq f(x) \leq |x|$. Since $|x| \to 0$ as $x \to 0$, the Squeeze Rule implies $\frac{x}{1+x^2} \to 0$ as $x \to 0$.

Restrictions

Let f be a function defined on D. Sometimes it is useful to consider f on some subset of its domain. This is called a restriction of f. If $E \subseteq D$, we define the *restriction* of f to E, to be the function $f\big|_E$ with domain E and having the same values as f on E. Hence $f\big|_E(x) := f(x)$ for all $x \in E$ and $f\big|_E(x)$ is not defined when $x \notin E$.

Exercise 1.4.18. Suppose $D = D_1 \cup D_2$.

(i) If a is an accumulation point of D_1 or of D_2, then a is an accumulation point of D.

(ii) If a is an accumulation point of D, then a is an accumulation point of D_1 or of D_2.

Theorem 1.4.19. *Suppose $D = D_1 \cup D_2$. Let a be an accumulation point of D_1 and of D_2. Let $f : D \to \mathbb{C}$ be a function.*

1. *If $f(x) \to L$ as $x \to a$, then $f\big|_{D_1}(x) \to L$ as $x \to a$.*
2. *If $f\big|_{D_1}(x) \to L$ as $x \to a$ and $f\big|_{D_2}(x) \to L$ as $x \to a$, then $f(x) \to L$ as $x \to a$.*

Proof. 1. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that

$$\forall x \in D, 0 < |x-a| < \delta \implies |f(x) - b| < \varepsilon.$$

For $x \in D_1$ we have $x \in D_1 \cup D_2 = D$ and

$$\begin{aligned} 0 < |x-a| < \delta &\implies |f(x) - b| < \varepsilon \\ &\implies \left|f\big|_{D_1}(x) - b\right| < \varepsilon -- \text{ since } f\big|_{D_1}(x) = f(x). \end{aligned}$$

Hence the δ that works for f also works for $f\big|_{D_1}$.

2. Let $\varepsilon > 0$ be given. For $j = 1, 2$, there are $\delta_j > 0$, such that

$$\forall x \in D_j, 0 < |x-a| < \delta_j \implies \left|f\big|_{D_j}(x) - b\right| < \varepsilon.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Let $x \in D$ with $0 < |x-a| < \delta$. Since $D = D_1 \cup D_2$ either $x \in D_1$ or $x \in D_2$.

If $x \in D_1$, then

$$\begin{aligned} 0 < |x-a| < \delta &\implies 0 < |x-a| < \delta_1 -- \text{ since } \delta \leq \delta_1 \\ &\implies \left|f\big|_{D_j}(x) - b\right| < \varepsilon -- \text{ construction of } \delta_1 \\ &\implies |f(x) - b| < \varepsilon -- \text{ since } f\big|_{D_1}(x) = f(x). \end{aligned}$$

The case $x \in D_2$ is similar. ☺

Corollary 1.4.20. *Suppose $D = D_1 \cup D_2$. Let a be an accumulation point of D_1 and of D_2. If $f : D \to \mathbb{C}$ is a function, then*

$$\lim_{x\to a} f \text{ exists}$$
$$\textit{iff}$$
$$\lim_{x\to a} f\big|_{D_j}(x),\ j = 1,2 \text{ both exists and are equal.}$$

For a set A let

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

The function $\mathbb{1}_A$ is called the *characteristic* function of A.

Example 1.4.21 (Dirichlet Function). Let $f : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the set of rational, $f := \mathbb{1}_{\mathbb{Q}}$. Let a be a real number. By density of the rationals and of the irrationals a is an accumulation point of $\mathbb{Q}$ and of $\mathbb{R} \setminus \mathbb{Q}$. Clearly,

$$\lim_{x\to a} f\big|_{\mathbb{Q}}(x) = \lim_{x\to a} 1 = 1, \text{ and}$$
$$\lim_{x\to a} f\big|_{\mathbb{R}\setminus\mathbb{Q}}(x) = \lim_{x\to a} 0 = 0.$$

Since $0 \neq 1$, Corollary 1.4.20 with $D_1 := \mathbb{Q}$ and $D_2 := \mathbb{R} \setminus \mathbb{Q}$, shows that f does not have a limit at the point a. Since a was arbitrary, f does not have a limit at any point.

1.5 Variations on Limits

In this section $D \subseteq \mathbb{R}$. We consider one-sided limits, limits at infinity, and infinite limits. The treatment is brief; most of the details are very similar to the ones for "ordinary" limits.

One-Sided Limits

When considering the limit of $\sqrt{x}$ at $a = 0$, we are necessarily only considering positive values for x. This is an example of a one-sided limit. But the real interest in one-sided limits is when f is defined on both sides of a. When considering one-sided limits, it is convenient to use one-sided neighborhoods

$$B_r^+(a) := B_r'(a) \cap]a, \infty[= \{x \mid a < x < a + r\}$$
$$B_r^-(a) := B_r'(a) \cap]-\infty, a[= \{x \mid a - r < x < a\}$$

Limits from *above* (from the *right*) are denoted by

$$\lim_{x \searrow a} f(x) = b \text{ or } f(x) \to b \text{ as } x \searrow a$$

and means that, a is an accumulation point of $D \cap]a, \infty[$ and

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x \in B^+_\delta(a) \implies |f(x) - a| < \varepsilon.$$

Sometimes $x \searrow a$ is written as $x \to a+$. Limits from *below* (from the *left*) are defined in a similar manner. The reader will not experience any difficulty in transcribing the results in Sects. 1.3 and 1.4 to the one-sided case. In fact, one-sided limits are really the same as limits of restrictions. For example, the considering $\lim_{x \searrow a} f(x)$ is the same as considering $\lim_{x \to a} f\big|_{D \cap]a,\infty[}(x)$.

Exercise 1.5.1. The two-sided limit of f exists at a iff both one-sided limits of f exists at a and are equal.

Example 1.5.2. As an example of one-sided limits consider

$$f(x) := \frac{|x-2|\,|x+2|}{x-2}$$

on $D := \{x \in \mathbb{R} \mid x \neq 2\}$. This is an example where f is not defined at the point of interest, in this case at $a = 2$. Since

$$f(x) = \begin{cases} |x+2| & \text{when } 2 < x \\ -|x+2| & \text{when } x < 2 \end{cases}$$

we have $\lim_{x \searrow 2} f(x) = 4$ and $\lim_{x \nearrow 2} f(x) = -4$. Consequently, $\lim_{x \to 2} f(x)$ does not exist.

Limits at Infinity

A limit at infinity is essentially a kind of one-sided limit. In fact,

$$\lim_{x \to \infty} f(x) = L \text{ or } f(x) \to L \text{ as } x \to \infty$$

means that D contains arbitrarily large real numbers (roughly ∞ is an accumulation point of D, precisely, $\forall N \in \mathbb{R}, \exists x \in D, N < x$) and

$$\forall \varepsilon > 0, \exists N \in \mathbb{R}, \forall x \in D, N < x \implies |f(x) - L| < \varepsilon.$$

That is, if we can arrange that $f(x)$ is as close to L as we please, by choosing x sufficiently large. Writing $x \nearrow \infty$ is consistent with writing $x \nearrow a$, but is redundant, since there in only one way to approach ∞. Similarly,

$$\lim_{x\to-\infty} f(x) = L \text{ or } f(x) \to L \text{ as } x \to -\infty$$

means that D contains arbitrarily small real numbers and

$$\forall \varepsilon > 0, \exists N \in \mathbb{R}, \forall x \in D, x < N \implies |f(x) - L| < \varepsilon.$$

Since the limits at infinity are very similar to one-sided limits, the reader will not experience any difficulty in transcribing the results in Sects. 1.3 and 1.4 to the cases of limits at infinity.

Example 1.5.3. $\frac{1}{x^2} \to 0$ as $x \to \infty$.

Proof. Let $\varepsilon > 0$ be given. We want to find N such that $N < x \implies \frac{1}{x^2} < \varepsilon$. Rewriting

$$\frac{1}{x^2} < \varepsilon \text{ as}$$
$$\frac{1}{\varepsilon} < x^2$$

we see $N := \frac{1}{\sqrt{\varepsilon}}$ does the job. ☺

Exercise 1.5.4. Let $n \in \mathbb{N}$. Prove

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ and}$$
$$\lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Infinite Limits

Suppose a is an accumulation point of D and $f : D \to \mathbb{R}$. We write

$$\lim_{x\to a} f(x) = \infty \text{ or } f(x) \to \infty \text{ as } x \to a$$

if

$$\forall N \in \mathbb{R}, \exists \delta > 0, \forall x \in D, 0 < |x-a| < \delta \implies N < f(x).$$

Clearly, it is sufficient to consider large N, e.g., $N \geq 1$. In this case, we will not say that f converges, rather we say f *diverges* to infinite as $x \to a$.

Exercise 1.5.5. If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Similarly, we can define $\lim_{x\to a} f(x) = -\infty$, as well as one-sided infinite limits and four variants involving infinite limits at infinity: $\lim_{x\to\pm\infty} f(x) = \pm\infty$. For example,

$\lim_{x\to\infty} f(x) = \infty$ means D contains arbitrary large numbers and

$$\forall N, \exists M, \forall x \in D, x > M \implies f(x) > N.$$

Appropriate versions of the rules for calculating with limits remain valid. Some care is needed, in addition to $0/0$, expressions of the forms $0 \cdot \infty$, ∞/∞, and $\infty - \infty$ should be avoided. While others, for example $\infty + \infty = \infty$, $a + \infty = \infty$, $0/\infty = 0$, and if $a > 0$, $a\infty = \infty$ are valid.

Example 1.5.6. We calculate the limits at infinity of polynomials. Let $p(z) := a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$ be a polynomial, suppose $n \geq 1$ and $a_n \neq 0$. For $z \neq 0$ we can rewrite $p(z)$ as

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right).$$

- Considering a real variable x we see

$$p(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

  Using $\lim_{x\to\pm\infty} 1/x^k = 0$, shows that the expression in the parenthesis $\to a_n$ as $x \to \pm\infty$. Hence, $|p(x)| \to \infty$ as $x \to \pm\infty$.
- For a complex variable z, we must modify this slightly:

$$\begin{aligned} |p(z)| &= |z|^n \left| a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \\ &\geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \frac{|a_{n-2}|}{|z|^2} - \cdots - \frac{|a_1|}{|z|^{n-1}} - \frac{|a_0|}{|z|^n} \right) \end{aligned}$$

  since the expression in the parenthesis $\to |a_n|$ as $|z| \to \infty$, we have $|p(z)| \to \infty$ as $|z| \to \infty$.

Example 1.5.7. The limits at infinity of rational functions can be calculated in a similar manner. For example, if

$$\begin{aligned} p(x) &:= a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x + a_0 \\ q(x) &:= b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0 \end{aligned}$$

where $a_m \neq 0$ and $b_n \neq 0$ are real numbers, then

$$\frac{p(x)}{q(x)} = \frac{x^m}{x^n} \cdot \frac{a_m + \frac{a_{m-1}}{x} + \frac{a_{m-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}{b_n + \frac{b_{n-1}}{x} + \frac{b_{n-2}}{x^2} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}}.$$

Consequently,

$$\frac{p(x)}{q(x)} \to \begin{cases} 0 & \text{if } m < n \\ \frac{a_m}{b_n} & \text{if } m = n \\ \pm\infty & \text{if } m > n \end{cases}$$

as $x \to \infty$. The sign in $\pm\infty$ equals the sign of a_m/b_n.

1.6 Sequences

In this section we will consider the limit $\lim_{x\to\infty} f$ in the case $D := \mathbb{N}$. Hence, essentially this section is a special case limits at infinity considered in Sect. 1.5. We include some proofs here since we left the corresponding proofs to the reader in Sect. 1.5.

A *sequence* of complex numbers is a function $x : \mathbb{N} \to \mathbb{C}$. When working with sequences we will usually write x_n in place of $x(n)$ and (x_n) or $(x_n)_{n=1}^{\infty}$ in place of x, or $x : \mathbb{N} \to \mathbb{C}$. The definition of a limit at infinite specializes to: (x_n) converges to x, if given any $\varepsilon > 0$, there is a K, such that $|x_n - x| < \varepsilon$ for all $n > K$.

A sequence (x_n) is *bounded*, if there is a K, such that $|x_n| \le K$ for all n.

Example 1.6.1. The sequence $(1 + (-1)^n)$ is bounded. For example, $K = 2$ and $K = 17$ both work.

Example 1.6.2. The sequence (n) is not bounded. Since, given any real number K there are integers larger than K.

Convergence of Sequences

We establish local boundedness and positivity for sequences. Essentially, this is contained in Sects. 1.3 and 1.5. We include proofs here for the readers convenience. The following is Theorem 1.3.12 (local boundedness) specialized to sequences.

Theorem 1.6.3. *A convergent sequence is bounded.*

Proof. Suppose $x_n \to x$. Since $\varepsilon := 1 > 0$ there is an N such that $n > N$ implies $|x_n - x| < 1$. Since

$$|x_n| = |x_n - x + x| \le |x_n - x| + |x| < 1 + |x|$$

when $n > N$, $K := \max\{|x_1|, |x_2|, \ldots, |x_N|, 1\}$ has the desired property. ☺

Specializing the exercise after Theorem 1.3.14 (local positivity) to sequences yields:

Theorem 1.6.4. *Suppose $x_n \to p$ and $p \neq 0$. There is an N, such that $n > N$ implies $|x_n| > |p|/2$.*

Proof. Since $|p|/2 > 0$, there is an N, such that $n > N$ implies $|x_n - p| < |p|/2$. Since

$$|x_n| = |p - (x_n - p)| \geq |p| - |x_n - p| > |p|/2$$

when $n > N$, we are done. ☺

Theorem 1.3.14 itself specializes to:

Exercise 1.6.5. Suppose (x_n) is a sequence of real numbers and $x_n \to p$. If $p > 0$, show there is an N, such that $n > N$ implies $x_n > p/2$.

Having established the two basic properties of convergent sequences, we can repeat the proofs about the algebra of limits of functions from Sect. 1.4 to establish:

Theorem 1.6.6. *Suppose $a_n \to a$, $b_n \to b$, and $k \in \mathbb{C}$. Then*

1. $ka_n \to ka$
2. $a_n + b_n \to a + b$
3. $a_n b_n \to ab$
4. $\frac{a_n}{b_n} \to \frac{a}{b}$, *provided* $b \neq 0$
5. *If* $a_n \leq b_n \leq c_n$ *for all* n, $a_n \to a$ *and* $c_n \to a$, *then* $b_n \to a$.

If (a_n) is a sequence of real numbers, then we will write $a_n \to \infty$, if given any real number K, there is an N, such that, for any integer n, $n > N \implies a_n > K$.

Similarly, we write $a_n \to -\infty$, if given any real number K, there is an N such that for any integer n, $n > N \implies a_n < K$.

As for limits of functions appropriate versions of the theorem above are valid for infinite limits. See Sect. 1.5.

Null Sequences

A sequence (x_n) is *null,* if given any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $n > N \implies |x_n| < \varepsilon$. Hence (x_n) is null iff $x_n \to 0$. Clearly, $x_n \to x$ iff $(x_n - x)$ is null.

Example 1.6.7. $(1/n)$ is null.

Proof. Let $\varepsilon > 0$. Let $N := \lceil 1/\varepsilon \rceil$ be the *ceiling* of $1/\varepsilon$, that is the integer integer satisfying $N - 1 < 1/\varepsilon \leq N$. Then $n > N$ implies

$$0 < \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

Hence $(1/n)$ is null. ☺

Exercise 1.6.8. $(1 + (-1)^n)$ is not null.

Theorem 1.6.9. *If $M \in \mathbb{C}$ and (x_n) is null, then (Mx_n) is null.*

Proof. Let $\varepsilon > 0$ be given. Since $\varepsilon/(1 + |M|) > 0$ and (x_n) is null there is an N, such that $n \geq N \implies |x_n| < \varepsilon/(1 + |M|)$. Hence, $n \geq N \implies |Mx_n| \leq (1 + |M|)|x_n| < \varepsilon$. Thus, (Mx_n) is null. ☺

1.7 Geometric Progression

An arithmetic progression is obtained by continually adding the same number, hence a, $a+b$, $a+2b$, ... is a *arithmetic progression*. A geometric progression is obtained by continually multiplying by the same number, hence a, ab, ab^2, ab^3,... is a *geometric progression*. Geometric progressions are also called geometric sequences.

Our first result about geometric progressions states that a certain class of geometric progressions are null sequences:

Theorem 1.7.1. *Let $x \in \mathbb{R}$. Suppose $0 < x < 1$. For any real number $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that $n \geq N \implies x^n \leq \varepsilon$.*

Exercise 1.7.2 (Outline of a Proof of the Previous Theorem). *Fix $0 < x < 1$ and let $y := \frac{1}{x} - 1$. Let $0 < \varepsilon < 1$ be given.*

1. *Prove $y > 0$.*
2. *Use Bernoulli's inequality to prove $\left(\frac{1}{x}\right)^n \geq 1 + ny$ for all $n \in \mathbb{N}$.*
3. *Prove $0 < x^n \leq \frac{1}{1+ny}$ for all $n \in \mathbb{N}$.*
4. *Let $N := 1 + \left\lfloor \frac{\varepsilon}{y} \right\rfloor$. Prove $n \geq N \implies x^n < \varepsilon$. [If t is a real number, then the* floor *of t is the integer $\lfloor t \rfloor$ satisfying $\lfloor t \rfloor \leq t < \lfloor t \rfloor + 1$.]*

Hence, if $0 < x < 1$, then the sequence (x^n) is null.

Corollary 1.7.3. *Let z be a complex number such that $|z| < 1$. Then (z^n) is null.*

Proof. By the theorem $|z^n| = |z|^n \to 0$. ☺

Example 1.7.4. Let x be a positive real number with infinite decimal representation $x = d_0.d_1d_2\cdots$. If $x_n := d_0.d_1d_2\cdots d_n$ is the corresponding sequence of finite decimals, then $x_n \to x$.

Proof. Let $\varepsilon > 0$. Since $(1/10^n)$ is null we can pick an integer N such that $1/10^N < \varepsilon$. Then

$$\begin{aligned} 0 \leq x - x_n &= 0.0\cdots 0d_{n+1}d_{n+2}\cdots \\ &\leq 0.0\cdots 099\cdots = 1/10^{n.} \end{aligned}$$

Hence, if $n \geq N$, then $|x - x_n| \leq 10^{-n} \leq 10^{-N} < \varepsilon$. ☺

Series

For a sequence $(x_n)_{n=0}^{\infty}$ let

$$\sum_{k=0}^{n-1} x_k := x_0 + x_1 + \cdots + x_{n-2} + x_{n-1}.$$

Then $\left(\sum_{k=0}^{n-1} x_k\right)_{n=1}^{\infty}$ is a sequence. We define $\sum_{k=0}^{\infty} x_k$ in terms of convergence of this sequence. Let

$$\sum_{k=0}^{\infty} x_k := \lim_{n\to\infty} \sum_{k=0}^{n-1} x_k$$

provided the limit exists, in that case we say $\sum_{k=0}^{\infty} x_n$ is *convergent* and the number $\lim_{n\to\infty} \sum_{k=0}^{n-1} x_k$ is called the *sum* of $\sum_{k=0}^{\infty} x_k$. An expression of the form $\sum_{k=0}^{\infty} x_k$ is called a *series* or an *infinite series*.

If $x_0, x_1, x_3, \cdots$ is a geometric progression, then $\sum_{k=0}^{\infty} x_k$ is a *geometric series*.

Theorem 1.7.5. *If $|z| < 1$, then the geometric series $\sum_{k=0}^{n} z^k$ is convergent and*

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}. \tag{1.9}$$

Proof. For a complex number z, let

$$s_n := 1 + z + z^2 + \cdots + z^{n-2} + z^{n-1} = \sum_{k=0}^{n-1} z^k.$$

Then $zs_n = z + z^2 + \cdots + z^{n-1} + z^n$, hence $zs_n - s_n = z^n - 1$. Consequently,

$$\sum_{k=0}^{n-1} z^k = \frac{z^n - 1}{z - 1} = \frac{1 - z^n}{1 - z}.$$

If $|z| < 1$, then (z^n) is null, hence

$$\frac{1 - z^n}{1 - z} \to \frac{1 - 0}{1 - z} = \frac{1}{1 - z} \text{ as } n \to \infty.$$

We conclude the geometric series $\sum_{k=0}^{\infty} z^k$ converges and (1.9) holds. ☺

1.8 Steinhaus' Three Distance Conjecture★

Let τ_ϕ be an irrational rotation of the unit circle. Meaning ϕ is irrational and $\tau_\phi(z)$ is obtained from ϕ by rotation z by $2\pi\phi$ radians in the counterclockwise direction. For any N, the points 1, $\tau_\phi(1)$, $\tau_{\phi^2} = \tau_\phi\left(\tau_\phi(1)\right)$, $\ldots$, $\tau_\phi^N(1)$ divides the circle into "subintervals". In this section it is shown that there are numbers a, b, and c such that the length of any of these "subintervals" is one of these three numbers. This was part of a group of conjectures Władysław Hugo Dionizy Steinhaus (14 January 1887 Jasło to 25 February 1972 Wrocław) made regarding the lengths of the intervals generated by irrational rotations. The corresponding problem in the closed unit interval $[0, 1]$ is solved in this section.

This section can be covered any time after Sect. 1.1.

The Problem

Let N be a positive integer and let ϕ be some positive irrational number. Pick m and M in $\{1,\dots,N\}$ such that

$$\{m\phi\} \leq \{j\phi\} \leq \{M\phi\} \text{ for all } j \in \{1,\dots,N\}.$$

Here $\{x\}$ denotes the fractional part of the real number x.

For $k = 0,1,\dots,N$ with $k \neq M$ let $k' \in \{0,\dots,N\}$ be such that

$$\begin{aligned}&\{k\phi\} < \{k'\phi\} \text{ and}\\ &\{k\phi\} < \{j\phi\} \implies \{k'\phi\} \leq \{j\phi\} \text{ for all } j \in \{1,\dots,N\}.\end{aligned}$$

Thus $\{k'\phi\}$ is the point "after" $\{k\phi\}$. Clearly, $0' = m$. If we set $M' = 0$, then $k \to k'$ is a bijection of $\{0,\dots,N\}$ onto itself. For irrational ϕ the points are dense in the closed interval $[0,1]$, see Sect. 12.6.

Theorem 1.8.1 (Steinhaus' Three Distance Conjecture). *For any $k = 0,\dots,N$ with $k \neq M$*

$$\{k'\phi\} - \{k\phi\} \in \{\{m\phi\}, 1 - \{M\phi\}, \{m\phi\} + 1 - \{M\phi\}\}.$$

That is, any one of the intervals $[\{k\phi\},\{k'\phi\}]$, has length $\{m\phi\}$, $1 - \{M\phi\}$, or $\{m\phi\} + 1 - \{M\phi\}$.

This is also true for negative irrationals ϕ and for rational ϕ. For negative numbers, the details depends to some extend on how the fractional part of a negative number is determined. Is $\{-1.23\} = -0.23$? or is $\{-1.23\} = 0.77$? The latter corresponds to the interpretation $\{x\} := x - \lfloor x \rfloor$, the former might seem more natural, just lop off the stuff before the decimal point.

Exercise 1.8.2. If α is irrational and $m \neq n$ are integers, then $\{m\alpha\} \neq \{n\alpha\}$.

If we order the numbers

$$\{0,1,\dots,N\} = \{k_0,k_1,\dots,k_N\}$$

such that $k'_j = k_{j+1}$ for $j = 0,1,\dots,N-1$, then $k_0 = 0$, $k_1 = m$, and $k_N = M$. And

$$0 = \{k_0\phi\} < \{k_1\phi\} < \cdots < \{k_N\phi\} < 1 \tag{1.10}$$

is a partition of the interval $[0,1]$ and the theorem states that this partition contains intervals of at most three different lengths (Fig. 1.9).

Fig. 1.9 Illustrating (1.10) in the case $\phi = 3/7$ and $N = 3$

Arithmetic Modulo One

Before presenting a proof of the Steinhaus Three Distance Conjecture we develop some useful properties of arithmetic of fractional parts. As an application we establish a criterion for the irrationality of a real number.

Let $\mathbb{N}_0 = [0,1,2,\ldots\}$. Denote the *fractional part* of a real number a by $\{a\}$. If $a \geq 0$ then

$$\{a\} = a - \lfloor a \rfloor$$

where $\lfloor a \rfloor = \max\{n \in \mathbb{N}_0 \mid n \leq a\}$ is the *floor* of a. Alternatively, $\{a\}$ is the unique real such that

$$0 \leq \{a\} < 1, \text{ and } a = \{a\} + n \text{ for some integer } n. \tag{1.11}$$

It is immediate that

$$\{a+b\} = \begin{cases} \{a\}+\{b\} & \text{if } \{a\}+\{b\} < 1 \\ \{a\}+\{b\}-1 & \text{if } \{a\}+\{b\} \geq 1 \end{cases}. \tag{1.12}$$

To see this write

$$a+b = \{a\}+\{b\}+\lfloor a \rfloor + \lfloor b \rfloor$$

and use the alternative characterization of the fractional part. Alternatively (1.12) can be written as

$$\{a\}+\{b\} = \begin{cases} \{a+b\} & \text{if } \{a\}+\{b\} < 1 \\ 1+\{a+b\} & \text{if } \{a\}+\{b\} \geq 1 \end{cases}. \tag{1.13}$$

Also, if $\{a\} > \{b\}$, then

$$\{a\}-\{b\} = \begin{cases} \{a-b\} & \text{if } a \geq b \\ 1-\{b-a\} & \text{if } a < b \end{cases}.$$

The first follows from

$$a-b = \{a\}-\{b\}+(\lfloor a \rfloor - \lfloor b \rfloor)$$

and the second from

$$b-a = 1+\{b\}-\{a\}+(\lfloor b \rfloor - \lfloor a \rfloor - 1)$$

and the alternative characterization (1.11) of the fractional part.

Exercise 1.8.3. Show $\{a+\{na\}\} = \{(n+1)a\}$ for all $n \in \mathbb{N}$.

Exercise 1.8.4. Show $\{m\{na\}\} = \{mna\}$ for all $m,n \in \mathbb{N}$.

As an application of fractional parts let us prove a very special case of a theorem, Theorem 6.5.3, due to Liouville.

Theorem 1.8.5 (Baby Liouville). *Let a be a real number. Then a is irrational iff for any $\varepsilon > 0$ there is an $m \in \mathbb{Z}$ and an $n \in \mathbb{N}$ such that $a \neq \frac{m}{n}$ and $\left|a - \frac{m}{n}\right| < \frac{\varepsilon}{n}$.*

Proof. Suppose a is irrational. Divide the closed interval $[0,1]$ into a finite number of subintervals each with length $< \varepsilon$. If $j \neq k$ are integers, then $\{ja\} \neq \{ka\}$, since a is irrational. Hence, the set $\{\{ka\} \mid k \in \mathbb{N}\}$ is infinite. Hence, one of the subintervals must contain an infinite number of the fractional parts $\{ka\}$, $k \in \mathbb{N}$. In particular, there are two integers $0 < m < n$ such that

$$|\{na\} - \{ma\}| < \varepsilon.$$

Hence,

$$|(na - \lfloor na \rfloor) - (ma - \lfloor ma \rfloor)| < \varepsilon.$$

Setting $k = \lfloor na \rfloor - \lfloor ma \rfloor$ and dividing by $n - m$ gives

$$\left|a - \frac{k}{n-m}\right| < \frac{\varepsilon}{n-m}.$$

Conversely, suppose a is rational. Then $a = \frac{j}{k}$ for some integers j, k with $k \geq 1$. Let $\varepsilon := \frac{1}{k}$. For any integers m, n such that $m \geq 1$ and $a \neq \frac{m}{n}$, we have

$$\left|a - \frac{m}{n}\right| = \left|\frac{j}{k} - \frac{m}{n}\right| = \frac{|jn - mk|}{kn} \geq \frac{1}{kn} = \frac{\varepsilon}{n}.$$

Where we used that $|jn - mk| \geq 1$, since $a \neq \frac{m}{n}$. ☺

A Proof of the Steinhaus Three Distance Conjecture

Since $\{(m+M)\phi\}$ is either $\{m\phi\} + \{M\phi\}$ or $\{m\phi\} + \{M\phi\} - 1$ either $\{(m+M)\phi\} > \{M\phi\}$ or $\{(m+M)\phi\} < \{m\phi\}$ hence

$$N < m + M.$$

Fix a k in $\{0, 1, \ldots, N\}$ with $k \neq M$. Recall k' is chosen to minimize $\{j\phi\} - \{k\phi\} > 0$.

Suppose $k < k'$

Then

$$\{k'\phi\} - \{k\phi\} = \{(k' - k)\phi\} \geq \{m\phi\}$$

where there is equality if $k' - k = m$. Thus $k' = k + m$ if $k + m \leq N$, that is

$$\text{if } 0 \leq k \leq N - m \text{ then } m \leq k' \leq N \text{ and } k' = k + m.$$

Suppose $k > k'$

Then

$$\{k'\phi\} - \{k\phi\} = 1 - \{(k-k')\phi\} \geq 1 - \{M\phi\}$$

where there is equality if $k - k' = M$. Thus $k' = k - M$ if $0 \leq k - M$, that is

$$\text{if } M \leq k \leq N \text{ then } 0 \leq k' \leq N - M \text{ and } k' = k - M.$$

Note this last line includes the case $k = M$. If $N - m + 1 = M$ then we are done. So for the remainder of the proof it is assumed that

$$N + 1 < m + M.$$

Suppose $N - m < k < M$

By the ranges established for k' above it follows that $N - M < k' < m$. In this case is it shown below that

$$\{k'\phi\} - \{k\phi\} = \{m\phi\} + 1 - \{M\phi\} \text{ and } k' = k + m - M.$$

Note $k \to k + m - M$ maps $\{x \mid N - m < x < M\}$ onto $\{y \mid N - M < y < m\}$.

Suppose $k < k'$

Then as above

$$\{k'\phi\} - \{k\phi\} = \{(k'-k)\phi\} > \{m\phi\}$$

and

$$\{(k'-k)\phi\} - \{m\phi\} = 1 - \{(m - k' + k)\phi\}$$

since $k' - k < k' < m$. Hence

$$\begin{aligned}\{k'\phi\} - \{k\phi\} &= \{m\phi\} + 1 - \{(m - k' + k)\phi\}\\ &\geq \{m\phi\} + 1 - \{M\phi\}\end{aligned}$$

with equality if $m - k' + k = M$, that is for $k' = k + m - M$.

Suppose $k > k'$

Then as above

$$\{k'\phi\} - \{k\phi\} = 1 - \{(k-k')\phi\} > 1 - \{M\phi\}$$

and

$$\{M\phi\} - \{(k-k')\phi\} = \{(M-k+k')\phi\}$$

since $k - k' < k < M$. Hence

$$\begin{aligned} \{k'\phi\} - \{k\phi\} &= 1 - \{M\phi\} + \{(M-k+k')\phi\} \\ &\geq 1 - \{M\phi\} + \{m\phi\} \end{aligned}$$

with equality if $M - k + k' = m$, that is for $k' = k + m - M$.

Summary

There are three cases

$$\begin{aligned} 0 \leq k \leq N-m &\implies k' = k+m \text{ and } \{k'\phi\} - \{k\phi\} = \{m\phi\} \\ N-m < k < M &\implies k' = k+m-M \text{ and} \\ &\qquad \{k'\phi\} - \{k\phi\} = \{m\phi\} + 1 - \{M\phi\} \\ M \leq k \leq N &\implies k' = k-M \text{ and } \{k'\phi\} - \{k\phi\} = 1 - \{M\phi\} \end{aligned}$$

Exercise 1.8.6. Prove Steinhaus' Three Distance Conjecture for rational $\phi > 0$.

Problems

Problems for Sect. 1.1

1. Show that $1.23\overline{9} = 1.24$.
2. Carry out the division algorithm for $17/7$.
3. Find the infinite decimal form of $1/7$. Show all the steps needed to perform the long division.
4. When calculating the decimal form of p/q there are q possible remainders. Why is the length of the repeating part at most $q-1$?
5. Show $23.6\overline{321}$ is rational.
6. Find integers p and q such that $3.14\overline{15} = \frac{p}{q}$.
7. Prove that any interval contains a rational number.
8. Show $\mathbb{Q} + i\mathbb{Q} := \{a + ib \mid a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$. *Hint*: A ball contains an open square with the same center.
9. Give a detailed proof of the density of irrationals in the case $x < 0$.
10. Let $\mathbb{F}$ be the set of finite decimals. Show that $\mathbb{F}$ is dense in $\mathbb{R}$.

Problems for Sect. 1.2.

In the spirit of Exercise 1.2.2, that is, when verifying that a point is an accumulation point, it is not necessary to check all $\varepsilon > 0$. This is explored in the following problem.

1. A number a is an accumulation point of the set D iff for all $k \in \mathbb{N}$, there is an $x \in D$, such that $0 < |x-a| < 1/10^k$. [Hint for one part: given any $\varepsilon > 0$, there is $k \in \mathbb{N}$, such that $\varepsilon > \frac{1}{10^k}$.]
2. If $D \subseteq \mathbb{R}$ and $\mathrm{Im}(c) \neq 0$, then c is not an accumulation point of D.
3. Prove 0 is the only accumulation point of $D := \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.
4. If c is an accumulation point of D and $r > 0$, then c is an accumulation point of $D \cap B_r(c)$.
5. Any real number is an accumulation point of $\mathbb{Q}$.
6. Show the imaginary unit i is not an accumulation point of the set of real numbers $\mathbb{R}$.
7. Show that if $B_r(z) \subseteq A \subseteq \overline{B}_r(z)$, then the set of accumulation points of A equals $\overline{B}_r(z)$.
8. Let $\mathbb{F}$ be the set of finite decimals. Show that every real number is an accumulation point of $\mathbb{F}$.

Problems for Sect. 1.3

1. Show $|x| \to 0$ as $x \to 0$.
2. Let $f : [0,\infty[\to \mathbb{R}$ be determined by $f(x) := \sqrt{x+3}$. Prove $f(x) \to 2$ as $x \to 1$.

   The following two problems are about limits of the function $f : [0,\infty[\to \mathbb{R}$ determined by $f(x) := \sqrt{x}$. Since $D = [0,\infty[$, only $x \geq 0$ play a role in Eq. (1.4).

3. Show $f(x) \to 2$ as $x \to 4$.
4. Show $f(x) \to 0$ as $x \to 0$.
5. If $f : \mathbb{R} \to \mathbb{R}$ is determined by $f(x) = x - x^2$ and $a = \frac{1}{2}$. Then $f(a) = f\left(\frac{1}{2}\right) = \frac{1}{4} > 0$. Find $\delta > 0$ satisfying the conclusions of Theorem 1.3.14.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be determined by $f(x) := 4 - x^2$. Then $f(0) = 4$. Find a $\delta > 0$, such that
$$0 < |x-0| < \delta \implies 2 < f(x).$$

7. If $|f(x)| \le 7$ for all x and $g(x) := (x-3)f(x)$ prove
$$g(x) \to 0 \text{ as } x \to 3.$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be determined by
$$f(x) := \begin{cases} x-3 & \text{when } x \text{ is rational} \\ 5-x & \text{when } x \text{ is irrational} \end{cases}.$$
If $x_0 \neq 4$, prove $\lim_{x\to x_0} f(x)$ does not exist.

9. If $\lim_{x\to a} f(x)$ exists and for any $\delta > 0$, there is an $x \in D$ with $0 < |x-a| < \delta$, such that $f(x) \ge 0$, then $\lim_{x\to a} f(x) \ge 0$.

   More generally, if $f(x) \to L$ as $x \to a$ and $M \neq L$, then f is not close to M when x is close to a. This is the content of the following problem.

10. Let $f : D \to \mathbb{C}$. If $f(x) \to L$ as $x \to a$ and $M \neq L$, then
$$\exists \delta > 0, \forall x \in D, 0 < |x-a| < \delta \implies |f(x) - M| > \frac{1}{2}|L-M|.$$

11. Prove $x^2 \to -1$ as $x \to i$.

12. Let D be a subset of $\mathbb{C}$. Let $f,g : D \to \mathbb{C}$. Suppose 0 is an accumulation point of D. If $|g(x| \le M$ and $f(x) = xg(x)$ for all $x \in D$, then $f(x) \to 0$ as $x \to 0$.

13. Let $f,g : D \to \mathbb{C}$. Suppose a is an accumulation point of D. If $f(x) \to 0$ as $x \to a$ and $|g(x)| \le 47$ for all $x \in D$, then $f(x)g(x) \to 0$ as $x \to a$.

14. Let $f : D \to \mathbb{C}$. Suppose a is an accumulation point of D. If $f(x) \to L$ as $x \to a$, then $\mathrm{Re}\,(f(x)) \to \mathrm{Re}\,(L)$ as $x \to a$.

15. Let $f : D \to \mathbb{C}$. Suppose x_0 is an accumulation point of D. If $\mathrm{Re}\,(f(x)) \to a$ and $\mathrm{Im}\,(f(x)) \to b$ as $x \to x_0$, then $f(x) \to a + ib$ as $x \to x_0$.

16. Prove or disprove: If $\lim_{x\to a} f(x)$ exists and $f(x) > 0$ for all $x \neq a$, then $\lim_{x\to a} f(x) > 0$.

Problems for Sect. 1.4

1. If A is a finite subset of $\mathbb{C}$, prove that any point in A is an accumulation point of $\mathbb{C} \setminus A$.

2. What is wrong with the Composition Rule proposed below?
"If $f : A \to B$, $g : B \to C$, a is an accumulation point of A, b is an accumulation point of B, $f(x) \to b$ as $x \to a$, and $g(x) \to c$ as $x \to b$, then $g \circ f(x) \to c$ as $x \to a$."

3. Find two functions $f.g : \mathbb{R} \to \mathbb{R}$ such that neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists, yet both $\lim_{x\to 0} f((x) + g(x)$ and $\lim_{x\to 0} f(x)g(x)$ exist.

4. Let σ be the pseudo-sine function. Let $f(x) := x\sigma(1/x)$, when $x \neq 0$. Prove $f(x) \to 0$ as $x \to 0$.

   In the following two problems we assume $x^{1/n}$ exists for all $x \geq 0$.

5. Let $D := [0, \infty[$. If $a > 0$ and $n \in \mathbb{N}$, then $x^{1/n} \to a^{1/n}$ as $x \to a$.

6. Let $D := [0, \infty[$. For any $n \in \mathbb{N}$, $x^{1/n} \to 0$ as $x \to 0$.

7. Suppose $\lim_{x\to a} f(x) = b$ and given any $r > 0$, the set $f(B'_r(a))$ contains both positive and negative real numbers, show that $b = 0$.

8. Suppose $\lim_{x\to a} f(x) = b$ and there is a $r > 0$ such that $f(B'_r(a)) \subseteq [0, \infty[$. Show that $b \geq 0$.

9. Suppose $\lim_{x\to a} f(x) = b$ and there is a $r > 0$ such that $f(B'_r(a)) \subseteq]0, \infty[$. Must $b > 0$?

10. Let $D := \mathbb{R}$. If $f(x) := \begin{cases} x & \text{when } x \in \mathbb{Q} \\ 1 - x & \text{when } x \notin \mathbb{Q} \end{cases}$, find all values of $a \in \mathbb{R}$, such that $\lim_{x\to a} f(x)$ exists.

11. Let $D_1 := \left\{ \frac{1}{2n} \mid n \in \mathbb{N} \right\}$ and $D_2 := \left\{ \frac{1}{2n-1} \mid n \in \mathbb{N} \right\}$. Let $D := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. Then $D = D_1 \cup D_2$. Consider the function $f : D \to \mathbb{R}$ determined by

$$f(x) := \begin{cases} 1 & \text{if } x \in D_1 \\ 0 & \text{if } x \in D_2 \end{cases}.$$

   Show 0 is an accumulation point of D_1 and of D_2 and show that $\lim_{x\to 0} f(x)$ does not exists.

Problems for Sect. 1.5

1. Let $f(x) = \frac{x}{|2x+3|}$. Find both limits at infinity: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
2. If $n \in \mathbb{N}$, prove $\lim_{x\to\infty} x^n = \infty$. What happens as $x \to -\infty$?
3. If f and g are real valued and $f(x) \leq g(x)$ for all x, then

$$\lim_{x\to a} f(x) = \infty \implies \lim_{x\to a} g(x) = \infty.$$

4. If f and g are real valued, then

$$\lim_{x\to a} f(x) = 4 \text{ and } \lim_{x\to a} g(x) = \infty \implies \lim_{x\to a} f(x)g(x) = \infty.$$

Give a proof based on the definitions. In particular, do not use the product rule for limits or use $4 \cdot \infty = \infty$.

5. If $f(x) > 0$ for all x and $f(x) \to 0$ as $x \to a$, then $\frac{1}{f(x)} \to \infty$ as $x \to a$. Give a proof based on the definitions.

Problems for Sect. 1.6

1. If (x_n) is bounded and (y_n) is null, then $(x_n y_n)$ is null.
2. If (y_n) is null and $|x_n| \le |y_n|$, then (x_n) is null.
3. If $x_n \le y_n$ and $x_n \to \infty$, then $y_n \to \infty$.
4. If $z_n \ne 0$ for all n, then (z_n) is null iff $1/|z_n| \to \infty$.
5. Find two sequences (x_n) and (y_n) such that $x_n \to \infty$, $y_n \to 0$ and $x_n y_n \to 1$.
6. Find two sequences (x_n) and (y_n) such that $x_n \to \infty$, $y_n \to 0$ and $x_n y_n \to 29$.
7. Find two sequences (x_n) and (y_n) such that $x_n \to \infty$, $y_n \to \infty$ and $x_n - y_n \to 1$.

Problems for Sect. 1.7

1. Let $x \in \mathbb{R}$. Suppose $1 < x$. Given any $M > 0$, prove there is an $N \in \mathbb{N}$, such that $n \ge N \implies M \le x^n$. That is prove $x^n \to \infty$ as $n \to \infty$.
2. Find the sum of the geometric series

$$0.999\cdots = \sum_{k=1}^{\infty} \frac{9}{10^k}.$$

Problems for Sect. 1.8

1. Prove Steinhaus' Three Distance Conjecture for $\phi = \frac{3}{7}$.
2. Investigate Steinhaus' Three Distance Conjecture on the unit circle (Fig. 1.10).
3. If ϕ and ψ are irrationals and M, N are positive integers, let

$$0 = \{j_0\phi\} < \{j_1\phi\} < \cdots < \{j_M\phi\} < \{j_{M+1}\phi\} = 1$$
$$0 = \{k_0\psi\} < \{k_1\psi\} < \cdots < \{k_N\psi\} < \{k_{N+1}\psi\} = 1$$

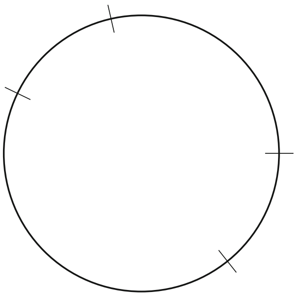

Fig. 1.10 Circle version of Fig. 1.9

as in (1.10). Here, $j_{M+1} := 1/\phi$ and $k_{N+1} := 1/\psi$ are notational devises introduced to simplify the notation below. One version of Steinhaus problem then is to consider the lengths of the sides of the rectangles whose vertices are at the points of intersections of the lines in Fig. 1.11. In this case the number of distances is at most six, by the theorem in Sect. 1.8. A linear ordering on the plane is introduced in the problems for Sect. E. This linear ordering imposes a linear ordering on the points $(\{j_m\phi\}, \{k_n\psi\})_{m,n}$. This introduces some additional distances into the problem. Find an upper bound on the number of length of "intervals" corresponding to this ordering.

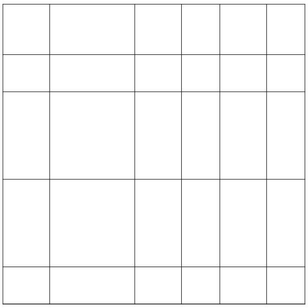

Fig. 1.11 A two dimensional Steinhaus problem. In the figure $M = 5$, $N = 4$, $\phi = e$, and $\psi = 5\pi$

4. A different version of the Steinhaus problem would be to consider the "intervals" obtained by using the linear order from the problems to Sect. E on the points

$$(\{k_m\phi\}, \{j_m\psi\})_{m=0}^{M+1}.$$

In particular, $N = M$, we are only considering M of the M^2 points in the open square, and of the points on the boundary only $(\{k_0\phi\}, \{j_0\psi\}) = (0,0)$ and $(\{k_{M+1}\phi\}, \{j_{M+1}\psi\}) = (1,1)$ are included in the list. [The author has not attempted to solve this problem, it may be easy or it may be very difficult.]

Solutions and Hints for the Exercises

Exercise 1.1.7. If $1/x$ is rational, then $1/x = p/q$, for integers $p, q \neq 0$. Hence $x = q/p$ contradicting that x is not rational. The other cases are similar.

Exercise 1.2.2. Let $\varepsilon > 0$ be given. If $\varepsilon \leq m$, then $D \cap B'_\varepsilon(c) \neq \emptyset$ by assumption. On the other hand, if $m < \varepsilon$, then

$$D \cap B'_\varepsilon(c) \supseteq D \cap B'_m(c)$$

and the right hand side is nonempty by assumption.

Exercise 1.3.9. If $|x - 1/2| < 1/4$, then $-1/4 < x - 1/2 < 1/4$, and therefore $1/4 < x < 3/4$. But $1/4 < x$, implies $0 < 2/x < 8$. Hence,

$$\left|x - \frac{1}{2}\right| < \frac{1}{4} \implies \frac{2}{x}\left|x - \frac{1}{2}\right| < 8\left|x - \frac{1}{2}\right|.$$

Consequently, if $|x - 1/2| < 1/4$ and $|x - 1/2| < \varepsilon/8$, then

$$\frac{2}{x}\left|x - \frac{1}{2}\right| < 8\left|x - \frac{1}{2}\right| < 8\frac{\varepsilon}{8} = \varepsilon.$$

Thus, $\delta := \min\{1/4, \varepsilon/8\}$ works.

Exercise 1.3.11. (*i*). It is easy to justify:

$$\begin{aligned} g\left(\{x \mid 0 < |x - 0| < \delta\}\right) &\supseteq \{g(x) \mid 0 < x < \delta\} \\ &= \{\sigma(1/x) \mid 0 < x < \delta\} \\ &= \{\sigma(t) \mid 1/\delta < t\} \\ &\supseteq \{-1, 1\}. \end{aligned}$$

(*ii*). For any real number L, either $|L - 1| \geq 1$ or $|L - (-1)| \geq 1$.

Exercise 1.3.15. This follows from

$$|f(x)| = |L - (L + f(x))| \geq |L| - |L - f(x)|.$$

An alternative argument is to show $f(x) \to L$ implies $|f(x)| \to |L|$ and then use Local Positivity on $g(x) = |f(x)|$.

Exercise 1.4.7. Use Example 1.3.5, the Product Rule, and induction on n.

Exercise 1.4.8. One way is to use the Sum Rule and induction on the degree.

Exercise 1.4.10. This is similar to the last part of the proof of the Product Rule. We need to control the size of the factor $|1/g(x)|$, hence we need a lower bound on $|g(x)|$. This is provided by Local Positivity in the form of Exercise 1.3.15.

Exercise 1.4.15. Comparing to the Composition Rule with $f(x) = 1 + x$ and $g(x) = \sqrt{x}$. As $x \to 3$ we have $f(x) \to 1 + 3 = 4$, and as $x \to 4$ we have $g(x) \to 2$, by assumption. To use the Composition Rule we need a $\gamma > 0$, such that $0 < |x-3| < \gamma$ implies $f(x) = 1 + x \neq 4$. Since $x \neq 3$ implies $1 + x \neq 4$ any choice, e.g., $\gamma = 1$ works.

Exercise 1.4.18. (*i*) $D_1 \cap B'_r(a) \subseteq D \cap B'_r(a)$.
(*ii*) If $D_1 \cap B'_r(a) = D_2 \cap B'_r(a) = \emptyset$, then $(D_1 \cup D_2) \cap B'_r(a) = \emptyset$.

Exercise 1.5.1. This is a special case of the Corollary in Sect. 1.4.

Exercise 1.5.4. In one case set $N = 1/\varepsilon^{1/n}$ in the other set $N = -1/\varepsilon^{1/n}$.

Exercise 1.6.5. This is a special case of Theorem 1.3.14.

Exercise 1.6.8. Let $\varepsilon := 3/2$. Given N, there is an even integer $n > N$. Since n is even, $1 + (-1)^n = 2 > 3/2 = \varepsilon$.

Exercise 1.7.2. (1) $1 < 1/x$. (2) $\left(\frac{1}{x}\right)^n = (1+y)^n$. (3) Rearrange (2). (4) Since $N > \varepsilon/y$ this follows from (3).

Chapter 2
Introduction to Continuity

In this chapter the algebra of continuous functions is established. A function that is continuous at each irrational number and discontinuous at each rational number is constructed. This function is know as the Riemann function, the Thomae function, the ruler function, or the raindrop function.

Establishing some of the more subtle properties of continuous functions requires properties of the set of real numbers that do not follow from the ordered field axioms. The relevant properties of the set of real numbers are contained in Chap. 3. We revisit continuity in Chap. 5, where we establish global properties of continuous functions.

2.1 Definition and Algebra

Let D be a subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *continuous* at a point $a \in D$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon. \tag{2.1}$$

Using neighborhoods this can also be written as

$$\forall \varepsilon > 0, \exists \delta > 0, f\left(D \cap B_\delta(a)\right) \subseteq B_\varepsilon\left(f(a)\right).$$

If a is not an accumulation point of D, then there is a $\delta > 0$, such that $\forall x \in D$, $|x-a| < \delta \implies x = a$. Such points are *isolated*. If a is isolated, the limit of $f(x)$ as $x \to a$ does not exist, but clearly, f is continuous at a. However, if a is an accumulation point of D, Eq. (2.1) means

$$f(x) \to f(a) \text{ as } x \to a.$$

Consequently, the algebra of limits leads to:

- if f is continuous at a and $k \in \mathbb{C}$, then kf is continuous at a;
- if f, g are continuous at a, then $f+g$ is continuous at a;

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- if f, g are continuous at a, then fg is continuous at a;
- if $f.g$ are continuous at a and $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

The first two of these are summarized by linearity, if f, g are continuous at x_0 and a, b are complex numbers, then $af + bg$ is continuous at x_0.

Theorem 2.1.1 (Composition Rule). *If f is continuous at a and g is continuous at $b := f(a)$, then $g \circ f$ is continuous at a.*

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at b, there is a $\gamma > 0$, such that

$$|y - b| < \gamma \implies |g(y) - g(b)| < \varepsilon. \tag{2.2}$$

Since f is continuous at a and $\gamma > 0$, there is a $\delta > 0$, such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \gamma,$$

since $b = f(a)$, Eq. (2.2) with $y = f(x)$ implies

$$|g(f(x)) - g(f(a))| = |g(y) - g(b)| < \varepsilon.$$

Thus $g \circ f$ is continuous at a. ☺

We say f is continuous *on* D, if f is continuous at every point in D. We showed in Sect. 1.4 that any rational function is continuous on the set of points where it is defined.

Example 2.1.2. $f(z) := |z|$ is continuous on $\mathbb{C}$.

Proof. Let $a \in \mathbb{C}$. Let $\varepsilon > 0$ be given. Let $\delta := \varepsilon$. Then $|z - a| < \delta$ implies

$$|f(z) - f(a)| = ||z| - |a|| \leq |z - a| < \delta = \varepsilon.$$

Hence Eq. (2.1) holds. ☺

Exercise 2.1.3. Explain why

$$f(x) := \frac{|3 + 7|x|| - |3x^2 - 8|^9}{5 - |1 - x^2|}$$

is continuous at 2.

Example 2.1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be determined by $f(x) := \begin{cases} 1 & \text{if } x \geq 3 \\ -1 & \text{if } x < 3 \end{cases}$. Then f is *discontinuous*, that is not continuous, at 3.

Proof. Since

$$\lim_{x \nearrow 3} f(x) = -1 \neq 1 = \lim_{x \searrow 3} f(x)$$

f does not have a limit at 3, in particular, f is not continuous at 3. ☺

Example 2.1.5. Let $f : \mathbb{C} \to \mathbb{R}$ be determined by $f(x) := \begin{cases} 8 & \text{if } x \neq 5 \\ 2 & \text{if } x = 5 \end{cases}$. Then

$$\lim_{x \to 5} f(x) = 8 \neq 2 = f(5),$$

hence f is discontinuous at 5.

In preparation for the next example we need to know more about approximating irrational numbers by rational numbers:

Exercise 2.1.6. Let a be some irrational number. Given any $M \in \mathbb{N}$, there is a $\gamma > 0$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \leq M \implies \left| a - \frac{p}{q} \right| \geq \gamma$.

Writing the contrapositive of the implication gives: Given any $M \in \mathbb{N}$, there is a $\gamma > 0$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $\left| a - \frac{p}{q} \right| < \gamma \implies q > M$.

Remark 2.1.7. Exercise 2.1.6 should be compared to Theorem 1.8.5.

The function in the following exercise is a modification of the Dirichlet function.

Exercise 2.1.8 (Riemann Function). Let $f : \mathbb{R} \to \mathbb{R}$ be determined by

$$f(x) := \begin{cases} 1/q & \text{when } x = p/q \text{ for some } p \in \mathbb{Z}, q \in \mathbb{N} \text{ in lowest terms} \\ 0 & \text{when } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Show f is discontinuous at every point in $\mathbb{Q}$ and continuous at every point in $\mathbb{R} \setminus \mathbb{Q}$.

We named this function after Georg Friedrich Bernhard Riemann (17 September 1826, Breselenz to 20 July 1866, Selasca). It is also called the Thomae function, after Carl Johannes Thomae (11 December 1840, Laucha an der Unstrut to 1 April 1921, Jena), the ruler function, the raindrop function among many other names.

Remark 2.1.9. Vito Volterra (3 May 1860, Ancona to 11 October 1940, Rome) showed that we cannot have a function that is continuous at the rational numbers and discontinuous at the irrationals. In fact, he showed that we cannot have two functions for which the points of discontinuity of one are the points of continuity of the other and vice versa (See Sect. 3.4). Thus, the roles of the rationals and irrationals in the previous exercise cannot be reversed by some clever choice of f.

2.2 Removable Discontinuity

Suppose f is discontinuous at a. Then, f has a *removable discontinuity* at a, if there is a function g, such that g is continuous at a and $g(x) = f(x)$ for all $x \neq a$. In the definition of removable discontinuity, it does not matter whether or not f is defined at a.

Exercise 2.2.1. Suppose f is discontinuous at a. Then $\lim_{x\to a} f(x)$ exists iff f has a removable discontinuity at a.

Example 2.2.2. $f(x) := \frac{x^2-1}{x-1}$ has a removable discontinuity at $a = 1$. Because, if $g(x) := x+1$, then $f(x) = g(x)$ for all $x \neq 1$ and g is continuos at $a = 1$.

2.3 One-Sided Continuity

In this section we assume $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{C}$.

Let $a \in D$. If

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, 0 < x - a < \delta \implies |f(x) - f(a)| < \varepsilon,$$

then we say f is *continuous from the right* at a. In particular, if a is an accumulation point of D, then $\lim_{x\searrow a} f(x)$ exists and equals $f(a)$ if and only if f is continuous from the right at a.

Similarly, f is *continuous from the left* at a, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, -\delta < x - a < 0 \implies |f(x) - f(a)| < \varepsilon.$$

If a is an accumulation point of D, this means $\lim_{x\nearrow a} f(x)$ exists and equals $f(a)$.

Example 2.3.1. Let $f(x) := \begin{cases} 2 & \text{when } x > 3 \\ 5 & \text{when } x \leq 3 \end{cases}$. Then f is continuous from the left at 3, since $\lim_{x\nearrow 3} f(x) = 5 = f(3)$. And f is not continuous from the right at 3 because $\lim_{x\searrow 3} f(x) = 2 \neq 5 = f(3)$.

Exercise 2.3.2. f is continuous at a iff f is both continuous from the right and left at a.

Problems

Problems for Sect. 2.1

1. If g is continuous at L and $f(x) \to L$ as $x \to a$, prove $g(f(x)) \to g(L)$ as $x \to a$.
2. Let $f : [0,1] \to [0,1]$ be determined by $f(0) = 0$, and for any $n \in \mathbb{N}$, $f(x) = 1/n$, when $\frac{1}{n+1} < x \leq \frac{1}{n}$. Since $\bigcup_{n=1}^{\infty} \left]\frac{1}{n+1}, \frac{1}{n}\right] =]0,1]$ and the union is disjoint, f is a function defined on the closed interval $[0,1]$.

   a. Prove that f is increasing, i.e., $x < y \implies f(x) \leq f(y)$.
   b. Prove that f is continuous at 0.
   c. Prove that f is continuous at 1.

d. Prove that f is continuous on $\left]\frac{1}{n+1}, \frac{1}{n}\right[$ for all $n \in \mathbb{N}$.
e. Prove that f is discontinuous at $x = 1/n$ for all $n \in \mathbb{N}$ with $n \geq 2$.

3. Let $D := [-1,1] \cup \{3\} \cup [5,7]$ and let $f : D \to \mathbb{R}$. Then f is continuous at 3.
4. If $f : \mathbb{R} \to \mathbb{C}$ is continuous on $\mathbb{R}$ and $f(x) = x^2$ for every rational x, show $f(x) = x^2$ for every real x.
5. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and
$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R},$$
then there is constant $c \in \mathbb{R}$, such that $f(x) = cx$, for all x in $\mathbb{R}$. [*Hint*: $f(2) = f(1+1) = 2f(1)$, and $f(1) = f(1/2) + f(1/2) = 2f(1/2)$, so $f(1/2) = \frac{1}{2}f(1)$].
6. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous at a. Suppose (x_n) is a sequence of complex numbers converging to a. Prove the sequence $(f(x_n))$ converges to $f(a)$.
7. Why does the composition rule for limits (Theorem 1.4.14) not imply the composition rule for continuity (Theorem 2.1.1)?

Problems for Sect. 2.2

1. Let σ be the pseudo-sine function. Let $f(x) := \sigma(1/x)$, when $x \neq 0$ and let $f(0) := 0$. Show that f is discontinuous at 0.
2. Let σ be the pseudo-sine function. Let $g(x) := x\sigma(1/x)$ for $x \neq 0$. Prove g has a removable discontinuity at 0.

Problems for Sect. 2.3

1. Prove the the function in Problem 2 for Sect. 2.1 is continuous from the left at every point in the half-open interval $]0,1]$.

Solutions and Hints for the Exercises

Exercise 2.1.6. For a fixed q, there are only finitely many p such that $a - \gamma \leq p/q \leq a + \gamma$. Alternatively, for any integer $k \geq 1$, the two integers closest to ka are $\lfloor ka \rfloor$ and $\lfloor ka \rfloor + 1$, in fact $\lfloor ka \rfloor < ka < \lfloor ka \rfloor + 1$. Hence, the largest γ satisfying the desired conclusion is the smallest of the numbers $a - \frac{\lfloor ka \rfloor}{k}$, $\frac{\lfloor ka \rfloor + 1}{k} - a$, $k = 1, 2, \ldots, M$.

Exercise 2.1.8. This is a consequence of Exercise 2.1.6 and Corollary 1.4.20.

Exercise 2.2.1. Let $L := \lim_{x\to a} f(x)$ exists, then

$$g(x) := \begin{cases} f(x) & \text{when } x \neq a \\ L & \text{when } x = a \end{cases}$$

is continuous at a.

Exercise 2.3.2. Similar to the corresponding result for one-sided limits.

Chapter 3
Sets of Real Numbers

Most of us believe we have a reasonable understanding of what a real number is. However, sets of real numbers have some deep and surprising properties. We will explore some of these properties in this chapter and in Chap. 4 on cardinality. Of notable interest for future applications are the order completeness of $\mathbb{R}$, the characterization of intervals in terms of the intermediate value property, and the nested interval theorem. Among other results we establish the existence of roots of positive real numbers and we introduce the amazing Cantor set as well as related functions.

3.1 Supremum and Infimum

In this section we investigate order completeness of sets of real numbers. These properties are not all shared by the set of rational numbers.

Let A be a subset of $\mathbb{R}$. A real number u is an *upper bound* for A, if $a \in A \implies a \leq u$. That is, if $\forall a \in A, a \leq u$.

Example 3.1.1. A few illustrative examples are:

1. 56 is an upper bound for $A := \{x \in \mathbb{R} \mid -34 \leq x \leq 56\}$ and $56 \in A$. Of course, any number greater than 56 is also an upper bound for A.
2. 56 is an upper bound for $B := \{x \in \mathbb{R} \mid x < 56\}$ and $56 \notin B$.
3. Consider the set $C := \left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\} = \{1/2, 2/3, \ldots\}$. Since $n < n+1$, we have $\frac{n}{n+1} < 1$, hence 1 is an upper bound for C and $1 \notin C$.

If u is an upper bound for A and $u \in A$, then u is the *maximum* of A, in symbols this is written: $u = \max(A)$.

Let A be some set of real numbers. Let u be a real number. If u is an upper bound for A and no number smaller than u is an upper bound for A, then u is the *least upper bound* for A. The least upper bound is also called the *supremum* and abbreviated by $\sup(A)$.

Example 3.1.2. Using the notation from Example 3.1.1.

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1. $\sup(A) = 56$.
2. $\sup(B) = 56$.
3. $\sup(C) = 1$.

Proof. We give a separate argument for each case.

1. We saw that 56 is an upper bound for A. No number smaller than 56 can be an upper bound since 56 is in A.
2. We saw that 56 is an upper bound for B. Let $v < 56$, we will show that v is not an upper bound for B, hence 56 is the smallest upper bound for B. To see that v is not an upper bound for B we must find $x \in B$, such that $v < x$. We claim that $x := v + \varepsilon$ works, when $\varepsilon := (56 - v)/2$. Now $v < 56$ implies $\varepsilon > 0$.

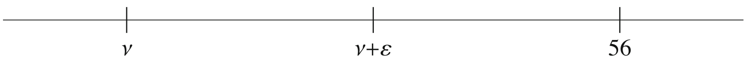

Fig. 3.1 $v + \varepsilon$ is the midpoint of $[v, 56]$

   Hence $v < v + \varepsilon = x$ and $x < x + \varepsilon$. It remains to check that $x + \varepsilon = 56$. But, $x + \varepsilon = v + 2\varepsilon = v + (56 - v) = 56$ (Fig. 3.1).
3. We saw that 1 is an upper bound for C. Suppose $v < 1$. We will show that v is not an upper bound for C. To see this we must show there is an integer N, such that $v < N/(N+1)$. Since $0 < 1 - v$ and $(1/n)$ is null, there is an integer N, such that $\frac{1}{n} < 1 - v$ for all $n > N$. Setting $n = N + 1$ yields $v < 1 - \frac{1}{N+1} = \frac{N}{N+1}$.

This completes the verification of the claims. ☺

Exercise 3.1.3. Let A be a subset of $\mathbb{R}$. Let u and v be real numbers. Suppose
(i) $a \in A \implies a \leq u$ and
(ii) $(a \in A \implies a \leq v) \implies u \leq v$.
Prove $u = \sup(A)$.

The following theorem establishes the existence of the supremum for any set of real numbers that has an upper bound. In the proof we will repeatedly use that if we have a nonempty finite set of integers with some property, then that set of integers has a largest element. This is a very special case of the well ordering property of the set of natural numbers.

Remark 3.1.4. For the set $A := \{x \in \mathbb{R} \mid x^2 < 2\}$ an outline of the proof of the theorem runs as follows: Using $1^2 < 2 < 2^2$ we see $d_0 = 1$ is not an upper bound for A and $d_0 + 1 = 2$ is an upper bound for A. Using $1.4^2 < 2 < 1.5^2$ we see $d_0.d_1 = 1.4$ is not an upper bound for A and $d_0.d_1 + 1/10 = 1.5$ is an upper bound for A. Since $1.41^2 < 2 < 1.42^2$ we see $d_0.d_1d_2 = 1.41$ is not an upper bound for A and $d_0.d_1d_2 + 1/10^2 = 1.42$ is an upper bound for A. Continuing in this manner we construct the least upper bound $d_0.d_1d_2\cdots$ for A as an infinite decimal.

For the set $A := \{x \in \mathbb{R} \mid x < 56\}$ this process yields the infinite decimal $55.\overline{9}$ as the least upper bound.

Theorem 3.1.5 (Order Completeness of $\mathbb{R}$). *Let A be a nonempty subset of $\mathbb{R}$. If A has an upper bound, then A has a least upper bound.*

Proof. **Step 1: Construction of a candidate for the** sup**.** Let u be an upper bound for A. If $u = \pm e_0.e_1e_2\cdots$, is an infinite decimal representation of u, then $u \leq e_0 + 1$. Hence, we have found an integer $N := e_0 + 1$ that is an upper bound for A. If $a \in A$, and $a = \pm f_0.f_1f_2\cdots$ is an infinite decimal representation of a, then $-(f_0+2) < a$. Hence, we have found an integer $M := -f_0 - 2$ that is not an upper bound for A. See Fig. 3.2. To begin the construction of $\sup(A)$, let

Fig. 3.2 Illustrating that $M = -(f_0+2)$ is not an upper bound for A and $N = e_0 + 1$ is an upper bound for A

$$B_0 := \{k \in \mathbb{Z} \mid M \leq k \text{ and } k \text{ is not an upper bound for } A\}.$$

Then $M \in B_0$ and $N \notin B_0$ is an upper bound for B_0. Hence, B_0 has at most $N - M$ elements, in particular, B_0 is finite. Let L be the largest element of B_0. Then L is not an upper bound for A and $L + 1$ is an upper bound for A. (A figure similar to Fig. 3.3, but with $N - M + 1$ tick marks illustrates this.) Assume $L \geq 0$. Let $d_0 = L$. Then d_0 is not an upper bound for A and $d_0 + 1$ is an upper bound for A.

Let

$$B_1 := \left\{k \in \mathbb{Z} \mid 0 \leq k \text{ and } d_0 + \tfrac{k}{10} \text{ is not an upper bound for } A\right\}.$$

Then $0 \in B_1$, since d_0 is not an upper bound for A, and $10 \notin B_1$, since $d_0 + 1$ is an upper bound for A. Hence, B_1 is a subset of $\{0, 1, \ldots, 9\}$, in particular, B_1 is finite. Let d_1 be the largest element of B_1. Then $0 \leq d_1 \leq 9$, $d_0.d_1$ is not an upper bound for A, and $d_0.d_1 + 1/10$ is an upper bound for A. See Fig. 3.3.

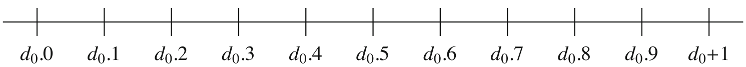

Fig. 3.3 Since $d_0.0$ is not an upper bound and $d_0 + 1$ is an upper bound, there is a k such that $d_0.k$ is not an upper bound and $d_0.k + 1/10$ is an upper bound

Let

$$B_2 := \left\{k \in \mathbb{Z} \mid 0 \leq k, \text{ and } d_0.d_1 + \tfrac{k}{10^2} \text{ is not an upper bound for } A\right\}.$$

Then $0 \in B_2$, since $d_0.d_1$ is not an upper bound for A, and $10 \notin B_2$, since $d_0.d_1 + 1/10$ is an upper bound for A. Hence, B_2 is a subset of $\{0, 1, \ldots, 9\}$, in particular, B_2 is finite. Let d_2 be the largest element of B_2. Then $0 \le d_2 \le 9$, $d_0.d_1d_2$ is not an upper bound for A, and $d_0.d_1d_2 + 1/10^2$ is an upper bound for A.

Continuing in this manner (i.e., by induction) we end up with a real number

$$u := d_0.d_1d_2\cdots$$

such that for each $n \in \mathbb{N}$, $d_0.d_1d_2\cdots d_n$ is not an upper bound for A and $d_0.d_1d_2\cdots d_n + 1/10^n$ is an upper bound for A.

Step 2: Showing the candidate works. To complete the proof we must show (i) that u is an upper bound for A and (ii) that no $v < u$ is an upper bound for A.

(i) Let $a \in A$. For any n, $d_0.d_1d_2\cdots d_n + 10^{-n}$ is an upper bound for A. Hence $a \le d_0.d_1d_2\cdots d_n + 1/10^n$ for all n. Consequently,

$$a - \frac{1}{10^n} \le d_0.d_1d_2\ldots d_n \le u.$$

Since $(1/10^n)$ is null, we conclude $a \le u$. Thus u is an upper bound for A.

(ii) Suppose v is an upper bound for A. For any n, $d_0.d_1d_2\cdots d_n$ is not an upper bound for A. Hence there is an $a_n \in A$ such that $d_0.d_1d_2\cdots d_n \le a_n$. Now v is an upper bound for A, in particular, $a_n \le v$. By the transitive property of inequalities, $d_0.d_1d_2\cdots d_n \le v$ for all n. Letting $n \to \infty$, we get $u \le v$, since $d_0.d_1d_2\cdots d_n \to u$. ☺

Exercise 3.1.6. Complete the proof of the order completeness of $\mathbb{R}$ by modifying the proof above such that it works in the case where $L < 0$.

Define $\sup(\emptyset) = -\infty$. Set $\sup(A) = \infty$, if A does not have an upper bound. Then any subset of $\mathbb{R}$ has a supremum $\sup(A)$. The previous theorem can therefore be restated as

$$\text{if } A \neq \emptyset \text{ has an upper bound, then } \sup(A) \in \mathbb{R}.$$

Infimum

Let A be a subset of $\mathbb{R}$ and let $m \in \mathbb{R}$. If $m \le a$ for all $a \in A$, then m is a *lower bound* for A. If m is an lower bound for A and $m \in A$, then m is the *minimum* of A in symbols this is written: $U = \min(A)$. A number m is a *greatest lower bound* for A, if m is a lower bound and no larger number is a lower bound for A. A greatest lower bound is also called an *infimum* and denoted by $\inf(A)$. Let $-A := \{-a \mid a \in A\}$.

Exercise 3.1.7. (i) Prove m is a lower bound for A iff $-m$ is an upper bound for $-A$. (ii) Prove $\inf(A) = \sup(-A)$.

Theorem 3.1.8. *If A has a lower bound, then A has a greatest lower bound.*

Proof. One approach to the proof is to imitate the proof that the supremum is a real number above. On the other hand it might be simpler to use that theorem to prove this one. This is the proof outlined in the previous exercise. ☺

Define $\inf(\emptyset) = \infty$. If A does not have a lower bound we will write $\inf(A) = -\infty$.

Having established that sets with upper bounds have a least upper bound and sets with lower bounds have a greatest lower bound we can, and will, from now on, for the most part forget about infinite decimals.

3.2 Intervals

In this section, we give a characterization of intervals in terms of a property that is useful in our study of continuous functions. We will establish this as a first consequence of the order completeness of the set of all real numbers.

A *closed interval* is a set of the form

$$\begin{aligned}
[a,b] &:= \{x \in \mathbb{R} \mid a \leq x \leq b\}, \\
[a,\infty[&:= \{x \in \mathbb{R} \mid a \leq x\} = \{x \in \mathbb{R} \mid a \leq x < \infty\}, \\
]-\infty,b] &:= \{x \in \mathbb{R} \mid x \leq b\} = \{x \in \mathbb{R} \mid -\infty < x \leq b\}, \text{ or} \\
]-\infty,\infty[&:= \{x \in \mathbb{R} \mid -\infty < x < \infty\} = \mathbb{R}.
\end{aligned}$$

Where $a \leq b$ are real numbers. An *open interval* is a set of the form

$$\begin{aligned}
]a,b[&:= \{x \in \mathbb{R} \mid a < x < b\}, \\
]a,\infty[&:= \{x \in \mathbb{R} \mid a < x\} = \{x \in \mathbb{R} \mid a < x < \infty\}, \\
]-\infty,b[&:= \{x \in \mathbb{R} \mid x < b\} = \{x \in \mathbb{R} \mid -\infty < x < b\}, \text{ or} \\
]-\infty,\infty[&:= \{x \in \mathbb{R} \mid -\infty < x < \infty\} = \mathbb{R}.
\end{aligned}$$

Where $a < b$ are real numbers. A *half-open interval* is a set of the form

$$\begin{aligned}
]a,b] &:= \{x \in \mathbb{R} \mid a < x \leq b\} \text{ or} \\
[a,b[&:= \{x \in \mathbb{R} \mid a \leq x < b\}.
\end{aligned}$$

Where $a < b$ are real numbers. A half-open interval is also called half-closed. An *interval* is a subset of $\mathbb{R}$ that is either an open, a closed, or a half-open interval. Note, the interval $]-\infty,\infty[$ is both open and closed.

A subset A of $\mathbb{R}$ has the *intermediate value propert*y, if for any x and y in A and any t in $\mathbb{R}$,

$$x < t < y \implies t \in A.$$

It is easy to see that any interval has the intermediate value property. Conversely, any subset of $\mathbb{R}$ that has the intermediate value property is an interval:

Theorem 3.2.1 (Interval Theorem). *Let A be a nonempty subset of $\mathbb{R}$. Then A has the intermediate value property if and only if A is an interval.*

Proof. Clearly, any interval has the intermediate value property. Conversely, suppose A has the intermediate value property. Let $a := \inf(A)$ and $b := \sup(A)$. Assume a and b both are finite. Then a is a lower bound for A and b is an upper bound for A, consequently $A \subseteq [a,b]$. Let $t \in \mathbb{R}$ be such that $a < t < b$. Since $a < t$ and $a = \inf(A)$ is the greatest lower bound for A, we see t is not a lower bound for A. Hence, there is an $x \in A$, such that $\inf(A) \leq x < t$. Similarly, there is a $y \in A$, such that $t < y \leq \sup(A)$. We have found x, y in A, such that $x < t < y$, hence, using the intermediate value property, $t \in A$. Since $t \in]a,b[$ was arbitrary, the open interval $]a,b[$ is a subset of A. The points a and b may or may not be in A, hence A is one of the intervals, $]a,b[$, $[a,b[$, $]a,b]$, or $[a,b]$. ☺

Exercise 3.2.2. Complete the proof of the Interval Theorem by considering the cases where $a = -\infty$ and/or $b = \infty$.

3.3 The Nested Interval Theorem

When we say that $\pi = 3.14159\cdots$, we usually interpret this to mean that we know $d_0 = 3, d_1 = 1, d_2 = 4, d_3 = 1, d_4 = 5, d_5 = 9$ and we do not know anything about the digits d_k with $k > 5$. If those digits are 0's then $\pi = 3.14159$, if those digits are 9's then $\pi = 3.14159 + 1/10^5 = 3.14160$. In all other cases π is somewhere in between those two extremes.

More formally, given a positive integer d_0 and digits $d_n \in \{0, 1, \ldots, 9\}$ let

$$I_k := \left[d_0.d_1d_2\cdots d_k, d_0.d_1d_2\cdots d_k + \frac{1}{10^k} \right]. \tag{3.1}$$

Then I_k is the set of real numbers that have a decimal expansion beginning with $d_0.d_1d_2\cdots d_k$. In particular, $I_{k+1} \subset I_k$ and

$$\bigcap_{k=1}^{\infty} I_k = \{d_0.d_1d_2\cdots\}.$$

The Nested Interval Theorem is a generalization of this to intervals whose endpoints need not be finite decimals.

The following results states that if $a \in A$ and $b \in B$ implies $a \leq b$, then there is a real number t separating A and B, in the sense that $a \leq t \leq b$ for all $a \in A$ and all $b \in B$.

Proposition 3.3.1. *Let A and B be nonempty subsets of $\mathbb{R}$. If any number in A is less than or equal to any number in B, then there is a number t, such that any number in A is less than or equal to t and any number in B is greater than or equal to t.*

Proof. Any number b in B is an upper bound for A, hence, $\sup(A) \leq b$ for any $b \in B$. So $\sup(A)$ is a lower bound for B, and therefore $\sup(A) \leq \inf(B)$. Consequently, any number t between $\sup(A)$ and $\inf(B)$ works. ☺

A sequence of intervals $(I_n)_{n\in\mathbb{N}}$, such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, is *nested*. If

Fig. 3.4 A few nested intervals $[a_0, b_0] \supset [a_1, b_1] \supset \cdots$

the lengths of the nested intervals form a null sequence, it appears that the intervals have exactly one point in common, see Fig. 3.4. This is the content of the following theorem. We repeat the idea of the proof of Proposition 3.3.1 in the proof below.

Theorem 3.3.2 (Nested Interval Theorem, Cantor's Principle). *Suppose the closed intervals $([a_n, b_n])_{n\in\mathbb{N}}$ are nested and the sequence of lengths of these intervals $(b_n - a_n)$ is null, then there is an $x \in \mathbb{R}$, such that*

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x\}.$$

That is, the intersection contains exactly one point.

Proof. Since the intervals are nested we have

$$a_i \le a_{i+1} \le a_{i+2} \le \cdots \le a_j \le b_j \le \cdots \le b_{i+2} \le b_{i+1} \le b_i$$

for all $i \le j$. Hence, for $i \le j$,

$$a_i \le a_j \le b_j \le b_i.$$

Consequently, $i \le j$ implies $a_i \le b_j$ and $a_j \le b_i$. Since i and j are arbitrary we have shown that

$$\forall m, n \in \mathbb{N}, a_m \le b_n.$$

Hence, if $A := \{a_m \mid m \in \mathbb{N}\}$ and $B := \{b_n \mid n \in \mathbb{N}\}$, then any number in A is less than or equal to any number in B. Consequently, $a := \sup(A)$ and $b := \inf(B)$ are finite and $a \le b$.

If $a < b$, then $0 < b - a \le b_n - a_n$ contradicts that $(b_n - a_n)$ is null. Hence, $a = b$.

It remains to establish that $\{a\} = \bigcap_{n=1}^{\infty} [a_n, b_n]$. We will show $\subseteq$ and $\supseteq$.

$\subseteq$ For all $n \in \mathbb{N}$, $a_n \le a$, since a is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. Similarly, $b \le b_n$ for all $n \in \mathbb{N}$, since b is a lower bound for the set $\{b_n \mid n \in \mathbb{N}\}$. Hence, $a_n \le a = b \le b_n$ for all $n \in \mathbb{N}$. So, $a \in [a_n, b_n]$ for all $n \in \mathbb{N}$. Thus $a \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ and consequently $\{a\} \subseteq \bigcap_{n=1}^{\infty} [a_n, b_n]$.

$\supseteq$ Conversely, if $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, then $a_n \le x \le b_n$ for all $n \in \mathbb{N}$. Hence, x is an upper bound for the set $\{a_n \mid n \in \mathbb{N}\}$. Since a is the smallest upper bound for $\{a_n \mid n \in \mathbb{N}\}$ it follows that $a \le x$. Similarly, $x \le b$. By transitivity of inequality, $a \le x \le b$. Hence, $b = a$ implies $x = a$. Consequently, $\bigcap_{n=1}^{\infty} [a_n, b_n] \subseteq \{a\}$. ☺

The following result will be used repeatedly when we use the Nested Interval Theorem.

Exercise 3.3.3. Adopt the notation from the statement of the Nested Interval Theorem above. Let $\delta > 0$ be given. Prove, there is an N, such that

$$[a_N, b_N] \subset]x-\delta, x+\delta[.$$

The following version of the Nested Interval Theorem, called the Binary Nested Interval Theorem, reflects the way we often construct the intervals we apply the Nested Interval Theorem to. We use the term "binary" because at each stage we pick the left or right half of the previous interval.

Theorem 3.3.4 (Binary Nested Interval Theorem). *Let $a < b$ be real numbers. Let $a_0 := a$ and $b_0 := b$. For each $n \geq 0$, let $c_n := (a_n + b_n)/2$ and suppose either $a_{n+1} = a_n$ and $b_n = c_n$, or $a_{n+1} = c_n$ and $b_{n+1} = b_n$, then*

$$\bigcap_{n=0}^{\infty} [a_n, b_n] = \{x\}$$

for some real number x.

Proof. Since $a_n < c_n < b_n$ the intervals $[a_n, b_n]$ are nested. By construction $b_n - a_n = (b-a)/2^n$. It remains to check $((b-a)/2^n)$ is null. But this follows from the fact that $(1/2^n)$ is null (Theorem 1.7.1). ☺

Exercise 3.3.5 (Nested Rectangle Theorem). For each $k \in \mathbb{N}$, let $[a_k, b_k] \times [c_k, d_k]$ be a rectangle in $\mathbb{R}^2 = \mathbb{C}$. Suppose for each $k \in \mathbb{N}$, $[a_{k+1}, b_{k+1}] \times [c_{k+1}, d_{k+1}] \subset [a_k, b_k] \times [c_k, d_k]$ and the sequence of lengths of the diameters

$$|(c_k, d_k) - (a_k, b_k)| = \sqrt{(c_k - a_k)^2 + (d_k - b_k)^2}$$

forms a null sequence. Prove there is a point $(x, y) \in \mathbb{R}^2$, such that

$$\{(x,y)\} = \bigcap_{k=1}^{\infty} [a_k, b_k] \times [c_k, d_k].$$

3.4 Sets of Continuity★

As an interesting application of the Nested Interval Theorem we prove Volterra's observation regarding sets of continuity. The proof is essentially Volterra's.

Theorem 3.4.1 (Volterra). *Let I be an interval and let f and g be functions $I \to \mathbb{C}$. Let $A \subset I$ be the set of points where f is continuous and let $B \subset I$ be the set of points where g is continuous. If A and B are dense subsets of I, then the intersection of A and B is nonempty.*

The following lemma helps organize the proof. For a closed interval $J = [a, b]$ let $\overset{\circ}{J} :=]a, b[$ be the corresponding open interval.

Lemma 3.4.2. *Let I, f, g, A, and B be as in the theorem. If J is a closed and bounded subinterval of I and $\varepsilon > 0$, then there is a closed interval $K \subset \overset{\circ}{J}$ such that the length of K is less than ε and*

$$|f(x) - f(y)| < \varepsilon \text{ and } |g(x) - g(y)| < \varepsilon$$

for all x and y in K.

Proof. Fix two points $a < b$ in I such that $J = [a,b]$. Let $x_0 \in A \cap (a,b)$. Since f is continuous at x_0, there is a $\delta' > 0$, such that

$$\forall x \in J, |x - x_0| \leq \delta' \implies |f(x) - f(x_0)| < \varepsilon/2. \tag{3.2}$$

Let $a' := \max\{a, x_0 - \delta'\}$ and $b' := \min\{b, x_0 + \delta'\}$. Consider the closed interval $I' := [a', b']$, then $x_0 \in \overset{\circ}{I'}$, since $a < x_0 < b$ and $I' \subseteq J$, since $a \leq a'$ and $b' \leq b$. If x and y are in I', then (3.2) yields

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

because $|x - x_0| \leq \delta'$ and $|y - x_0| \leq \delta'$. Hence,

$$\forall x, y \in I', |f(x) - f(y)| < \varepsilon, \tag{3.3}$$

Similarly, let $y_0 \in B \cap (a', b')$. Since g is continuous at y_0, there is a $\delta'' > 0$, such that

$$\forall x \in I', |x - y_0| \leq \delta'' \implies |g(x) - g(y_0)| < \tfrac{\varepsilon}{2}. \tag{3.4}$$

Let $a'' := \max\{a', y_0 - \delta''\}$ and $b'' := \min\{b', y_0 + \delta''\}$. Consider the closed interval $I'' := [a'', b'']$, then $y_0 \in \overset{\circ}{I''}$, since $a' < y_0 < b'$ and $I'' \subseteq I'$, since $a' \leq a''$ and $b'' \leq b'$. If x and y are in I'', then (3.4) yields

$$\forall x, y \in I'', |g(x) - g(y)| < \varepsilon.$$

because $|x - y_0| \leq \delta''$ and $|y - y_0| \leq \delta''$.

By construction $I'' \subseteq I' \subseteq J$. Let K be a closed subinterval of I'' with length at most ε such that $K \subset \overset{\circ}{J}$. ☺

The idea of the proof of the theorem is to apply the lemma inductively to construct a nested sequence of close intervals I_n such that $|f(x) - f(y)| < \frac{1}{n}$ for all x, y in I_n.

Proof. [Proof of the theorem] Let $J := [a,b]$ be any closed and bounded subinterval of I and let $\varepsilon = 1$. By the lemma there is a closed interval $I_1 \subset \overset{\circ}{J}$ with length at most 1, such that

$$|f(x) - f(y)| < 1 \text{ and } |g(x) - g(y)| < 1$$

for all $x, y \in I_1$. Let $\varepsilon := 1/2$. By the lemma with $J = I_1$, there is a closed interval $I_2 \subset \overset{\circ}{I_1}$ with length at most $1/2$, such that

$$|f(x) - f(y)| < \tfrac{1}{2} \text{ and } |g(x) - g(y)| < \tfrac{1}{2}$$

for all $x, y \in I_2$. Continuing in this manner we get a sequence of closed intervals $I_n = [a_n, b_n]$ such that (a) the length $b_n - a_n$ of I_n is at most $1/n$, (b) $I_{n+1} \subset \overset{\circ}{I_n}$, and (c)

$$|f(x) - f(y)| < \tfrac{1}{n} \text{ and } |g(x) - g(y)| < \tfrac{1}{n} \tag{3.5}$$

for all $x, y \in I_n$. By the Nested Interval Theorem there is a z_0 real number such that

$$\{z_0\} = \bigcap_{n=1}^{\infty} I_n.$$

We claim f and g are continuous at z_0. To verify this claim, let $\varepsilon' > 0$ be given. Fix and integer $n > 1/\varepsilon'$. By construction $z_0 \in I_{n+1} \subset \overset{\circ}{I_n}$. Let

$$\delta := \min\{z_0 - a_n, b_n - z_0\}. \tag{3.6}$$

Then $\delta > 0$, since $z_0 \in \overset{\circ}{I_n} =]a_n, b_n[$. Suppose x is a real number and $|x - z_0| < \delta$. By (3.6), $x \in I_n$, consequently (3.5) implies

$$|f(x) - f(z)| < \tfrac{1}{n} < \varepsilon' \text{ and } |g(x) - g(z)| < \tfrac{1}{n} < \varepsilon'.$$

Establishing continuity of f and g at z_0. Hence, z_0 is in the intersection of A and B. Thus $A \cap B$ is nonempty. ☺

Combining the theorem with the Riemann function (Exercise 2.1.8) we conclude:

Corollary 3.4.3. *There does not exist an interval I and a function $f : I \to \mathbb{C}$, such that f is continuous at each rational in I and discontinuous at each irrational in I.*

This claim is part of Remark 2.1.9.

Corollary 3.4.4. *Consider an interval I and a function $f : I \to \mathbb{C}$. If f is continuous at each rational in I then f is continuous at least one irrational in I.*

3.5 Roots

In this section, we show that if x is a positive real number and n is a positive integer, then x has an n^{th}−root in the sense that $y^n = x$ for some real number y. We also present a method that can be used to show that certain roots are not rational numbers.

Existence of Roots

We will use the Binary Nested Interval Theorem to establish the existence the n^{th} root $x^{1/n}$ of a positive real number x.

Suppose we want to construct $\sqrt{2}$. Well, $1 < 2 < 4$. So taking square roots: $1 < \sqrt{2} < 2$. Repeatedly dividing the relevant interval, starting with $[1,2]$, in half gives

$$\begin{aligned} 1 < \sqrt{2} < \tfrac{3}{2}, &\quad \text{since } 2 < \left(\tfrac{3}{2}\right)^2; \\ \tfrac{5}{4} < \sqrt{2} < \tfrac{3}{2}, &\quad \text{since } \left(\tfrac{5}{4}\right)^2 < 2; \\ \tfrac{11}{8} < \sqrt{2} < \tfrac{3}{2}, &\quad \text{since } \left(\tfrac{11}{8}\right)^2 < 2; \\ \tfrac{11}{8} < \sqrt{2} < \tfrac{23}{16}, &\quad \text{since } 2 < \left(\tfrac{23}{16}\right)^2. \end{aligned}$$

Continuing this process by induction is the essence of our proof that 2 has a square root.

Theorem 3.5.1. *Let $x > 0$ be a real number and $k \in \mathbb{N}$. There is a real number $y > 0$, such that $y^k = x$. Thus $x^{1/k} = y$.*

Proof. If $k = 1$ or $x = 1$ we may set $y := x$. Hence, we will assume $k > 1$ and $x \neq 1$. If $0 < x < 1$, let $a_0 = x$ and $b_0 = 1$. If $1 < x$, let $a_0 = 1$ and $b_0 = x$. In either case $a_0^k < x < b_0^k$.

Let $c_0 := (a_0 + b_0)/2$. Then $a_0^k < c_0^k < b_0^k$. If $c_0^k = x$, let $y := c_0$. If $a_0^k < x < c_0^k$, let $a_1 := a_0$ and $b_1 := c_0$. If $c_0^k < x < b_0^k$ let $a_1 := c_0$ and $b_1 := b_0$. In either case, $a_1^k < x < b_1^k$.

Let $c_1 := (a_1 + b_1)/2$. Then $a_1^k < c_1^k < b_1^k$. If $c_1^k = x$, let $y := c_1$. If $a_1^k < x < c_1^k$, let $a_2 := a_1$ and $b_2 := c_1$. If $c_1^k < x < b_1^k$ let $a_2 := c_1$ and $b_2 := b_1$. In either case, $a_2^k < x^k < b_2^k$.

Continuing in this manner, either the process is stopped at some point because $c_n^k = x$, or we get a_n and b_n as in the Binary Nested Interval Theorem with $a_n^k < x < b_n^k$ for all $k \in \mathbb{N}$. Let y be the real number satisfying

$$\bigcap_{n=0}^{\infty} [a_n, b_n] = \{y\}.$$

It remains to check that $y^k = x$. Using $a_n^k < x < b_n^k$ and $a_n \leq y \leq b_n$ we conclude

$$\{x, y^k\} \subseteq \bigcap_{n=0}^{\infty} [a_n^k, b_n^k].$$

The intervals $[a_n^k, b_n^k]$ are nested, since the intervals $[a_n, b_n]$ are nested. Also,

$$b_n^k - a_n^k = (b_n - a_n) \sum_{j=0}^{k-1} b_n^j a_n^{(k-1)-j} \leq (b_n - a_n)M$$

where $M := \sum_{j=0}^{k-1} b_0^j b_0^{(k-1)-j} = k\, b_0^k$. Hence, $\left(b_n^k - a_n^k\right)$ is null. So by the Nested Interval Theorem $x = y^k$. ☺

Remark 3.5.2. The proof is an algorithm that allows us to estimate $y = x^{1/k}$, because $a_n \le y \le b_n$ implies $|c_n - y| \le (b_n - a_n)/2$, since c_n is the midpoint of $[a_n, b_n]$. So $b_n - a_n = (b_0 - a_0)/2^n$, tell us that $y \approx c_n$ with error at most $(b_0 - a_0)/2^{n+1} = |x-1|/2^{n+1}$.

Exercise 3.5.3. Prove $b^k - a^k = (b-a)\sum_{j=0}^{k-1} b^j a^{k-1-j}$.

Irrationality of Roots

We use the case $\sqrt{2}$ to illustrate a method that can be used to show that $(p/q)^{1/n}$ is irrational for many choices of positive integers p, q, and n.

Theorem 3.5.4. *$\sqrt{2}$ is irrational.*

Proof. The proof is by contradiction. Suppose $\sqrt{2}$ is rational. Let a, b be positive integers such that $\sqrt{2} = a/b$. For any $n \in \mathbb{N}_0$

$$\sqrt{2}^{2n} b = 2^n b \in \mathbb{N} \text{ and}$$
$$\sqrt{2}^{2n+1} b = 2^n \sqrt{2} b = 2^n a \in \mathbb{N}$$

hence, for all $k \in \mathbb{N}_0$, $\sqrt{2}^k b \in \mathbb{N}$.

Since $1 < \sqrt{2} < 2$ we have $0 < \sqrt{2} - 1 < 1$. Since $0 < \sqrt{2} - 1$ we have $0 < \left(\sqrt{2}-1\right)^n b$ for any $n \in \mathbb{N}$. But

$$\left(\sqrt{2}-1\right)^n b = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sqrt{2}^k b$$

is a sum of integers, hence an integer. Hence, $\left(\sqrt{2}-1\right)^n b \in \mathbb{N}$ for all $n \in \mathbb{N}$.

Since $0 < \sqrt{2} - 1 < 1$, the sequence $\left(\left(\sqrt{2}-1\right)^n\right)$ is null. Consequently, the sequence $\left(\left(\sqrt{2}-1\right)^n b\right)$ is null. But a convergent sequence in $\mathbb{N}$ has limit ≥ 1, hence cannot be null. This contradiction completes the proof. ☺

Alternatively, a contradiction can be obtained using well ordering of $\mathbb{N}$ in place of null sequences as follows. Since,

$$\left\{\left(\sqrt{2}-1\right)^n b \mid n \in \mathbb{N}\right\}$$

is a nonempty subset of $\mathbb{N}$, it has a smallest element. That is, there is an $n_0 \in \mathbb{N}$, such that

$$\left(\sqrt{2}-1\right)^{n_0} b \le \left(\sqrt{2}-1\right)^n b \text{ for all } n \in \mathbb{N}.$$

But $\sqrt{2}-1<1$ implies $\left(\sqrt{2}-1\right)^{n_0+1} b < \left(\sqrt{2}-1\right)^{n_0} b$ contradicting the choice of n_0.

Remark 3.5.5. The argument illustrates a general method for proving that numbers are irrational or even transcendental: (a) suppose the number of interest is rational or algebraic, (b) use (a) to set up equalities $x_n = y_n$, (c) establish $|x_n| < 1$, and (d) show (y_n) is a sequence of nonzero integers. See the proofs that e and π are irrational and e is transcendental for other examples of this strategy in action.

3.6 Cantor Set

Not everyone likes to work with numbers in base 10. Some like base 2. In this text popular choices are base 2, base 3, and base 10.

A finite decimal is a number of the form $d_0.d_1\cdots d_n = d_0 + d_1/10 + d_2/10^2 + \cdots + d_n/10^n$, where d_0 is an integer and d_k is in $\{0,1,\dots,9\}$ for $1 \le k$. Given an integer $b > 1$ we can replace the 10 by b. That is, we interpret the finite "decimal" $d_0 \underset{b}{.} d_1 \cdots d_n$ where $d_0 \in \mathbb{N}_0$ and $d_j \in \{0,1,\dots,b-1\}$ with respect to base b as

$$d_0 \underset{b}{.} d_1 \cdots d_n = d_0 + \frac{d_1}{b} + \frac{d_2}{b^2} + \cdots + \frac{d_n}{b^n}.$$

We can then expand this to infinite base b representations $d_0 \underset{b}{.} d_1 d_2 \cdots$. When $b = 2$ we say *binary number*, when $b = 3$ *ternary number*, and when $b = 10$ decimal number.

The discussion of nonunique expansions for decimal numbers carry over without major changes to base b representations. The details are left for the reader to work out. For example, any $0 \le x < 1$ can be represented as

$$x = 0 \underset{b}{.} d_1 d_2 \dots$$

and this representation is unique, if we do not allow representations ending in $\overline{b-1}$. The representation of $0 < x < 1$ is also unique, if we do not allow representations ending in $\overline{0}$. By long division any rational number has a representation in any base.

It remains to check that the set of real numbers does not depend on the choice of base. That is, for example, that set of ternary numbers coincides with the set of decimal numbers. Let

$$x := d_0 \underset{3}{.} d_1 d_2 \cdots$$

be a ternary number. Then

$$x \in \bigcap_{k=1}^{\infty} \left[d_0 \underset{3}{.} d_1 d_2 \cdots d_k, d_0 \underset{3}{.} d_1 d_2 \cdots d_k + \frac{1}{3^k} \right].$$

By the Nested Interval Theorem x is a decimal number. The proof that any decimal number is a ternary number is similar. However, we proved the Nested Interval Theorem for decimal numbers. So to establish that any decimal number is a ternary number we have to establish the Nested Interval Theorem for ternary numbers. Clearly, this can be done using the method we used to establish the Nested Interval Theorem for decimal numbers.

The Cantor set was introduced by Georg Ferdinand Ludwig Philipp Cantor (3 March 1845 Saint Petersburg - to January 1918 Halle) in 1883 while he was working on a problem related to Fourier series. The *Cantor set*, more precisely the middle thirds Cantor set, or ternary Cantor set, is the set

$$C := \left\{ 0\underset{3}{.}d_1 d_2 \ldots \,\middle|\, d_j \in \{0,2\} \right\}.$$

That is, C is the set of base three numbers, in the interval $[0,1]$, that can be constructed using only the digits 0 and 2.

The term middle thirds comes from an alternative construction of C. Let $C_0 := [0,1]$. Let C_1 be obtained from C_0 by removing the middle third of the interval in C_0. Inductively, let C_{k+1} be obtained from C_k by removing the open middle third of each intervals in C_k.

Fig. 3.5 The sets C_0, C_1, C_2, C_3

For example,

$$\begin{aligned} C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \text{ and} \\ C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]. \end{aligned}$$

See Fig. 3.5. Then C_1 is the ternary numbers having a ternary representation with $d_1 \neq 1$, C_2 is the ternary numbers having a ternary representation with $d_1 \neq 1$ and $d_2 \neq 1$, and so on. Consequently,

$$C = \bigcap_{k=0}^{\infty} C_k.$$

Exercise 3.6.1. The *Cantor function* $f : C \to [0,1]$ determined by

$$f\left(0\underset{3}{.}d_1 d_2 \ldots\right) = 0\underset{2}{.}\frac{d_1}{2}\frac{d_2}{2}\ldots \tag{3.7}$$

is surjective, but not injective.

Exercise 3.6.2. The Cantor type function $g : C \to [0,1]^2 = [0,1]\times[0,1]$ determined by

$$g\left(0._3 d_1 d_2 \dots\right) = \left(0._2 \frac{d_1}{2}\frac{d_3}{2}\dots, 0._2 \frac{d_2}{2}\frac{d_4}{2}\dots\right)$$

is onto and not one-to-one.

Problems

Problems for Sect. 3.1

1. Any finite set has a maximum.
2. If A has a maximum, then $\sup(A) = \max(A)$.
3. If r is irrational and $A :=]-\infty, r[$, then $\sup(A) = r$. In particular, $\sup(A)$ is not rational.
4. Let A be a subset of $\mathbb{R}$. Let U be a real numbers such that
   Prove $U = \sup(A)$, if

   a. $A \cap]U, \infty[= \emptyset$ and
   b. $A \cap]V, \infty[\neq \emptyset$, for any $V < U$.

5. Let A be a set of real numbers and let M be a real number. Suppose $A \cap]M, \infty[= \emptyset$ and
   $$\forall \varepsilon > 0, A \cap]M - \varepsilon, \infty[\neq \emptyset.$$
   Prove M is the least upper bound of A.
6. Let A be a set and let $g : A \to [0, \infty[$ be some function. Suppose there exists $M \geq 0$, such that
   $$\sum_{b \in B} g(b) \leq M$$
   for all finite subsets B of A. Show there is a real number L, such that
   $$\forall \varepsilon > 0, \exists \text{finite } B \subseteq A, \forall \text{finite } C \subseteq A, B \subseteq C \implies 0 \leq L - \sum_{c \in C} g(c) < \varepsilon.$$
   This is usually abbreviated as $\sum_{a \in A} g(a) = L$. [*Hint*: By assumption the set
   $$S := \left\{ \sum_{b \in B} g(b) \mid B \text{ a finite subset of } A \right\}$$
   has M as an upper bound. Let L be the supremum of S.]

Problems for Sect. 3.2

1. Prove $A :=]0,1[\cap \mathbb{Q}$ does not have the intermediate value property.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing, i.e., $x \leq y \implies f(x) \leq f(y)$. Fix a real number y_0. Let
$$A := \{x \in \mathbb{R} \mid y_0 \leq f(x)\}.$$
Suppose A is nonempty. Prove A is an interval. [*Hint*: f need not be onto.]

Problems for Sect. 3.3

1. Find two different sequences of closed intervals I_k as in (3.1) whose intersection equals $\{1.24\overline{0}\}$.
2. Use the Nested Interval Theorem to prove: If $0 \leq x$ and $x \leq 1/n$ for all $n \in \mathbb{N}$, then $x = 0$.
3. Consider the open intervals $I_n := \left]0, \frac{1}{n}\right[$. Since $\frac{1}{n+1} < \frac{1}{n}$ the intervals are nested: $I_{n+1} \subset I_n$. Find $\bigcap_{n=1}^{\infty} I_n$.
4. Consider the closed intervals $I_n := [n, \infty[$. Since $n < n+1$ the intervals are nested: $I_{n+1} \subset I_n$. Find $\bigcap_{n=1}^{\infty} I_n$.
5. Consider the open intervals $I_n := \left]-\frac{1}{n}, \frac{1}{n}\right[$. Since $\frac{1}{n+1} < \frac{1}{n}$ the intervals are nested: $I_{n+1} \subset I_n$. Find $\bigcap_{n=1}^{\infty} I_n$.

Problems for Sect. 3.4

1. Fill in the details of the induction in the proof of Theorem 3.4.1.
2. Give an example of a function $f : [0,1] \to \mathbb{R}$ that f is continuous on $\left[0, \frac{1}{2}\right]$ and discontinuous on $\left]\frac{1}{2}, 1\right]$ or prove that such a function does not exist.
3. Give an example of a function $f : [0,1] \to \mathbb{R}$ such that f is continuous on $\left[0, \frac{1}{2}\right[$ and discontinuous on $\left[\frac{1}{2}, 1\right]$ or prove that such a function does not exist.

Problems for Sect. 3.5

The strategy of our proof that $\sqrt{2}$ is irrational is roughly: Let x be a real number and m be an integer such that $m < x < m+1$. Suppose x is rational, use this and properties of x, to find a real number A such that $x^k A \in \mathbb{N}$ for all $k \in \mathbb{N}$. Then $((x-m)^n A)$ is a null sequence in $\mathbb{N}$, contradiction.

1. Use the strategy above to show that $\sqrt[3]{2} = 2^{1/3}$ is irrational.
2. Why can the strategy above not be used to show that $\frac{5}{3}$ is irrational? More precisely, why does there not exist an A such that $\left(\frac{5}{3}\right)^k A \in \mathbb{N}$ for all $k \in \mathbb{N}$.
3. Let $n \in \mathbb{N}$. Show that any $x > 0$ has at most one n^{th} root.

Problems for Sect. 3.6

Another way to construct the sets C_k is to use the functions, $f_0(x) := \frac{x}{3}$ and $f_2(x) := \frac{x}{3} + \frac{2}{3}$. Clearly, $C_{k+1} = f_0(C_k) \cup f_2(C_k)$. In terms of ternary numbers $f_m\left(0\underset{3}{.}d_1 d_2 \dots\right) = 0\underset{3}{.}m d_1 d_2 \dots$. The iterated function system approach to the Cantor set is based on the first problem below.

1. $C = f_0(C) \cup f_2(C)$.
2. If $x = 0\underset{3}{.}d_1 d_2 \dots$ where $d_j \in \{0, 2\}$ for all j. For each k, the point x is a left-hand endpoint of one of the intervals in C_k iff $d_j = 0$ for all $j > k$.
3. If $x = 0\underset{3}{.}d_1 d_2 \dots$ where $d_j \in \{0, 2\}$ for all j. For each k, the point x is a right-hand endpoint of one of the intervals in C_k iff $d_j = 2$ for all $j > k$.
4. Prove $1/4 \in C$. [*Hint*: $1/4 = 0\underset{3}{.}\overline{02}$.]
5. Prove $1/4$ is not an endpoint of an interval in C_k for any $k \in \mathbb{N}$.
6. Prove $1/5 \notin C$.
7. The Cantor function is continuous.
8. Prove that any point x_0 in the Cantor set C is an accumulation point of C.

Solutions and Hints for the Exercises

Exercise 3.1.3. (*i*) States that u is an upper bound for A. (*ii*) States that any upper bound for A is larger than u. Thus u is the smallest upper bound.

Exercise 3.1.6. Inductively, construct $x_n = d_0.d_1 d_2 \cdots d_n$ such that $-x_n - 1/10^n$ is not an upper bound for A and $-x_n$ is an upper bound for A.

Exercise 3.2.2. If $a = -\infty$, then A does not have a lower bound. In particular, t is not a lower bound for A. Hence there is $x \in A$, such that $x < t$.

Exercise 3.3.3. Recall $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{x\}$, where $x = \sup\{a_n \mid n \in \mathbb{N}\}$. Since $x-\delta < x$, $x-\delta$ is not an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. Hence, for some n_0 we have $x-\delta < a_{n_0}$. Similarly, there is an m_0 such that $b_m < x+\delta$. Setting $N := \max\{n_0, m_0\}$ completes the proof.

Exercise 3.3.5. The intervals $[a_k, b_k]$ are nested and $(b_k - a_k)$ is null. Hence there is real number x, such that $\bigcap_{k=1}^{\infty}[a_k, b_k] = \{x\}$. Similarly, there is a real number y, such that $\bigcap_{k=1}^{\infty}[c_k, d_k] = \{y\}$.

Exercise 3.5.3. Simplify the right-hand side.

Exercise 3.6.1. Any point y in $[0,1]$ has a binary representation

$$y = 0 \underset{2}{.} d_1 d_2 \ldots.$$

If $e_k := 2d_k$, then $x := 0 \underset{3}{.} e_1 e_2 \ldots$ is a point in C and $f(x) = y$. Thus f is onto.

That f is not one-to-one is a consequence of the nonuniqueness of the binary representation of some numbers in $[0,1]$. For example, $f\left(0 \underset{3}{.} 0\overline{2}\right)$ and $f\left(0 \underset{3}{.} 2\overline{0}\right)$ both equal $0 \underset{2}{.} 1\overline{0}$.

Exercise 3.6.2. Similar to Exercise 3.6.1.

Chapter 4
Counting

This chapter contains a brief introduction to cardinality. The focus is on countable sets. Of course, Cantor's results that an interval is not countable and that the set of irrational numbers is uncountable are also included. In fact, most of the results in this chapter are due to Cantor.

4.1 Countable Sets

Any set that can be arranged in a list $a_1, a_2, a_3, \ldots$, possibly with repetitions, is *countable*.

Example 4.1.1. $\{1,2,3\}$ can be arranged as $1,2,3,1,2,3,1,2,3,\ldots$ Hence, the finite set, $\{1,2,3\}$ is countable.

Similarly, any finite set is countable. An infinite set that is countable is *countably infinite*. The basic countably infinite set is

$$\mathbb{N} = \{1,2,3,\ldots\}.$$

Lemma 4.1.2. *The union of two countable sets is a countable set.*

Proof. Suppose A and B are countable. If $a_1, a_2, \ldots$ is a listing of A and $b_1, b_2, \ldots$ is a listing of B, then

$$a_1, b_1, a_2, b_2, a_3, b_3, \cdots$$

is the required list of the elements of the union $A \cup B$. ☺

Repeating this argument shows that a finite union of countable sets is countable.

Example 4.1.3. The set of integers $\mathbb{Z}$ is countable, because we can list the integers as

$$0, -1, 1, -2, 2, \ldots.$$

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Alternatively, we can write $\mathbb{Z}$ as the union of

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

and

$$-\mathbb{N} = \{-1, -2, \ldots\}.$$

Lemma 4.1.4. *If A_k is finite for all k in $\mathbb{N}$, then $\bigcup_{k=1}^{\infty} A_k$ is countable.*

Proof. Writing the elements of A_1 (in some order), then the elements of A_2, and so on gives the required list. ☺

This argument does not work, if at least one of the A_k's is countable infinite. For example, if A_1 is infinite we will never get to list the elements of A_2. However, it is true that a countable union of countable sets is a countable set:

Theorem 4.1.5. *If A_k is countable for all $k \in \mathbb{N}$, then $\bigcup_{k=1}^{\infty} A_k$ is countable.*

Proof. Write $A_k := \{a_{k,1}, a_{k,2}, a_{k,3}, \ldots\}$. Let $B_m := \{a_{i,j} \mid i + j = m\}$. See Fig. 4.1. Then B_m has $m-1$ element, hence is finite. Consequently, $\bigcup_{k=1}^{\infty} A_k = \bigcup_{m=2}^{\infty} B_m$ is countable by the lemma. ☺

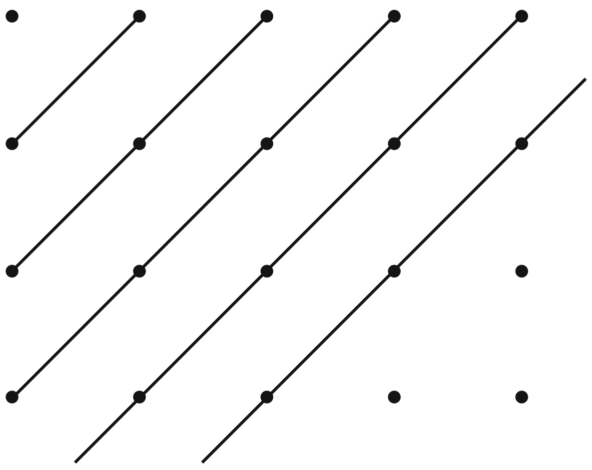

Fig. 4.1 Instead of counting along the rows we count along the diagonals. Using the notation from the proof of Theorem 4.1.5 A_1 is the top row, A_2 is the second row, etc. and B_2 is the diagonal with one dot, B_3 is the diagonal with two dots, and so on

Exercise 4.1.6. Use the theorem to prove that the set of rational numbers is countable.

Theorem 4.1.7 (Cantor). *The interval $[0, 1]$ is not countable.*

Proof. [First proof that $[0,1]$ is not countable.] Suppose $[0,1]$ is countable. Let $\{y_1, y_2, \ldots\} = [0,1]$. Let $x := 0.d_1 d_2 \cdots$ where

$$d_n := \begin{cases} 8 & \text{if the } n\text{th digit of } y_n \text{ is } \leq 4 \\ 3 & \text{if the } n\text{th digit of } y_n \text{ is } \geq 5 \end{cases}.$$

For any n, $x \neq y_n$, because they have different n^{th} digits. Consequently, x is not on the list. ☺

The choice of 3 and 8 as the digits is mostly arbitrary. We just wanted to avoid possible complications resulting from repeating 9's. This proof is an example of the Cantor Diagonal Argument.

Proof. [Second proof that $[0,1]$ is not countable.] Suppose $[0,1]$ is countable. Let $\{y_1, y_2, \ldots\} = [0,1]$. Let $[a_0, b_0] := [0,1]$. Inductively, for $k \in \mathbb{N}$, let $[a_k, b_k]$ be the left or right third of $[a_{k-1}, b_{k-1}]$ that does not contain y_k. If neither the left nor the right third of $[a_{k-1}, b_{k-1}]$ contains y_k we let $[a_k, b_k]$ be the left third of $[a_{k-1}, b_{k-1}]$. By construction $y_k \notin [a_k, b_k]$ for all k and the intervals $[a_k, b_k]$ are nested. Since $b_k - a_k = 1/3^k$ is null, it follows from the Nested Interval Theorem that

$$\bigcap_{k=0}^{\infty} [a_k, b_k] = \{x\},$$

for some $x \in [0,1]$.

It remains to check that x is not one of the $y_n's$. But for any n, $x \in [a_n, b_n]$ and $y_n \notin [a_n, b_n]$. Thus, $x \neq y_n$ for all n. ☺

We used thirds instead of halves in the proof, because $1/2$ is in the left and in the right half of $[0,1]$.

Exercise 4.1.8. The Cantor set C is not countable.

A set that is not countable is *uncountable*. A listing $a_1, a_2, a_3, \ldots$ of a countable set is an *enumeration* of that set. An enumeration of a countably infinite set can be assumed not to include duplicates. Such an enumeration can be constructed by starting at the beginning of the list and deleting duplicates as they are encountered when we move through the list.

Exercise 4.1.9. A subset of a countable set is countable.

Exercise 4.1.10. If A is uncountable and B is any set, then $A \cup B$ is uncountable.

Exercise 4.1.11. $\mathbb{R}$ is uncountable.

Exercise 4.1.12. If A and B are countable, so is $A \times B := \{(a,b) \mid a \in A, b \in B\}$.

4.2 Uncountable Sets★

We saw above that a set A is countably infinite iff it can be written as an infinite list $a_1, a_2, a_3, \ldots$ without duplicates. That is, a set A is countably infinite iff there is a bijection $f : \mathbb{N} \to A$. Specifically, when A is listed without duplicates the bijection is determined by $f(n) := a_n$. This idea can be used to compare the size of any two sets A and B. We say A and B have the *same number of elements* if there is a bijection $f : A \to B$. Similarly, A has *fewer elements* than B if there is a one-to-one may $f : A \to B$ and A has *more elements* than B if there is a one-to-one map $g : B \to A$. The following results say that this terminology makes sense.

Proposition 4.2.1. *Let A and B be sets. There is a one-to-one map $f : A \to B$ iff there is a surjective map $g : B \to A$.*

Proof. If $f : A \to B$ is $1-1$. Fix a point a_0 in A. Define

$$g(b) := \begin{cases} a & \text{if } b = f(a) \\ a_0 & \text{if } b \notin f(A) \end{cases}.$$

Then $g : B \to A$ is surjective.

Conversely, suppose $g : B \to A$ is surjective. For each $a \in A$, the set

$$g^{-1}(\{a\}) := \{b \in B \mid g(b) = a\}$$

is nonempty, since g maps B onto A. Let $f(a)$ be one of the elements of $g^{-1}(\{a\})$. Then $g(f(a)) = a$, so f is an injection. In fact, if $f(a_1) = f(a_2)$, then $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. ☺

Theorem 4.2.2 (Cantor–Bernstein–Schroeder Theorem). *If A both has fewer and more elements than B, then A and B have the same number of elements.*

Proof. Let $f : A \to B$ and $g : B \to A$ be $1-1$ functions. Let

$$\mathscr{A} := \{C \subseteq A \mid A \setminus g(B) \subseteq C \text{ and } g(f(C)) \subseteq C\}.$$

Note $A \in \mathscr{A}$. Let X be the intersection of all the sets in $\mathscr{A}$. That is

$$X = \cap\mathscr{A} = \bigcap_{C \in \mathscr{A}} C.$$

Clearly $A \setminus g(B)$ is a subset of X and $g(f(X)) \subseteq X$, hence $X \in \mathscr{A}$. By construction $C \in \mathscr{A} \implies X \subseteq C$. We aim to show that

$$h(x) = \begin{cases} f(t) & \text{if } t \in X \\ g^{-1}(t) & \text{if } t \in A \setminus X \end{cases}$$

is a bijection of A onto B. We need $A \setminus X \subseteq g(B)$ to make sense out of the second part of the definition. Once we have that, we need to verify that the resulting map is $1-1$ and onto.

Let $Y = g(f(X)) \cup (A \setminus g(B))$, then $Y \subseteq X \cup (A \setminus g(B)) \subseteq X$. Hence, $g(f(Y)) \subseteq g(f(X)) \subseteq X$, consequently, $Y \in \mathscr{A}$. So $X \subseteq Y$ and therefore $X = Y$. We have shown that

$$X = g(f(X)) \cup (A \setminus g(B)) \tag{4.1}$$

In particular, $A \setminus X \subseteq g(B)$.

If h is not $1-1$ then for some $s \in X$ and $t \in A \setminus X$ we have $f(s) = g^{-1}(t)$. That is $g(f(s)) = t$, but then $g(f(s)) \in X$ and $t \notin X$ gives a contradiction, hence h is $1-1$.

To show that h is onto pick $b \in B$, either $g(b) \in X$ or $g(b) \in A \setminus X$. In the second case $b = h(g(b)) \in h(A)$. In the first case $g(b) \in g(f(X))$ by (4.1), so $b \in f(X) \subseteq h(A)$. In both cases $b \in h(A)$, thus h is onto. ☺

The theorem is named after Cantor of Cantor set fame, Felix Bernstein (February 24, 1878, Halle to December 3, 1956, Zurich) and Friedrich Wilhelm Karl Ernst Schröder (25 November 1841 in Mannheim to 16 June 1902, Karlsruhe).

We discuss some examples suggesting that using bijections to decide if two sets have the same size is not always appropriate. The map $f(x) := 2x$ determines a bijection of $[0,1]$ onto $[0,2]$. So the two intervals have the same number of elements. Of course they have different lengths.

Example 4.2.3. (a) Exercise 3.6.1 constructs a surjection $C \to [0,1]$ mapping the Cantor set onto the interval $[0,1]$, hence (by the proposition) there is an injective map $f_1 : [0,1] \to C$. Clearly $f_2(x) = x$ is an injective may $C \to [0,1]$, so by Cantor–Bernstein–Schroeder there is a bijection $f : C \to [0,1]$.

(b) Similar to part (a) it follows from Exercise 3.6.2 that there is a bijection $g : C \to [0,1]^2$ between the Cantor set C and the square $[0,1]^2 = [0,1] \times [0,1]$.

(c) Combining (a) and (b) we see that, $h := g \circ f^{-1} : [0,1] \to [0,1]^2$ is a bijection between the interval $[0,1]$ and the square $[0,1]^2$. Hence, both the interval $[0,1]$ and the square $[0,1]^2$ both have the same number of elements as C.

To distinguish the size of the closed interval $[0,1]$ from the size of the square $[0,1]^2$ we might consider area or some notion of dimension, e.g., "topological" dimension.

ZFC is an abbreviation of: Zermelo–Fraenkel set theory with the axiom of choice, the standard set of axioms most mathematics is based on. ZFC is named after Ernst Friedrich Ferdinand Zermelo (27 July 1871 Berlin to 21 May 1953 Freiburg im Breisgauand) and Abraham Halevi (Adolf) Fraenkel (17 February 1891 Munich to 15 October 1965 Jerusalem). The C is for the axiom of choice.

The *continuum hypothesis*, due to Cantor, states that any set of real numbers is either countable or bijective to $[0,1]$. Less formally, no subset of $\mathbb{R}$ is both more infinite than $\mathbb{N}$ and less infinite than $[0,1]$.

In 1939 Kurt Friedrich Gödel (28 April 1906 Brno to 14 January 1978 Princeton) showed that the continuum hypothesis cannot be disproved based on ZFC. In 1963 Paul Joseph Cohen (2 April 1934 Long Branch to 23 March 2007 Stanford) showed that the continuum hypothesis cannot be proved based on on ZFC. Thus the continuum hypothesis is independent of "normal" mathematics in the sense that it can neither be proven nor disproven.

We have seen that there are problems with only considering the size of infinite sets using comparisons based on the existence of injective or bijective maps and that in some cases it seems more sensible to consider length, area, or volume as a measure of size. That difficulties must remain is shown by Stefan Banach (30 March 1892 Kraków to 31 August 1945 Lviv) and Alfred Tarski (14 January 1901 Warsaw to 26 October 1983 Berkeley):

Example 4.2.4 (Banach-Tarski Paradox, 1924). In $\mathbb{R}^3$ the unit ball $B := \{x \in \mathbb{R}^3 \mid |x| \leq 1\}$ can be partitioned into five disjoint sets E_j, $j = 1,2,3,4,5$ in such a way that $D_1 \cup D_2 = B$ and $B = D_3 \cup D_4 \cup D_5$ where each D_j is obtained from the corresponding E_j by a rigid motion. A rigid motion is a combination of rotations and translations.

Proof. You must be joking. This is well beyond the scope of this course. ☺

Despite all these difficulties the influential mathematician David Hilbert (23 January 1862 Königsberg to 14 February 1943 Göttingen) stated: "No one will drive us from the paradise which Cantor created for us" Hilbert (1926).

Problems

Problems for Sect. 4.1

1. Is the set of all enumerations of $\mathbb{N}$ countable?
2. The set of all subsets of $\mathbb{N}$ is not countable.

   A real number a is *algebraic* if it solves $p(x) = 0$ for some polynomial with integer coefficients. A real number that is not algebraic is *transcendental*.
3. The set of polynomials with integer coefficients is countable.
4. The set of algebraic numbers is countable.
5. The set of transcendental numbers is uncountable. In particular, some real numbers are transcendental.
6. Let R be the set of numbers of the form

$$0.a_1b_1 0a_2b_2 00a_3b_3 000a_4b_4 \ldots,$$

   where $\{a_k, b_k\} = \{0,1\}$ for all $k \in \mathbb{N}$.
   (a) Show each number in R is irrational.
   (b) Show the set R is uncountable.

7. Let A be a set, for example, A could be an interval. Consider a function $g : A \to \mathbb{R}$. Suppose there is a real number M, such that

$$-M \le \sum_{b \in B} g(b) \le M$$

for all *finite subsets* B *of* A. *Prove*

$$\{a \in A \mid g(a) \neq 0\}$$

*is countable. [*Hint*: for each integer* $n \ge 1$, *the set* $G_n := \left\{a \in A \mid g(a) > \frac{1}{n}\right\}$ *is finite, in fact has at most* Mn *elements.]*

8. *Let* $f :]0,1[\to \mathbb{R}$ *be determined by* $f(x) := \frac{1}{n}$ *for* $\frac{1}{n+1} \le x < \frac{1}{n}$ *and all* $n \in \mathbb{N}$. *Show that the set of points where* f *is discontinuous is an infinite countable set.*

Problems for Sect. 4.2

1. Find a bijection $f : [0,1] \to [0,1[$. [*Hint*: verify

$$f(x) := \begin{cases} 1/(n+1) & \text{when } x = 1/n, \text{ for some } n \in \mathbb{N} \\ x & \text{when } x \neq 1/n, \text{ for all } n \in \mathbb{N} \end{cases}$$

works.]

2. Find a bijection $g : [0,1[\to]0,1[$.

3. Find a bijection $h :]0,1[\to \mathbb{R}$.

Combining the three problems we conclude the map $h \circ g \circ f$ is a bijection $[0,1] \to \mathbb{R}$.

Solutions and Hints for the Exercises

Exercise 4.1.6 Let A_k be the rational numbers with denominator k.

Exercise 4.1.8 Mimic the proof that the interval $[0,1]$ is uncountable.

Exercise 4.1.9 Suppose $A \subseteq B$ and B is countable. If A is finite we are done. Suppose A is infinite. Enumerate the elements of B. Deleting the elements from the list that are not in A gives an enumeration of the elements of A.

Exercise 4.1.10 If $A \cup B$ is countable, then A is countable by Exercise 4.1.9.

Exercise 4.1.11 Use Cantor's Theorem and Exercise 4.1.10.

Exercise 4.1.12 Mimic the proof of Theorem 4.1.5

Chapter 5
Continuity

We discuss continuity and limit of monotone functions, the intermediate value theorem and show that a continuous image of a compact interval is a compact interval and that a continuous function defined on a compact interval is uniformly continuous. These results are all global in the sense that they depend on the function being continuous on an interval; the pointwise (local) results about continuity are contained in Chap. 2. The main tool used in the proofs is the Nested Interval Theorem.

5.1 Monotone Functions

Let I be some interval. A function $f : I \to \mathbb{R}$ is *increasing*, if for all x, y in I,

$$x < y \implies f(x) \leq f(y).$$

We say f is *strictly increasing*, if

$$x < y \implies f(x) < f(y).$$

Similarly, f is *decreasing* if $x < y$ implies $f(x) \geq f(y)$ and *strictly decreasing* if $x < y$ implies $f(x) > f(y)$. A function is *monotone* if it is increasing or decreasing. The same is the case for strictly monotone. Clearly, a function f is increasing iff $-f$ is decreasing, so any theorem about increasing functions has a companion theorem for decreasing functions.

Exercise 5.1.1. If $f : I \to \mathbb{R}$ is increasing, $x, y \in I$, and $f(x) < f(y)$, then $x < y$.

Exercise 5.1.2. If $f : I \to \mathbb{R}$ is increasing and $a \in I$, then $f(a)$ is an upper bound for $\{f(x) \mid x < a\}$ and $\lim_{x \nearrow a} f(x) = \sup\{f(x) \mid x < a\}$.

Similarly, if f is increasing, then $f(a)$ is a lower bound for $\{f(x) \mid a < x\}$ and $\lim_{x \searrow a} f(x) = \inf\{f(x) \mid a < x\}$. Consequently,

$$\lim_{x \nearrow a} f(x) \leq f(a) \leq \lim_{x \searrow a} f(x).$$

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Hence, an increasing function is continuous at a iff

$$\lim_{x\nearrow a} f(x) = \lim_{x\searrow a} f(x).$$

As mentioned above, a similar discussion applies to decreasing functions.

If f is a (not necessarily increasing) function, the one-sided limits at a exist, and

$$\lim_{x\nearrow a} f(x) \neq \lim_{x\searrow a} f(x),$$

then f has a *jump discontinuity* at a. Endpoints of an interval are special, because at an endpoint there only in *one* one-sided limit. For example, if $f : [a,b] \to \mathbb{R}$, the one-sided limit at a exists and

$$f(a) \neq \lim_{x\searrow a} f(x),$$

then f has a jump discontinuity at a. The previous discussion shows

Theorem 5.1.3. *Let I be an interval and suppose $f : I \to \mathbb{R}$ is monotone. Let $a \in I$. Then f is discontinuous at a iff f has a jump discontinuity at a.*

Corollary 5.1.4. *A monotone function has a countable, perhaps empty, set of discontinuities.*

Proof. Suppose f is increasing. Let A be the set of discontinuities. If $a \in A$, then $\lim_{x\nearrow a} f(x) < \lim_{x\searrow a} f(x)$, since f is increasing and discontinuous at a. Consider the open interval $I_a := \left]\lim_{x\nearrow a} f(x), \lim_{x\searrow a} f(x)\right[$. If $a' < a$, then $\lim_{x\searrow a'} f(x) \leq f\left(\frac{a+a'}{2}\right) \leq \lim_{x\nearrow a} f(x)$, hence the intervals $I_{a'}$ and I_a are disjoint. By density of the rationals, each I_a contains a rational number r_a. If $a \neq a'$, then $r_a \neq r_{a'}$, since $I_a \cap I_{a'}$ is empty. Consequently, an enumeration of the rationals $\{r_a \mid a \in A\}$ gives an enumeration of A. Thus A is countable. ☺

Example 5.1.5. Let a_k, $k = 1, 2, \ldots$ be an enumeration of the rationals in the open interval $]0,1[$. Consider the function determined by $f(0) = 0$ and if $0 < x \leq 1$ then

$$f(x) = \sum_{k, a_k < x} \frac{1}{2^k} = \sum_{j=1}^{\infty} \frac{1}{2^{k_j}},$$

where $k_j = k_j(x)$ is the subsequence of subscripts k for which $a_k < x$. Then, f is an increasing function mapping the closed interval $[0,1]$ into itself and f is discontinuous at every rational and is the open interval with endpoints 0 and 1. Note, $f(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

Proof. If $x < y$, then $f(y) - f(x) = \sum_{k, x \leq a_k < y} \frac{1}{2^k}$. Clearly, $\lim_{x\nearrow a_j} f(x) \leq f(a_j) = \sum_{k, a_k < a_j} \frac{1}{2^k}$ and $\lim_{x\searrow a_j} f(x) \geq \sum_{k, a_k \leq a_j} \frac{1}{2^k}$. Hence, $\lim_{x\searrow a_j} f(x) - \lim_{x\nearrow a_j} f(x) \geq \frac{1}{2^j}$. It is easy to show that this is an equality, but we do not need that. ☺

The following result, characterizing continuity of a monotone function, is the main result in this section.

Theorem 5.1.6 (Continuity Theorem for Monotone Functions). *A monotone function defined on an interval is continuous iff its range is an interval.*

Proof. Let I be an interval and let $f : I \to \mathbb{R}$ be increasing.

$\Longleftarrow$: Suppose the range of f is an interval J. Let $x_0 \in I$. Let $\varepsilon > 0$ be given. Suppose x_0 is not the left endpoint of I.

(a) If $f(x_0) - \varepsilon \in J$, then there is an $x_1 \in I$ such that $f(x_1) = f(x_0) - \varepsilon$. $f(x_1) < f(x_0)$ implies $x_1 < x_0$. Hence, $x_1 < x < x_0$ implies $f(x_0) - \varepsilon \le f(x) \le f(x_0)$. (b) If $f(x_0) - \varepsilon \notin J$, then $f(x_0) - \varepsilon < f(x) \le f(x_0)$ for all $x \in I$ with $x < x_0$. Fix any $x_1 < x_0$ in I, then $x_1 < x < x_0$ implies $f(x_0) - \varepsilon < f(x) \le f(x_0)$. Hence, both in case (a) and in (b) we have shown that $\lim_{x \nearrow x_0} f(x) = f(x_0)$.

Similarly, if x_0 is not the right hand endpoint of I, then $\lim_{x \searrow x_0} f(x) = f(x_0)$. Consequently, f is continuous at x_0.

$\Longrightarrow$: Suppose f is continuous. We will show that the range of f has the intermediate value property. Let $x_0, y_0 \in I$ and let $t \in \mathbb{R}$ such that $f(x_0) < t < f(y_0)$. We must show that t is in the range of f. Since $f(x_0) < f(y_0)$ and f is increasing $x_0 < y_0$. Let

$$A := \{a \in I \mid f(a) < t\}$$
$$B := \{b \in I \mid t \le f(b)\}.$$

Clearly, $x_0 \in A$, $y_0 \in B$, $A \cap B = \emptyset$, and $A \cup B = I$. Since f is increasing, y_0 is an upper bound for A. Hence, $p := \sup(A)$ is finite. Similarly, $q := \inf(B)$ is finite.

Exercise 5.1.7. A and B have the intermediate value property.

By the Interval Theorem A and B are intervals. Hence A is a subinterval of I and $p = \sup(A)$, so $A = I \cap]-\infty, p]$ or $A = I \cap]-\infty, p[$. Similarly, $B = I \cap [q, \infty[$ or $B = I \cap]q, \infty[$. Since $A \cup B = I$, and I has the intermediate value property $p = q$. By definition of A, $\lim_{x \nearrow p} f(x) \le t$ and by definition of B, $\lim_{x \searrow p} f(x) \ge t$. So continuity of f at p, implies $f(p) = t$.

The case where f is decreasing is most simply dealt with by observing that $g = -f$ is increasing. ☺

The $\Longrightarrow$ part of the Continuity Theorem for Monotone Functions is a special case of the Intermediate Value Theorem (Theorem 5.2.2).

By the Interval Theorem:

Corollary 5.1.8. *A monotone function defined on an interval is continuous iff its range has the intermediate value property.*

If f is a strictly monotone function, then f is $1-1$, hence f has an inverse function f^{-1}.

Exercise 5.1.9. The inverse of a strictly increasing function is strictly increasing.

Corollary 5.1.10. *The inverse of a strictly monotone continuous function defined on an interval is continuous.*

Proof. Let I be an interval and suppose $f : I \to \mathbb{R}$ is a strictly increasing and continuous. The range of f is an interval J. The inverse function f^{-1} is strictly increasing and maps J onto the interval I. Hence, f^{-1} is continuous. ☺

The following exercise gives another proof of the existence of roots, Theorem 3.5.1.

Exercise 5.1.11. Let $n \in \mathbb{N}$. $f : [0,\infty[\to \mathbb{R}$ be determined by $f(x) := x^n$. Prove

1. f is strictly increasing
2. The range of f is the interval $[0,\infty[$
3. $g(x) := x^{1/n}$ is continuous: $[0,\infty[\to [0,\infty[$

5.2 The Intermediate Value Theorem

Solving equations of the form $f(x) = 0$ plays a large role in mathematics. For example, the existence of the nth root of $a > 0$ is equivalent to the equation $x^n - a = 0$ having a solution. The Intermediate Value Theorem leads to another proof of the existence of roots. In fact, the proof of the Intermediate Value Theorem is remarkably similar to our proof, that roots exists, i.e., to the proof of Theorem 3.5.1.

The Intermediate Value Theorem is a slight extension of a theorem due to Bernhard Placidus Johann Nepomuk Bolzano (5 October 1781 Prague to 18 December 1848 Prague). Contrary to most mathematicians of his era, Bolzano believed intuitive ideas, for example time and motion, do not belong in mathematics. Consequently, he was one of the first mathematician to insist on rigor in mathematics. Bolzano's notion of a limit was similar to the modern one. Since Bolzano's work predates Weierstrass' by some 50 years, perhaps we should have credited our definition of limits to Bolzano.

Theorem 5.2.1 (Bolzano's Intermediate Value Theorem). *Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous function. If there are points a and b in I, such that $f(a) < 0$ and $f(b) > 0$, then there is a point c in the open interval with endpoints a and b, such that $f(c) = 0$.*

Proof. Suppose $a < b$. Let $a_0 := a$ and $b_0 := b$. Then, $f(a_0) < 0$ and $f(b_0) > 0$. Let $c_0 := (a_0 + b_0)/2$ be the midpoint of $[a_0, b_0]$. If $f(c_0) = 0$, we are done. If $f(c_0) \le 0$, let $a_1 := c_0$ and $b_1 := b_0$. If $f(c_0) > 0$, let $a_1 := a_0$ and $b_1 := c_0$. In either case, $f(a_1) \le 0$ and $f(b_1) > 0$. Continuing in this manner, we get a_n and b_n such that, for all n, $f(a_n) \le 0$, $f(b_n) > 0$, and for all n, $c_n := (a_n + b_n)/2$, and $a_{n+1} = a_n$ and $b_n = c_n$ or $a_{n+1} = c_n$ and $b_{n+1} = b_n$. Hence, for some real number c

$$\bigcap_{n=0}^{\infty} [a_n, b_n] = \{c\}.$$

Since $f(a_n) \le 0$ for all n, $f(c) = \lim_{n\to\infty} f(a_n) \le 0$. Similarly, $f(b_n) > 0$ for all n, implies $f(c) = \lim_{n\to\infty} f(b_n) \ge 0$. Consequently, $f(c) = 0$. ☺

The proof gives a method for approximating the value of c. At any stage c is between a_n and b_n. Hence, $|c-c_n| \leq (b-a)/2^{n+1}$. See the remark following Theorem 3.5.1.

Theorem 5.2.2 (Intermediate Value Theorem). *If a real valued function f is continuous on some interval I, then the image $f(I)$ of that interval has the intermediate value property.*

Proof. Let $a < b$ be in I. Let y_0 be between $f(a)$ and $f(b)$. If $f(a) < f(b)$ apply Bolzano's Intermediate Value Theorem to $g(x) := f(x) - y_0$. If $f(a) > f(b)$ apply Bolzano't Intermediate Value Theorem to $g(x) := y_0 - f(x)$. ☺

Corollary 5.2.3. *If a real valued function f is a continuous on some interval I, then $f(I)$ is an interval.*

This gives another proof of one part of the Continuity Theorem for Monotone Functions (Theorem 5.1.6).

Exercise 5.2.4. Let I be an interval and $f : I \to \mathbb{R}$ be continuous. Supposing f is $1-1$, prove f is strictly monotone on I.

5.3 Continuous Images of Compact Intervals

Suppose $a < b$ are real numbers. Let $I := [a,b]$ be the corresponding closed interval. We will call intervals of this form as *compact intervals*. If $f : [a,b] \to \mathbb{R}$ is continuous, then it follows from the Intermediate Value Theorem that $J := f(I)$ is an interval. The endpoints of the interval $f(I)$ are

$$\begin{aligned} \inf(f(I)) &= \inf\{f(x) \mid x \in I\} \text{ and} \\ \sup(f(I)) &= \sup\{f(x) \mid x \in I\}. \end{aligned}$$

In this section we will show that $\inf(f(I))$ and $\sup(f(I))$ are real numbers in the range of f, consequently

$$f([a,b]) = [\inf(f([a,b]), \sup(f([a,b]))].$$

In particular, continuous functions map compact intervals onto compact intervals. The proof is divided into two steps. The first step, The Global Boundedness Theorem, shows that $\inf(f(I))$ and $\sup(f(I))$ are real numbers. The second step, The Extreme Value Theorem, shows that $\inf(f(I))$ and $\sup(f(I))$ are contained in the interval $f(I)$.

The following examples illustrate some of the reasons we consider compact intervals I.

Example 5.3.1. If $f(x) := \frac{1}{x}$ and $I :=]0,1[$, then $f(I) =]1,\infty[$. While the endpoints of I are real numbers, the right hand endpoint of $f(I)$ is not a real number.

Example 5.3.2. If $f(x) := x^2$ and $I :=]-2,1[$, then $f(I) = [0,4[$. A continuous image of a an open interval need not be an open interval.

Global Boundedness Theorem

A function $f : D \to \mathbb{C}$ is *bounded* on $E \subseteq D$, if there is an M, such that $|f(z)| \leq M$ for all $z \in E$. If f is real valued this means that $-M \leq f(z) \leq M$ for all $z \in E$. The Local Boundedness Theorem for limits states that if $\lim_{x\to a} f(x)$ exists, then f is bounded near a. Consequently, if f is continuous at a, then f is bounded near a.

Theorem 5.3.3 (Global Boundedness Theorem). *Let I be compact interval. If $f : I \to \mathbb{C}$ is continuous on I, then f is bounded on I.*

Proof. Suppose f is not bounded. Bisect I. If f is bounded in both halves of I, then f is bounded on I. Hence, f is unbounded in at least one of the two halves of I.

Repeating this argument we get a sequence of nested intervals $I_{n+1} \subset I_n$, such that f is unbounded in each of the intervals I_n. By the Nested Interval Theorem, the intersection $\bigcap I_n$ of these intervals contains exactly one point, call this point x_0.

We will show f is bounded on one of the intervals I_n obtained by bisection. Note f is continuous at x_0. By Local Boundedness, f is bounded on the open interval $]x_0 - \delta, x_0 + \delta[$, for some δ. For sufficiently large n, the interval I_n is a subset of $]x_0 - \delta, x_0 + \delta[$, this is Exercise 3.3.3. On these intervals f is both bounded (by choice of δ) and unbounded (by construction of I_n). This contradiction completes the proof. ☺

Exercise 5.3.4. Let I and J be compact intervals. Suppose $f : I \times J \to \mathbb{C}$ is continuous. Show f is bounded.

Extreme Value Theorem

If a function has a largest value, then that value is called the maximum of the function. Similarly, a smallest value, if it exists, is called a minimum. These notions are important enough that we will also write them out using symbols. Let $f : D \to \mathbb{R}$ be a function. If $x_{\max} \in D$ is a point such that

$$f(x) \leq f(x_{\max}) \text{ for all } x \in D,$$

then $f(x_{\max})$ is a *global maximum* of f. Similarly, $f(x_{\min})$ is a *global minimum* of f, if

$$f(x_{\min}) \leq f(x) \text{ for all } x \in D.$$

Theorem 5.3.5 (Extreme Value Theorem). *Let I be a compact interval. If $f : I \to \mathbb{R}$ is continuous on I, then there are $x_{\min}$ and $x_{\max}$ in I, such that*

$$f(x_{\min}) \le f(x) \le f(x_{\max}), \textit{ for all } x \in I.$$

Proof. Let $M := \sup(f(I))$. Since f is bounded, M is a real number. Bisect I. If the supremum of f on both halves is $< M$, then the supremum of f over all of I is $< M$. Hence, we can choose a half of I, such that the supremum of f over that half is M.

Repeating this argument we get a nested sequence of intervals I_n, such that the supremum of f over each interval I_n is M. By the Nested Interval Theorem, the intersection of these intervals contains exactly one real number. Call this number x_0.

We will show that $f(x_0) = M$. That is $x_{\max} = x_0$. We know $f(x) \le M$, for all $x \in I$. In particular, $f(x_0) \le M$. Suppose $f(x_0) < M$. Let $\varepsilon := (M - f(x_0))/2$. Then $\varepsilon > 0$ and $f(x_0) + \varepsilon < M$. Since, f is continuous at x_0, there is a $\delta > 0$, such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Hence, $x \in]x_0 - \delta, x_0 + \delta[$ implies $f(x) < f(x_0) + \varepsilon < M$. (Compare the argument in this paragraph to the proof of Local Positivity.)

Pick N, such that $I_N \subset]x_0 - \delta, x_0 + \delta[$. Then

$$\begin{aligned} M &= \sup\{f(x) \mid x \in I_N\} \\ &\le \sup\{f(x) \mid x \in]x_0 - \delta, x_0 + \delta[\} \\ &\le f(x_0) + \varepsilon \\ &< M. \end{aligned}$$

This contradiction shows that $f(x_0) = M$. The existence of $x_{\min}$ can be established in a similar fashion, or by applying the existence of $x_{\max}$ to $g = -f$. ☺

Corollary 5.3.6. *If $f : [a,b] \to \mathbb{R}$ is continuous and $x_{\min}$ and $x_{\max}$ are point in $[a,b]$, such that $f(x_{\min}) \le f(x) \le f(x_{\max})$ for all $x \in [a,b]$, then*

$$f([a,b]) = [f(x_{\min}), f(x_{\max})].$$

Proof. By assumption $f([a,b]) \subseteq [f(x_{\min}), f(x_{\max})]$. The reverse inclusion $\supseteq$ is a consequence of the Intermediate Value Theorem. ☺

Exercise 5.3.7. Let I and J be compact intervals. Suppose $f : I \times J \to \mathbb{R}$ is continuous. Show there are $(x_{\min}, y_{\min})$ and $(x_{\max}, y_{\max})$ in $I \times J$, such that

$$f(x_{\min}, y_{\min}) \le f(x,y) \le f(x_{\max}, y_{\max}), \text{ for all } (x,y) \in I \times J.$$

5.4 Uniform Continuity

A function $f : D \to \mathbb{C}$ is *uniformly continuous on D*, if given any $\varepsilon > 0$, there is a $\delta > 0$, such that for all x, y in D, $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. In symbols,

$$\forall \varepsilon > 0, \exists \delta > 0. \forall x, y \in D, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Note $\delta = \delta(\varepsilon)$ only depends on ε.

Comparing this to f being continuous on D :

$$\forall x \in D, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in D, |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

We see that here $\delta = \delta(x, \varepsilon)$ can depend on both x and ε. So uniform continuity means that the same δ works for all x, the choice of δ is *uniform* in x.

Remark 5.4.1. At this point uniform continuity is a concept without applications. However, we will need uniform continuity to show that continuous functions are integrable later.

Example 5.4.2. Let $f(z) := z^2$. Then f is uniformly continuous on $D := \{z \in \mathbb{C} \mid |z| < 7\}$.

Proof. Let $\varepsilon > 0$ be given. For any $w, z \in D$,

$$|f(w) - f(z)| = |w^2 - z^2| = |w+z|\,|w-z| < 14|w-z|$$

hence $|w-z| < \varepsilon/14$ implies $|f(w) - f(z)| < \varepsilon$. Consequently, $\delta := \varepsilon/14$. ☺

Example 5.4.3. $f :]0,1[\to \mathbb{R}$ determined by $f(x) := 1/x$ is not uniformly continuous.

Proof. Let $\varepsilon := 1$. Let $\delta > 0$ be small, say $< 1/2$. For $0 < x < 1/2$, let $y := x + \delta$. Note x and y both are in $]0,1[$. We want to choose x, such that

$$f(x) - f(y) = \frac{1}{x} - \frac{1}{x+\delta} \geq 1.$$

Now

$$\frac{1}{x} - \frac{1}{x+\delta} = \frac{\delta}{x(x+\delta)}.$$

Hence, we just need x such that $\delta \geq x(x+\delta)$, that is, such that

$$0 \geq x^2 + \delta x - \delta.$$

One way to do this is to solve the equality for x, using the quadratic formula or equivalently, by factoring the quadratic. Another is trial and error. Guessing $x = \delta/2$, and using $0 < \delta < 1 \implies \delta^2 < \delta$, we get

$$x^2 + \delta x - \delta = \frac{3}{4}\delta^2 - \delta < \frac{3}{4}\delta - \delta < 0.$$

☺

Example 5.4.4. Let A be a subset of $\mathbb{C}$. Let $D_A(x) := \inf\{|x-a| \mid a \in A\}$ be the distance from $x \in \mathbb{C}$ to A. Then D_A is uniformly continuous on $\mathbb{C}$.

Proof. By the triangle inequality

$$D_A(x) \leq |x-a| \leq |x-y| + |y-b| + |b-a|.$$

for all a, b in A. Setting $a = b$ on the right hand side, gives

$$D_A(x) \leq |x-y| + |y-b|$$

for all $b \in A$. Taking the infimum over $b \in A$ gives

$$D_A(y) \leq |x-y| + D_A(y).$$

Hence $D_A(x) - D_A(y) \leq |x-y|$. Interchanging the roles of x and y leads to

$$|D_A(x) - D_A(y)| \leq |x-y|.$$

Consequently, setting $\delta := \varepsilon$ verifies uniform continuity of D_A. ☺

Lemma 5.4.5. *Let $f : [0,1] \to \mathbb{C}$ be continuous. Suppose f is uniformly continuous on $I_1 := [0,1/2]$, on $I_2 := [1/4,3/4]$, and on $I_{3:=}[1/2,1]$. Then f is uniformly continuous on $[0,1]$.*

Proof. Let $\varepsilon > 0$ be given. For each $j = 1,2,3$, pick a uniform continuity $\delta_j > 0$ corresponding to ε on I_j. Let $\delta := \min\{\delta_1, \delta_2, \delta_3, 1/2\}$. Let $x, y \in [0,1]$ with $|x-y| < \delta$. $|x-y| < \delta \leq 1/2$ implies $x, y \in I_j$ for, at least, one of the three j's. For that j, $|x-y| < \delta \leq \delta_j$ implies $|f(x) - f(y)| < \varepsilon$. ☺

The main result is this section is:

Theorem 5.4.6 (Uniform Continuity Theorem). *Let $f : [a,b] \to \mathbb{C}$ be continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.*

Proof. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Suppose f is not uniformly continuous on $[a,b]$. Let $\varepsilon_0 > 0$ be such that there is no corresponding uniform continuity δ on $[a,b]$. By the proof of Lemma 5.4.5, there is no uniform continuity δ corresponding to ε_0 on one, or more, of the three "halves" of $[a,b]$. Setting $a_1 := a$ and $b_1 := b$ and repeating this argument, we get, for each $n \in \mathbb{N}$, a_n and b_n such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, $b_{n+1} - a_{n+1} = (b_n - a_n)/2$, and there is no uniform continuity δ corresponding to ε_0 on $[a_n, b_n]$.

Since the the intervals $[a_n, b_n]$ are nested and $b_n - a_n = (b-a)/2^{n-1}$ is null, the Nested Interval Theorem produces an x_0 such that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x_0\}.$$

By assumption, f is continuous at x_0. So since $\varepsilon_0/2 > 0$, there is a $\delta_0 > 0$, such that

$$\forall x \in [a,b], |x - x_0| < \delta_0 \implies |f(x) - f(x_0)| < \varepsilon_0/2.$$

Let N be large enough that $[a_N, b_N] \subset]x_0 - \delta_0, x_0 + \delta_0[$. Then for all $x, y \in [a_N, b_N]$, $|x - x_0| < \delta_0$ and $|y - x_0| < \delta_0$, hence

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0.$$

Therefore, $x, y \in [a_N, b_N] \implies |f(x) - f(y)| < \varepsilon_0$. Thus any $\delta > 0$, for example $\delta := 1$, is a uniformly continuity δ corresponding to ε_0 on $[a_N, b_N]$. But this contradicts the construction of $[a_n, b_n]$.

Exercise 5.4.7. If $f : [a,b] \times [c,d] \to \mathbb{C}$ is continuous on $[a,b] \times [c,d]$, then f is uniformly continuous on $[a,b] \times [c,d]$.

Problems

Problems for Sect. 5.1

1. Modify the definition of f in Problem 2 for Sect. 2.1 in such at way that the modified f is strictly increasing, yet still has the same continuity/discontinuity properties as the original f.
2. Let $f : [-1/2, 1/2] \to \mathbb{R}$ be determined by $f(x) = \sigma(x)$, where σ is the pseudo-sine function. Show that f is strictly increasing and the range of f is $[-1,1]$. Consequently, there is a continuous $g : [-1,1] \to [-1/2, 1/2]$ such that $g \circ f(x) = x$ and $f \circ g(y) = y$ for $x \in [-1/2, 1/2]$ and $y \in [-1,1]$. Give a formula for g.
3. Let $f : [a,b] \to \mathbb{R}$ satisfy $x < y \implies f(x) \le f(y)$. For each $x \in [a,b[$ let $f(x+) := \lim_{t \searrow x} f(t)$, for each $x \in]a,b]$ let $f(x-) := \lim_{t \nearrow x} f(t)$, $f(b+) := f(b)$, and $f(a-) := f(a)$. For each x in $[a,b]$, let
$$J(x) := f(x+) - f(x-).$$
Completing the following steps gives an alternative proof of the fact that a monotone functions is discontinuous on a countable set.
   a. Prove $J(x) \ge 0$ for all $x \in [a,b]$.
   b. f is discontinuous at x iff $J(x) > 0$.
   c. If $a \le x < y \le b$, then $f(x+) \le f(y-)$.
   d. For each $x \in [a,b]$, let $S_x := \{y \in \mathbb{R} \mid f(x-) < y < f(x+)\}$. Thus, S_x is either empty or an open interval. Show S_x is a subset of the closed interval $[f(a), f(b)]$ and that if $x \ne y$, then $S_x \cap S_y = \emptyset$.
   e. $\sum_{x \in [a,b]} J(x) \le f(b) - f(a)$.
   f. The set $\{x \in [a,b] \mid J(x) > 0\}$ is countable.
4. The Cantor function (defined in Exercise 3.6.1) is increasing and continuous.

The *Devil's Staircase* is the continuous function $g : [0,1] \to [0,1]$ which agrees with the Cantor function on the Cantor set and is constant on the intervals that are deleted in the construction of the Cantor set. Below are two constructions of the Devil's Staircase:

Let f denote the Cantor function.

1. The function $g:[0,1]\to[0,1]$ is determined by $g(x):=f(x)$ when $x\in C$. If $x\notin C$, then $x\notin C_k$ for some k. Hence, x is in one of the deleted intervals, let y be the left hand endpoint of that interval and set $g(x):=f(y)$.
2. Alternatively, suppose $x=0\underset{3}{.}d_1d_2\ldots$. If all $d_j\neq 1$, let $y:=x$. If some $d_k=1$, let $y=0\underset{3}{.}d_1d_2\ldots d_{n-1}\overline{2}$, where $d_j\neq 1$ for $1\le j<n$, and $d_n=1$. Then $g(x):=f(y)$.

Problems for Sect. 5.2

1. Let $f:[0,1]\to[0,1]$ be continuous. There is a $c\in[0,1]$, such that $f(c)=c$. [*Hint*: Consider $g(x):=x-f(x)$.] A point c such that $f(c)=c$ is called a *fixed point*.
2. Let $p(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ where the a_k's are real and $n\in\mathbb{N}$ is odd. Prove $p(c)=0$ for some real number c.
3. Let $a>0$ be a real number and let $n\in\mathbb{N}$. Use Bolzano's Intermediate Value Theorem to show that $a^{1/n}$ exists.
4. If I and J are intervals and $f:I\times J\to\mathbb{R}$ is continuous, then the range of f is an interval. [*Hint*: (a) One way to prove this is to imitate the proof above: Suppose $f(a,b)<0$ and $f(c,d)>0$. Construct nested intervals $[a_n,b_n]$ and $[c_n,d_n]$ such that $f(a_n,c_n)\le 0$ and $f(b_n,d_n)>0$. Then show $f(x_0,y_0)=0$ if $\bigcap_{n=1}^{\infty}[a_n,b_n]=\{x_0\}$ and $\bigcap_{n=1}^{\infty}[c_n,d_n]=\{y_0\}$. ($b$) Another way it to use Bolzano's Intermediate Value Theorem: Suppose $f(a,b)<0$ and $f(c,d)>0$. Let $\phi:[0,1]\to\mathbb{R}^2$ be determined by $\phi(t):=(1-t)(a,b)+t(c,d)$. Apply Bolzano's Intermediate Value Theorem to $g:=f\circ\phi$.]
5. Prove that a rectangle is pathwise connected.
6. Prove that any pathwise connected subset of $\mathbb{R}$ is an interval.
7. Suppose D is pathwise connected and $f:D\to\mathbb{R}$ is continuous. Prove $f(D)$ is an interval.
8. Let $f:[0,2]\to\mathbb{R}$ be continuous. Suppose $f(1)<f(0)<f(2)$. Prove f is not one-to-one.

Problems for Sect. 5.3

1. Let $f:\mathbb{R}\to]0,\infty[$ be continuous. Suppose

$$\lim_{x\to-\infty}f(x)=0 \text{ and } \lim_{x\to\infty}f(x)=0.$$

a. Prove f does not have a minimum on $\mathbb{R}$.
b. Prove f has a maximum on $\mathbb{R}$.

2. Let p be a polynomial of even degree > 0. Show that p has a maximum or a minimum, but not both.
3. If $f : [a,b] \to \mathbb{C}$ is continuous, then $g(x) := |f(x)|$ has a maximum on $I := [a,b]$.
4. Let $I_0 := [0,1]$ and $0 < c < 1$. Suppose $f : I_0 \to I_0$ is a function satisfying

$$|f(x) - f(y)| \leq c\,|x-y|\,, \text{ for all } x, y \in I_0.$$

Inductively, let $I_{n+1} := f(I_n)$ for integers $n \geq 0$. Prove:

a. $I_{n+1} \subseteq I_n$ for all $n \geq 0$.
b. The length of I_n is $\leq c^n$.
c. By the Nested Interval Theorem $\bigcap_{n=0}^{\infty} I_n$ contains exactly one point. If $\{x_0\} = \bigcap_{n=0}^{\infty} I_n$, then $f(x_0) = x_0$.

Comment on this problem: The existence of a fixed points $f(x_0) = x_0$ is a simple consequence of the Intermediate Value Theorem, see the Problems for Sect. 5.2. The point of this problem is: if y_0 is any point in I_0, and inductively $y_{n+1} := f(y_n)$, then $|y_n - x_0| \leq c^n$. In particular, the sequence (y_n) converges to the fixed point x_0 and the rate of convergence is controlled by c^n.

Problems for Sect. 5.4

1. Show $f(x) := \sqrt{x}$ is uniformly continuous on $[0,1]$.
2. Show $f(x) := \sqrt{x}$ is uniformly continuous on $[1,\infty[$.
3. Let f be continuous on some interval I. Let a be a point in I and let $I_1 := I \cap]-\infty, a]$ and $I_2 := [a, \infty[$. Suppose f is uniformly continuous on the intervals I_1 and I_2. Show f is uniformly continuous on I.

   Combining the previous three problems it follows that $x \to \sqrt{x}$ is uniformly continuous on $[0,\infty[$.
4. Show $f(x) := x^2$ is not uniformly continuous on $[1,\infty[$.
5. Let $f : D \to \mathbb{C}$. If $\operatorname{Re} f$ and $\operatorname{Im} f$ both are uniformly continuous on D, then f is uniformly continuous on D.

   A function f satisfies a *Hölder condition* of order α, if there are constants $\alpha > 0$ and $M > 0$, such that

$$|f(x) - f(y)| \leq M|x-y|^{\alpha} \text{ for all } x \text{ and } y.$$

The function is said to be $\alpha-$Hölder with constant M. We say f is Hölder, if f is α-Hölder for some $\alpha > 0$.

6. If f is Hölder, then f is uniformly continuous.
7. If $f :]a,b[\to \mathbb{R}$ is uniformly continuous, then f is bounded.
8. If $f :]a,b[\to \mathbb{R}$ is uniformly continuous, then there is a continuous function $g : [a,b] \to \mathbb{R}$, such that $f(x) = g(x)$ for all $x \in]a,b[$.
9. Suppose $f : \mathbb{R} \to \mathbb{C}$ is continuous and $f(x) = 0$ for all $|x| > 1$. Show f is uniformly continuous on $\mathbb{R}$.

Solutions and Hints for the Exercises

Exercise 5.1.1. Contraposition on the definition of increasing. Since $x = y$ implies $f(x) = f(y)$.

Exercise 5.1.2. Let $L := \sup\{f(x) \mid x < a\}$. Let $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{f(x) \mid x < a\}$, hence there is a $y < a$ such that $L - \varepsilon < f(y)$. Let $\delta := a - y$, then $\delta > 0$ and $a - \delta < x < a$ implies $|f(x) - L| < \varepsilon$.

Exercise 5.1.7. Suppose $x < y < z$ and x, z are in A. We must show y is in A. Since f is increasing and z is in A, we have $f(y) \le f(z) < t$. Thus, y is in A. Similarly, B has the intermediate value property.

Exercise 5.1.9. This is Exercise 5.1.1.

Exercise 5.1.11. (1) Exercise D.1.20. (2) Follows from $x^n \to \infty$ as $x \to \infty$ and the Continuity Theorem for Monotone Functions. (3) Corollary 5.1.10.

Exercise 5.2.4. Since f is $1-1$, $x \ne y \implies f(x) \ne f(y)$. Fix points $a < b$ in I. Suppose $f(a) < f(b)$. Let x be in I. If $x < a$, then $f(x) > f(a)$ and the Intermediate Value Theorem leads to a contradiction that f is $1-1$. Hence, $x < a$ implies $f(x) < f(a) < f(b)$. Similarly $a < x < b$ implies $f(a) < f(x) < f(b)$ and $b < x$ implies $f(a) < f(b) < f(x)$. Suppose $y < z$. (i) If $y < z < a$, setting $x = z$ shows $f(z) < f(a)$. So replacing $x < a < b$ by $y < z < a$ shows $f(y) < f(z)$. (ii) If $y < a < z$, considering $x = y$ and $x = z$ in the argument above shows $f(y) < f(a) < f(z)$, hence $f(y) < f(z)$. The last case (iii) $a < y < z$ is similar. Hence, if $f(a) < f(b)$, then f is strictly increasing. Similarly, if $f(a) > f(b)$, then f is strictly decreasing.

Exercise 5.3.4. If f is not bounded on $I \times J$ divide the rectangle $I \times J$ into four sub-rectangles, f must be unbounded on at least one of these sub-rectangles ...

Exercise 5.3.7. Divide the rectangle $I \times J$ into four sub-rectangles, the supremum of f over at least one of these sub-rectangle must equal the supremum of f over $I \times J$ ...

Exercise 5.4.7. Begin by extending Lemma 5.4.5 to rectangles. Then mimic the proof of the Uniform Continuity Theorem.

Chapter 6
Derivatives and Their Applications

Local properties algebra derivatives and the relationship between the sign of the derivative at a point and the function being monotone at a point. Global properties of the derivative include Rolle's Theorem, the Mean Value Theorem (MVT), and Darboux's Theorem. Applications include Taylor polynomials, l'Hopital's rule, Liouville's theorem about transcendental numbers, and some aspects of convex function theory.

6.1 Definition

Let D be a subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *differentiable* at $a \in D$, if a is an accumulation point of D and there is a complex number $f'(a)$, such that

$$\frac{f(x)-f(a)}{x-a} \to f'(a) \text{ as } x \to a.$$

The number $f'(a)$ is called the *derivative* of f at a. We can rewrite the definition of derivative as: f is differentiable at a with derivative b, if given any $\varepsilon > 0$, there is a $\delta > 0$, such that

$$0 < |x-a| < \delta \implies \left| \frac{f(x)-f(a)}{x-a} - b \right| < \varepsilon.$$

When f is differentiable at a with derivative b we set $f'(a) := b$. Anticipating the application of limits to derivatives is one of the reasons we used $0 < |x-a| < \delta$ and not $|x-a| < \delta$ in the definition of limits.

The line $y = f(a) + f'(a)(x-a)$ is called the *tangent* to the curve $y = f(x)$ at the point $(a, f(a))$. See Fig. 6.1.

Example 6.1.1. Let $f(x) := \sqrt{x}$ and $a > 0$. Then f is differentiable at a and $f'(a) = \frac{1}{2\sqrt{a}}$.

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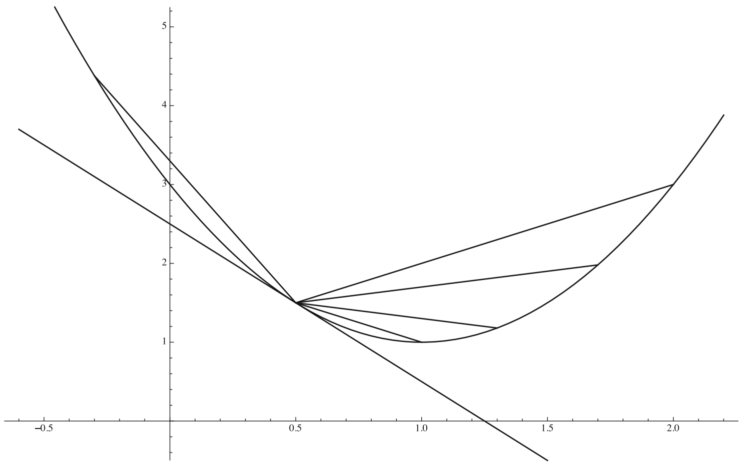

Fig. 6.1 The curve $y = f(x)$, line segments connecting the point $(a, f(a))$ to the point $(x, f(x))$ for several values of x, and the tangent line $y = f(a) + f'(a)(x - a)$

Proof. Essentially, this is a calculation verifying $g(x) := \frac{f(x)-f(a)}{x-a}$ has a removable discontinuity at a.

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{a}}{x-a} &= \frac{\sqrt{x}-\sqrt{a}}{x-a}\,\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\\ &= \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}\\ &= \frac{1}{\sqrt{x}+\sqrt{a}}\\ &\to \frac{1}{\sqrt{a}+\sqrt{a}} \text{ as } x \to a.\end{aligned}$$

Hence, $f'(a) = 1/(2\sqrt{a})$. ☺

We say that f is *differentiable on D*, if $f'(a)$ exists for all $a \in D$. In particular, $f(x) = \sqrt{x}$ is differentiable on $]0, \infty[$ and its derivative is $f'(x) = \frac{1}{2\sqrt{x}}$.

Example 6.1.2. Let $f(x) := 1/x$ and let a be any complex number $\neq 0$. Then $f'(a) = -1/a^2$.

Proof. This is similar to the previous example. In fact,

$$\frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \frac{\frac{a-x}{xa}}{x-a} = -\frac{1}{xa} \to -\frac{1}{a^2}$$

as $x \to a$. ☺

Using one-sided limits (Sect. 1.5) we can similarly define *one-sided derivatives*. The right-hand derivative (the derivative from the right) $f'^{+}(a)$ is

$$\frac{f(x)-f(a)}{x-a} \to f'^{+}(a) \text{ as } x \searrow a.$$

Similarly, the left-hand derivative (the derivative from the left) $f'^{-}(a)$ is

$$\frac{f(x)-f(a)}{x-a} \to f'^{-}(a) \text{ as } x \nearrow a.$$

Using a basic fact about one-sided limits: $f'(a)$ exists iff $f'^{-}(a)$ and $f'^{+}(a)$ both exists and are equal.

Example 6.1.3. Let $f(x) := \sqrt{x}$ and $a := 0$. Then

$$\begin{aligned} \frac{\sqrt{x}-\sqrt{0}}{x-0} &= \frac{\sqrt{x}}{x} \\ &= \frac{1}{\sqrt{x}} \\ &\to \infty \text{ as } x \searrow 0. \end{aligned}$$

Hence, $f'^{+}(0)$ does not exist.

Example 6.1.4. The pseudo-sine function σ from Example 1.3.10 is differentiable at 0 with $\sigma'(0) = 4$.

Proof. Clearly,

$$\frac{\sigma(x)-\sigma(0)}{x-0} = 4(1-x) \text{ when } 0 < x < 1.$$

and $4(1-x) \to 4$ as $x \searrow 0$. Hence $\sigma'^{+}(0) = 4$. Similarly,

$$\frac{\sigma(x)-\sigma(0)}{x-0} = 4(1+x) \to 4 \text{ as } x \nearrow 0,$$

hence, $\sigma'^{-}(0) = 4$. Since, $\sigma'^{+}(0) = 4 = \sigma'^{-}(0)$, we conclude σ is differentiable at 0 and $\sigma'(0) = 4$. ☺

6.2 Local Properties

The following result shows that a differentiable function is continuous. We establish this at a point. Hence, if f is differentiable at all points in some set D, then f is continuous on D.

Theorem 6.2.1. *If f is differentiable at a, then f is continuous at a.*

Proof. This is a consequence of the Product Rule for limits and continuity of $g(x) := x$.

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a) \to f'(a) \cdot 0 = 0 \text{ as } x \to a.$$

Hence, $f(x) \to f(a)$ as $x \to a$. ☺

In the remaining part of this section we assume f be a real valued function defined in some open interval containing a.

We say f is *increasing at a point* a, if there is a $\delta > 0$, such that $f(x) \leq f(a)$, when $a - \delta < x < a$ and $f(a) \leq f(x)$, when $a < x < x + \delta$. *Strictly increasing at* a is defined in the same way using $<$ in place of $\leq$. *Decreasing* and *strictly decreasing* at a point are defined in a similar manner. Example 6.2.5 contains a function that is both increasing at 0 and not strictly increasing at 0.

The following are direct consequences of the limit definition of the derivative. The results illustrate why points where the tangent line is a horizontal line are important, i.e., why the roots of $f'(x) = 0$ are important.

Exercise 6.2.2. Suppose $f'(a)$ exists and f is increasing at a. Prove $f'(a) \geq 0$.

Exercise 6.2.3. If $f'(a)$ exists and $f'(a) > 0$, then f is strictly increasing at a.

$f(a)$ is a *local maximum* of f, if there is a $\delta > 0$, such that

$$a - \delta < x < a + \delta \implies f(x) \leq f(a).$$

A *local minimum* is defined similarly, replace $\leq$ by $\geq$. $f(a)$ is a *local extremum*, if $f(a)$ is a local maximum or a local minimum.

We say a is a *critical point* of f, if either $f'(a) = 0$ or f is not differentiable at a. The following exercise shows that the "candidates" for local extrema are the critical points.

Exercise 6.2.4. If $f(a)$ a local extremum and $f'(a)$ exists, then $f'(a) = 0$.

We conclude this section with an example explaining why we distinguish between increasing and strictly increasing at a point.

Example 6.2.5. Let σ be the pseudo-sine function. Let $f(x) := x|\sigma(1/x)|$ if $x \neq 0$ and $f(0) := 0$. Then $f(x) \geq 0$ when $x > 0$, $f(x) \leq 0$ when $x \leq 0$, so f is increasing at 0. Moreover, $f(\pm 1/n) = 0$ for all $n \in \mathbb{N}$. Hence, f is not strictly increasing at 0. Consequently, f is increasing but not strictly increasing at 0.

6.3 Calculating with Derivatives

If f is differentiable at a, then $\frac{f(x)-f(a)}{x-a} \to f'(a)$ as $x \to a$, and if g is differentiable at b, then $\frac{g(y)-g(b)}{y-b} \to g'(b)$ as $y \to b$. The calculations underlying the proofs in this section are motivated by the desire to find expressions of the form $\frac{f(x)-f(a)}{x-a}$ and

$\frac{g(y)-g(b)}{y-b}$ as appropriate. That is, we are looking for expression we know converge and we hope to somehow deal with whatever turns up as a result.

Algebra

We establish the standard rules for working with derivatives.

Theorem 6.3.1 (Constant Rule). *Suppose $k \in \mathbb{C}$, a is an accumulation point of D, and $f : D \to \mathbb{C}$ is differentiable at a, then kf is differentiable at a and*

$$(kf)'(a) = kf'(a).$$

Proof. By assumption $(f(x) - f(a))/(x-a) \to f'(a)$ as $x \to a$, hence

$$\frac{(kf)(x) - (kf)(a)}{x-a} = k\frac{f(x) - f(a)}{x-a} \to kf'(a) \text{ as } x \to a,$$

by the constant rule for limits. Thus, kf is differentiable at a and $(kf)'(a) = kf'(a)$. ☺

Theorem 6.3.2 (Sum Rule). *Suppose a is an accumulation point of D and $f, g : D \to \mathbb{C}$ are differentiable at a, then $f+g$ is differentiable at a and*

$$(f+g)'(a) = f'(a) + g'(a).$$

Proof. By assumption $(f(x) - f(a))/(x-a) \to f'(a)$ and $(g(x) - g'(a))/(x-a) \to g'(a)$ as $x \to a$. Hence,

$$\begin{aligned}\frac{(f+g)(x) - (f+g)(a)}{x-a} &= \frac{f(x)-f(a)}{x-a} + \frac{g(a)-g(a)}{x-a} \\ &\to f'(a) + g'(a) \text{ as } x \to a,\end{aligned}$$

by the sum rule for limits. Thus, $f+g$ is differentiable at a and $(f+g)'(a) = f'(a) + g'(a)$. ☺

A transformation $f \to L(f)$ where $L(f)$ is some other function depending on f, is called *linear*, if $L(kf) = kL(f)$ and $L(f+g) = L(f) + L(g)$ where k is a constant and f and g are functions. Using this terminology, we can restate the Constant and Sum Rules as: $f \to f'$ is linear.

The following result is sometimes called *Leibniz' Rule* after Gottfried Wilhelm Leibniz (1 July 1646, Leipzig to 14 November 1716, Hanover.)

Theorem 6.3.3 (Product Rule). *If a is an accumulation point of D and $f, g : D \to \mathbb{C}$ are differentiable at a, then fg is differentiable at a and*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. By assumption $(f(x)-f(a))/(x-a) \to f'(a)$ and $(g(x)-g'(a))/(x-a) \to g'(a)$ as $x \to a$. Hence,

$$\begin{aligned}\frac{(fg)(x)-(fg)(a)}{x-a} &= \frac{f(x)g(x)-f(a)g(a)}{x-a} \\ &= \frac{f(x)-f(a)}{x-a}g(x)+f(a)\frac{g(x)-g(a)}{x-a} \\ &\to f'(a)g(a)+f(a)g'(a).\end{aligned}$$

Where we used that g is continuous at a and the Product and Sum Rules for Limits. Thus, fg is differentiable at a and $(fg)'(a) = f'(a)g(a)+f(a)g'(a)$. ☺

Having dealt with addition and multiplication, essentially inherited from arithmetic, we turn to an operation intrinsic to functions.

Theorem 6.3.4 (Chain Rule). *Suppose $g : A \to B$ is differentiable at a and $f : B \to C$ is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. Let $h(y) := (f(y)-f(g(a))/(y-g(a))$ for $y \in B$. By our assumptions $h(y) \to f'(g(a))$ as $y \to g(a)$ and $(g(x)-g(x))/(x-a) \to g'(a)$ as $x \to a$. Let

$$j(x) := \frac{f \circ g(x) - f \circ g(a)}{x-a} = \frac{f(g(x))-f(g(a))}{x-a} \text{ on } A \setminus \{a\}.$$

We must show that $j(x) \to f'(g(a))g'(a)$ as $x \to a$.

We will use restrictions, see Sect. 1.4. Let

$$\begin{aligned}D_1 &:= \{x \in A \mid g(x) \neq g(a)\} \text{ and} \\ D_2 &:= \{x \in A \mid g(x) = g(a)\}.\end{aligned}$$

Since a is an accumulation point of A, either a is an accumulation point of D_1 or a is an accumulation point of D_2, or both.

Case 1: Suppose a is an accumulation point of D_1. For $x \in D_1$

$$\begin{aligned}j\big|_{D_1}(x) = j(x) &= \frac{f(g(x))-f(g(a))}{g(x)-g(a)}\,\frac{g(x)-g(a)}{x-a} \\ &= h(g(x))\,\frac{g(x)-g(a)}{x-a}.\end{aligned}$$

Since g is differentiable at a, g is continuous at a, equivalently $g(x) \to g(a)$ as $x \to a$. Hence, by the Composition Rule for Limits $h \circ g\big|_{D_1}(x) \to f'(g(a))$ as $x \to a$. Consequently, the Product Rule for Limits tells us that

$$j\big|_{D_1}(x) \to f'(g(a))g'(a) \text{ as } x \to a.$$

Case 2: Suppose a is an accumulation point of D_2. For $x \in D_2$

$$\frac{g(x)-g(a)}{x-a} = \frac{g(a)-g(a)}{x-a} = 0$$

hence $g'(a) = 0$. So, for $x \in D_2$

$$\begin{aligned} j\big|_{D_2}(x) = j(x) &= \frac{f(g(x)) - f(g(a))}{x-a} \\ &= \frac{f(g(a)) - f(g(a))}{x-a} = 0 \end{aligned}$$

Since $g'(a) = 0$, we have

$$j\big|_{D_2}(x) \to 0 = f'(g(a))g'(a) \text{ as } x \to a.$$

If a is not an accumulation point of D_1 we can ignore Case 1, and if a is not an accumulation of D_2 we can ignore Case 2. If a is an accumulation point of D_1 and of D_2, then Case 2 shows $g'(a) = 0$. Hence, both $j\big|_{D_1}(x) \to 0$ and $j\big|_{D_2}(x) \to 0$ as $x \to a$. Consequently,

$$j(x) \to 0 = f'(g(a))g'(a).$$

In all cases, the limit of $j(x)$ as $x \to a$ exists and equals $f'(g(a))g'(a)$. Hence $j(x) \to f'(g(a))g'(a)$ as $x \to a$. ☺

The following result can be proven in a manner similar to our proof of the Product Rule. However, we will establish it as a consequence of the Product and Chain Rules.

Theorem 6.3.5 (Quotient Rule). *Suppose a is an accumulation point of D, $f, g : D \to \mathbb{C}$ are differentiable at a, and $g(a) \neq 0$, then f/g is differentiable at a and*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof. The Quotient Rule is a consequence of the Product Rule, the Chain Rule, and Example 6.1.2. Suppose g is differentiable at a and that $g(a) \neq 0$. Let $h(x) := 1/x$, then h is differentiable at $g(a)$ and $h'(g(a)) = -1/g(a)^2$. Hence, by the Chain Rule, $1/g(x) = h \circ g(x)$ is differentiable at a and

$$\left(\frac{1}{g}\right)'(a) = h'(g(a))g'(a) = \frac{-1}{g(a)^2} g'(a).$$

Using f is differentiable at a and

$$\left(\frac{f}{g}\right)(x) = \left(f \frac{1}{g}\right)(x)$$

the Product Rule tells us f/g is differentiable at a and

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)(x) \\ &= f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a) \\ &= \frac{f'(a)}{g(a)} - f(a)\frac{g'(a)}{g(a)^2} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.\end{aligned}$$

This calculation completes the proof. ☺

Inverse Function Rule

Suppose f has an inverse function f^{-1}. If f is differentiable at a and f^{-1} is differentiable at $b = f(a)$, then the Chain Rule implies that $f^{-1} \circ f$ is differentiable at a, and

$$\left(f^{-1} \circ f\right)'(a) = \left(f^{-1}\right)'(f(a)) \cdot f'(a).$$

But $f^{-1} \circ f(x) = x$ for all x, so $\left(f^{-1} \circ f\right)'(a) = 1$. Hence,

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}$$

provided $f'(a) \neq 0$. We will show that this formula works without assuming f^{-1} is differentiable.

Theorem 6.3.6 (Inverse Function Rule). *Let I be an interval. Suppose $f : I \to \mathbb{R}$ is continuous and one-to-one. If f is differentiable at a and $f'(a) \neq 0$, then the inverse function f^{-1} is differentiable at $b := f(a)$, and*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)} = \frac{1}{f'(a)}.$$

Proof. The discussion prior to the statement of the theorem suggests we want to use the Composition Rule for Limits.

Let

$$h(y) := \frac{f^{-1}(y) - f^{-1}(b)}{y - b} \text{ for } y \neq b$$

we need to show that $h(y)$ converges as $y \to b$, the limit then equals $\left(f^{-1}\right)'(b)$. Anticipating the formula for $\left(f^{-1}\right)'(b)$, let

$$g(x) := \frac{1}{\frac{f(x)-f\left(f^{-1}(b)\right)}{x-f^{-1}(b)}} = \frac{x-f^{-1}(b)}{f(x)-b} \text{ for } x \neq f^{-1}(b). \tag{6.1}$$

Note $x \neq f^{-1}(b)$ implies $f(x) \neq f\left(f^{-1}(b)\right) = b$. Since f is differentiable at $a = f^{-1}(b)$,

$$\frac{f(x)-f\left(f^{-1}(b)\right)}{x-f^{-1}(b)} \to f'\left(f^{-1}(b)\right) \neq 0 \text{ as } x \to f^{-1}(b).$$

Using the Quotient Rule for Limits we conclude $g(x) \to 1/f'\left(f^{-1}(b)\right)$ as $x \to f^{-1}(b)$.

The second expression for $g(x)$ in (6.1) shows that $h(y) = g\left(f^{-1}(y)\right)$. Since $f^{-1}(y) \neq f^{-1}(b)$ for all $y \neq b$, the Composition Rule for Limits yields

$$h(y) = g\left(f^{-1}(y)\right) \to 1/f'\left(f^{-1}(b)\right) \text{ as } x \to f^{-1}(b),$$

provided $f^{-1}(y) \to f^{-1}(b)$ as $y \to b$. Hence we need to show that f^{-1} is continuous at b.

But we know that a $1-1$ continuous function on an interval must be strictly monotone (Exercise 5.2.4), and that the inverse of a continuous strictly monotone function defined on an interval must be continuous (Corollary 5.1.10). ☺

Examples of Differentiable Functions

Example 6.3.7. $f(x) := 1$ is differentiable at every point a and $f'(a) = 0$.

Proof.

$$\frac{f(x)-f(a)}{x-a} = \frac{1-1}{x-a} = 0 \to 0$$

as $x \to a$. ☺

Example 6.3.8. $f(x) := x$ is differentiable at every point x and $f'(x) = 1$.

Proof.

$$\frac{f(x)-f(a)}{x-a} = \frac{x-a}{x-a} = 1 \to 1$$

as $x \to a$. ☺

Exercise 6.3.9. If $n \in \mathbb{N}$, then x^n is differentiable at every x and $(x^n)' = nx^{n-1}$.

Exercise 6.3.10. Any polynomial is differentiable at every point.

Exercise 6.3.11. Any rational function is differentiable at every point where it is defined.

Exercise 6.3.12. If $n \in \mathbb{N}$, then $x^{1/n}$ is differentiable at every $x > 0$ and $(x^{1/n})' = \frac{1}{n}x^{\frac{1}{n}-1}$.

6.4 Global Properties of Derivatives

We will now change our viewpoint slightly. Instead of considering the derivative at a single point a, we will consider f' as a function. $f' : \{\text{all } a \text{ where } f'(a) \text{ exists}\} \to \mathbb{C}$, or $\to \mathbb{R}$ as appropriate. We can then ask questions about this function.

Let $f : [a,b] \to \mathbb{R}$ be continuous. By the Extreme Value Theorem, there are $x_{\max}$ and $x_{\min}$ in $[a,b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{max}) \text{ for all } x \in [a,b].$$

If $x_{\min}$ is in $]a,b[$, then $f(x_{\min})$ is a local minimum, hence $f(x_{\min})$ is a critical point. So, if f is differentiable at $x_{\min}$, then $f'(x_{\min}) = 0$ (by Exercise 6.2.4). Consequently, if f is differentiable on $]a,b[$, then either $f'(x_{\min}) = 0$ or $x_{\min}$ is an endpoint of $[a,b]$. A similar discussion applies to $x_{\max}$.

Exercise 6.4.1. If $f(x) := x^3 - 6x^2 + 7$ on $[-2,2]$, find $x_{\min}$ and $x_{\max}$.

The Intermediate Value Theorem for Derivatives

Derivatives defined on intervals have the intermediate value property, this result is named after Jean-Gaston Darboux (14 August 1842, Nîmes to 23 February 1917, Paris).

Theorem 6.4.2 (Darboux's Intermediate Value Theorem). *Let f be a real valued function. Suppose f is differentiable on some interval I, then $f'(I)$ has the intermediate value property.*

Proof. Let $a < b$ be in I and let k be between $f'(a)$ and $f'(b)$. If $f'(a) < k < f'(b)$, let $g(x) := f(x) - kx$. (If $f'(b) < k < f'(a)$, let $g(x) := kx - f(x)$.) Then g is continuous on $[a,b]$ and differentiable on $[a,b]$. Let $x_{\min} \in [a,b]$ be such that $g(x_{\min}) \leq g(x)$ for all $x \in [a,b]$. Since $g'(a) = f'(a) - k < 0$, g is strictly decreasing at a. In particular, $a \neq x_{\min}$. Similarly, g is increasing at b, hence $b \neq x_{\min}$. So $a < x_{\min} < b$ and consequently, $g'(x_{\min}) = 0$. ☺

Remark 6.4.3. Because of the Intermediate Value Theorem for f', it is natural to ask "If f' exists on some interval, must f' be continuous on that interval?" The answer is no. The most natural way to see this is to use the integral, discussed in Chap. 7, and set

$$f(x) := \int_0^x \sigma(1/t)\,dt$$

where σ is the pseudo-sine function, or accepting a more involved calculation, the usual sine function (Sect. 11.2).

The following exercise uses a "clever guess" to get a simple example of a function with a discontinuous derivative.

Exercise 6.4.4. Let

$$f(x) := \begin{cases} x\sigma(1/x) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases},$$

where σ is the pseudo-sine function. Then f is not differentiable at 0, because $\lim_{x\to 0} f(x)/x$ does not exist. Prove that $g(x) := xf(x)$ is differentiable on $\mathbb{R}$ and g' is not continuous at 0.

The Mean Value Theorem

The essence of this important result is named after Michel Rolle (21 April 1652 Ambert to 8 November 1719 Paris) who gave the first formal proof. The geometric content of the result is that, if $f(a) = f(b)$, then some tangent to the curve $y = f(x)$, $a < x < b$ is horizontal, see Fig. 6.2.

Theorem 6.4.5 (Rolle's Theorem). *Suppose* $g : [a,b] \to \mathbb{R}$ *is differentiable on the open interval* $]a,b[$ *and continuous on the closed interval* $[a,b]$. *If* $g(a) = g(b)$, *then* $g'(c) = 0$ *for some* $c \in]a,b[$.

Proof. Let $f(x) := g(x) - g(a)$. Then $f(a) = f(b) = 0$. If $f(x) \neq 0$ for some $x \in [a,b]$, then $f(x_{\min}) \neq 0$ or $f(x_{\max}) \neq 0$. So at least one of $x_{\min}$ and $x_{\max}$ is in the open interval $]a,b[$. Consequently, $f'(x_{\max}) = 0$ or $f'(x_{\min}) = 0$. Hence, c is $x_{\min}$ or $x_{\max}$.

If $f(x) = 0$ for all x, then $c := (a+b)/2$. Of course any other value c in the open interval $]a,b[$ would also work. ☺

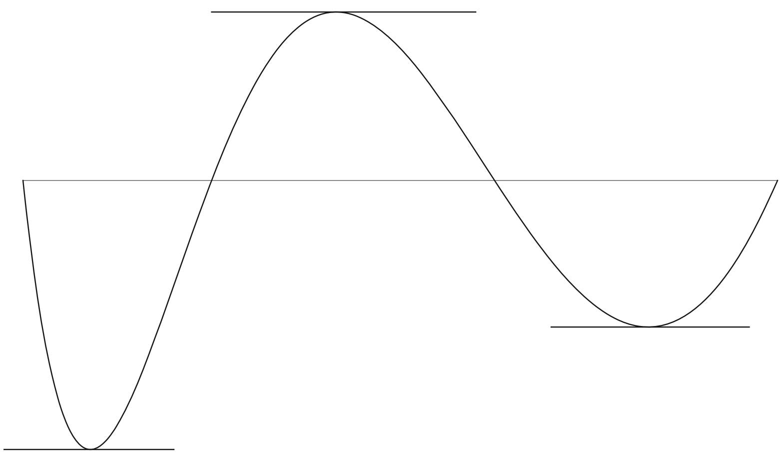

Fig. 6.2 A curve $y = f(x)$ with $f(a) = f(b) = 0$, the $x-$axis, and *three line segments* indicating the points c in Rolle's Theorem where $f'(x) = 0$, i.e., the points where the tangent line is horizontal

The full version of the MVT is obtained from Rolle's Theorem by a simple reduction. The reduction is illustrated in Fig. 6.3. Geometrically, the MVT states that some tangent to the curve $y = f(x)$, $a < x < b$ is parallel to the line though the points $(a, f(a))$ and $(b, f(b))$.

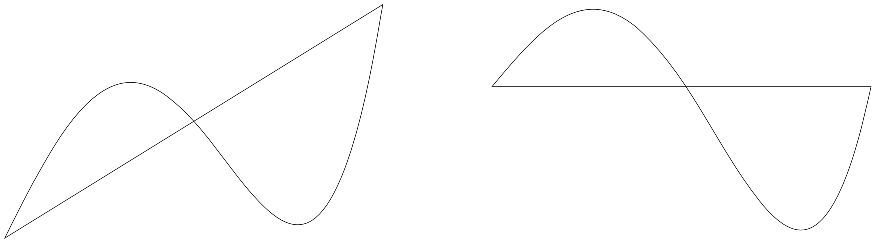

Fig. 6.3 The graphic on the *left* shows the curve $y = f(x)$ and the line $y = L(x)$ connecting the endpoints $(a, f(a))$ and $(b, f(b))$ of the curve. The graphic on the *right* shows $g(x) = f(x) - L(x)$ and the x–axis. In this illustration $a = 1$, $b = 4$, and $f(x) = x\left(2x^3 - 15x^2 + 33x - 16\right)/2$. Hence, $L(x) = 2x$ and $g(x) = x\left(2x^3 - 15x^2 + 33x - 20\right)/2$

Theorem 6.4.6 (The Mean Value Theorem). *Suppose $f : [a,b] \to \mathbb{R}$ is differentiable on the open interval $]a,b[$ and continuous on the closed interval $[a,b]$, then*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ for some } c \in]a,b[.$$

Proof. Let

$$L(x) := \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

then $L(a) = f(a)$ and $L(b) = f(b)$. So we can apply Rolle's Theorem to $g(x) := f(x) - L(x)$, to get a c in $]a,b[$ such that $g'(c) = 0$ for some $c \in]a,b[$. For that c

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}.$$

This completes the proof. ☺

The MVT has many application in analysis. The rest of this chapter is devoted to exploring some of the consequences of this important theorem.

We showed earlier that the derivative of a constant function $f(x) := k$ is $f'(x) = 0$. Conversely, the only solutions to the differential equation $f'(x) = 0$ on an interval are the constant functions.

Theorem 6.4.7. *If f is differentiable on some interval I and $f'(x) = 0$ for all x in I, then there is a constant k, such that $f(x) = k$ for all x in I.*

Proof. Let a be in I. If x is in I and $x \neq a$, then the Meal Value Theorem gives us a c between a and x such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0.$$

Consequently, $f(x) = f(a)$, so $k := f(a)$. ☺

Exercise 6.4.8. Let f and g be differentiable on some interval I. If $f' = g'$ on I, then there is a constant k, such that $f(x) = g(x) + k$ for all x in I.

Exercise 6.4.9. Let I be an interval, suppose f and g are differentiable on I. If $f'(x) \le g'(x)$ on $]a,b[$ and c is in I, then

$$f(x) - f(c) \le g(x) - g(c) \text{ for all } x \text{ in } I \text{ with } c < x$$

and

$$f(x) - f(c) \ge g(x) - g(c) \text{ for all } x \text{ in } I \text{ with } c > x.$$

Exercise 6.4.10. Let I be an interval and suppose $f : I \to \mathbb{R}$ is differentiable. Prove f is increasing on I iff $f'(x) \ge 0$ on I.

6.5 Transcendental Numbers★

In this section we use the MVT to solve a seemingly unrelated problem, more precisely we establish the existence of transcendental numbers.

A real number a is *algebraic*, if there is a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ with integer coefficients a_k, such that $p(a) = 0$. A real number a is algebraic of order n, if the smallest degree of such a polynomial is n. A real number that is not algebraic is *transcendental*.

In 1844, Joseph Liouville [24 March 1809, Saint-Omer to 8 September 1882, Paris] constructed a family of transcendental numbers. Prior to this construction, people suspected that numbers like e and π were transcendental, but the existence of even one transcendental was in doubt. Of course, later Cantor showed that most real numbers are transcendental by a nonconstructive argument. See the problems for Sect. 4.1 for an outline of Cantor's argument.

Example 6.5.1. $\frac{2}{3}$ is algebraic of order 1. To see this, let $p(x) := 3x - 2$, clearly $p(2/3) = 0$.

Similarly, any rational, other than 0, is algebraic of order 1. Clearly, 0 is algebraic of order 0.

Example 6.5.2. $\sqrt{3}$ is algebraic of order 2. Since, $\sqrt{3}$ is not rational it is not algebraic of order 1. Considering $p(x) = x^2 - 3$, shows that $\sqrt{3}$ is algebraic of order 2. Similarly, $2^{1/3}$ is algebraic of order 3. Let $p(x) := x^3 - 2$.

The $n = 1$ case of the following theorem is Theorem 1.8.5.

Theorem 6.5.3 (Liouville). *Let α be a real number. If*

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + a_1\alpha + a_0 = 0,$$

for some natural number n and some integers a_k with $a_n \neq 0$, then there exists $K > 0$ such that for all $p, q \in \mathbb{Z}$ with $q > 0$ either $\alpha = p/q$, or

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{K}{q^n}.$$

More poetically, an algebraic number is poorly approximated by rational numbers. This is closely related to what is called the "irrationality measure" of a real number.

Proof. We reproduce Liouville's proof. Let $f(t) := \sum_{k=0}^{n} a_k t^k$ where the a_k are integers. Then $f(\alpha) = 0$. If f has a root other than α, let

$$\delta := \min\{|\alpha - x| \mid x \neq \alpha, f(x) = 0\}.$$

The minimum exists and is > 0, since the set of roots is nonempty by assumption and finite by Theorem 1.4.11. If f does not have any root other than α, set $\delta := 1$. In either case, $\delta > 0$ and $f(t) \neq 0$ for all t satisfying $0 < |\alpha - t| < \delta$.

The derivative $f'(t)$ is a polynomial and therefore a continuous function. By the Global Boundedness Theorem for continuous functions, there is an $M > 0$ such that $|f'(t)| \leq M$ for all $t \in [\alpha - \delta, \alpha + \delta]$. It remains to show that $K := \min\{\delta, 1/M\}$ works.

Let $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ be such that $\alpha \neq p/q$. If $|\alpha - p/q| \geq \delta$, then $|\alpha - p/q| \geq \delta \geq \delta/q^n \geq K/q^n$. Hence, we are done.

So, suppose $|\alpha - p/q| < \delta$. By construction of δ, we have $f(p/q) \neq 0$. The product $q^n f(p/q)$ is a nonzero integer, in fact,

$$q^n f(p/q) = q^n \sum_{k=0}^{n} a_n (p/q)^k = \sum_{k=0}^{n} a_k p^k q^{n-k}$$

is a sum of integers. Hence, $q^n |f(p/q)| \geq 1$. That is

$$\frac{1}{q^n} \leq |f(p/q)|. \tag{6.2}$$

By the MVT, there is a c between α and p/q such that

$$f(p/q) = f(p/q) - f(\alpha) = f'(c)\left(\frac{p}{q} - \alpha\right). \tag{6.3}$$

Since $|\alpha - p/q| < \delta$ and c is between α and p/q, we conclude $c \in [\alpha - \delta, \alpha + \delta]$. Consequently, $|f'(c)| \leq M$, by our construction of M. Combining (6.2) and (6.3) yields

$$\frac{1}{q^n} \leq |f(p/q)| = \left|f'(c)\left(\frac{p}{q} - \alpha\right)\right| \leq M\left|\frac{p}{q} - \alpha\right|$$

as needed. ☺

Example 6.5.4 (Liouville 1851). Liouville's constant α is the infinite decimal $0.d_1 d_2 \cdots$ whose digits all are zeros, except the k!th digits are ones. Hence,

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}.$$

Liouville's constant is transcendental.

Proof. Suppose Liouville's constant α is algebraic. Let $f(t) := \sum_{k=0}^{n} a_k t^k$ be a polynomial with integer coefficients a_k such that $a_n \neq 0$ and $f(\alpha) = 0$. By Liouville's Theorem, there is a constant $K > 0$, such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{K}{q^n} \tag{6.4}$$

for all integers $p, q \geq 1$ with $\alpha \neq p/q$. Pick $m > n$ such that $10 \cdot 10^{-m!} < K$. Let $p := \sum_{k=1}^{m} 10^{m!-k!}$ and $q := 10^{m!}$, then p and q are integers and $\frac{p}{q} = \sum_{j=1}^{m} 10^{-k!}$. Hence,

$$\begin{aligned}\left|\alpha - \frac{p}{q}\right| &= \sum_{k=m+1}^{\infty} 10^{-k!} \leq \sum_{k=(m+1)!}^{\infty} 10^{-k} \\ &= \frac{10^{-(m+1)!}}{1-10^{-1}} = 10 \cdot \left(10^{-m!}\right)^{m+1} = 10 \cdot 10^{-m!} \cdot \left(10^{-m!}\right)^{m} \\ &= 10 \cdot 10^{-m!} \cdot q^{-m} < K \cdot q^{-n}.\end{aligned}$$

This contradicts (6.4). Consequently, α is transcendental. ☺

6.6 Taylor Polynomials

Taylor's formula is named after Brook Taylor (18 August 1685, Edmonton to 29 December 1731, London). Taylor is by far not the first to have used Taylor series, for example, Mādhava of Sañgamāgrama (c. 1350 to c. 1425) made use of Taylor series.

The remainder formula below is due to Joseph-Louis Lagrange (25 January 1736, Turin to 10 April 1813, Paris). Another formula for the remainder is in the problems for Sect. 7.5.

The nth derivative, $f^{(n)}$, of f is f differentiated n times. Inductively, $f^{(0)} := f$ and for $n \in \mathbb{N}_0$, $f^{(n+1)} := \left(f^{(n)}\right)'$. In particular, $f' = f^{(1)}$, $f'' = f^{(2)}$, and $f''' = f^{(3)}$. We say f is $\mathscr{C}^n$ on D, in symbols: $f \in \mathscr{C}^n(D)$, if $f^{(n)}$ exists at each point in D and $f^{(n)}$ is continuous on D.

Theorem 6.6.1 (Taylor's Formula with Lagrange Remainder). *Suppose f is real valued and $\mathscr{C}^{n+1}$ on the closed interval with endpoints x and x_0. Let*

$$T_n f(x) = (T_n f)(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{6.5}$$

Then $f(x) = T_n f(x) + R_n f(x)$, *where*

$$R_n f(x) = (R_n f)(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \tag{6.6}$$

for some c *between* x_0 *and* x.

$T_n f$ is the nth *Taylor polynomial* for f at x_0 and $R_n f := f - T_n f$ is the remainder. The formula (6.6) for the remainder is due to Lagrange. Note, the Lagrange form of $R_n f$ is not a polynomial in x of degree $n+1$, because c depends on x.

Taylor's Formula shows that

$$f(x) \approx T_n f(x) \text{ with error } |R_n f(x)|.$$

A different formula for the remainder $R_n f(x)$ can be found in the Problems for Sect. 7.5.

In most applications of Taylor's Formula we do not know the value of c. However, this does not matter, if we can show that $|R_n f(c)|$ is small by finding an estimate for $\left|f^{(n+1)}(c)\right|$.

Corollary 6.6.2. *If* f *is real valued and* $\mathscr{C}^{n+1}$ *on the closed interval* I *with end points* x *and* x_0, *then*

$$|R_n f(x)| \le \frac{M_{n+1}}{(n+1)!} |x - x_0|^{n+1}.$$

where M_{n+1} *is a constant satisfying* $\left|f^{(n+1)}(t)\right| \le M_{n+1}$ *for all* t *in the closed interval with endpoints* x *and* x_0.

Remark 6.6.3. A polynomial of degree n equals its Taylor polynomials of degree $m \ge n$. In fact, if f is a polynomial of degree n and $m \ge n$, then $f^{(m+1)} = 0$. Hence, $R_m f = 0$ and consequently $T_n f = f$.

Below are two proofs of Taylor's Formula. One proof is based on Exercise 6.4.9 and the other is based on Cauchy's MVT.

Proof of Taylor's Formula using Exercise 6.4.9

We give the proof for $n = 2$. Let I be the closed interval with endpoints x and x_0. Since $f^{(3)}$ is continuous there is are constants m and M such that

$$m \le f'''(t) \le M \tag{6.7}$$

for all $t \in I$ and such that $f'''(t_{\min}) = m$ and $f'''(t_{\max}) = M$ for some $t_{\min}, t_{\max} \in I$.

Suppose $x_0 < x$. By Exercise 6.4.9 and (6.7)

$$m(t-x_0) \le f''(t) - f''(x_0) \tag{6.8}$$

for all $t \in I$, since $(m(t-x_0))' = m \le f'''(t) = (f''(t) - f''(x_0))'$. Similarly, by Exercise 6.4.9 and (6.8)

$$\tfrac{1}{2}m(t-x_0)^2 \le f'(t) - f'(x_0) - f''(x_0)(t-x_0) \tag{6.9}$$

for all $t \in I$. Repeating the argument one more time gives

$$\tfrac{1}{6}m(t-x_0)^3 \le f(t) - f(x_0) - f'(x_0)(t-x_0) - \tfrac{1}{2}f''(x_0)(t-x_0)^2 \tag{6.10}$$

for all $t \in I$. Similarly, using the other half of (6.7), we get

$$f(t) - f(x_0) - f'(x_0)(t-x_0) - \tfrac{1}{2}f''(x_0)(t-x_0)^2 \le \tfrac{1}{6}M(t-x_0)^3 \tag{6.11}$$

for all $t \in I$. Setting $t = x$ in (6.10) and (6.11) we conclude

$$6\left(f(x) - f(x_0) - f'(x_0)(x-x_0) - \tfrac{1}{2}f''(x_0)(x-x_0)^2\right)/(x-x_0)^3 \tag{6.12}$$

is a number between $m = f'''(t_{\min})$ and $M = f'''(t_{\max})$. By the Intermediate Value Theorem there is a $c = c(t)$ between $t_{\min}$ and $t_{\max}$, hence between x and x_0, such that (6.12) equals $f'''(c)$. This completes the proof when $x_0 < x$. The case $x < x_0$ is similar.

Proof of Taylor's Formula using Cauchy's Mean Value Theorem

The following slight generalization of the MVT is named after Augustin-Louis Cauchy (21 August 1789, Paris to 23 May 1857, Sceaux). The usual MVT is obtained by setting $g(x) = x$. The proof of the usual MVT above has a simple geometric interpretation. While the proof of Cauchy's MVT is a clever trick, the result has a simple geometric interpretation.

Theorem 6.6.4 (Cauchy's Mean Value Theorem). *Suppose real valued functions f and g are differentiable on the open interval $]a,b[$ and continuous on the closed interval $[a,b]$, then there is a point c in $]a,b[$ such that*

$$f'(c)(g(b)-g(a)) = (f(b)-f(a))g'(c). \tag{6.13}$$

Proof. Let

$$\phi(x) := (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a)).$$

Then ϕ is continuous on $[a,b]$, differentiable on $]a,b[$, and $\phi(a) = \phi(b) = 0$. So by Rolle's Theorem $\phi'(c) = 0$ for some c in $]a,b[$. ☺

If at least one of $f'(c)$ and $g'(c)$ is nonzero, then (6.13) means that the tangent to the curve $\varphi(t) := (f(t), g(t))$, $a < t < b$ at the point $\varphi(c)$ is parallel to the line though the points $\varphi(a) = (f(a), g(a))$ and $\varphi(b) = (f(b), g(b))$.

Remark 6.6.5. The formula in the Cauchy's MVT remains true, if we interchange a and b.

Proof. [Proof of Taylor's Formula with Lagrange Remainder] We will give the proof for $n = 3$. Let

$$\begin{aligned} F(t) &:= \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \\ &= f(t) + f'(t)(x-t) + \tfrac{1}{2} f''(t)(x-t)^2 + \tfrac{1}{6} f^{(3)}(t)(x-t)^3. \end{aligned}$$

When $t = x$, only the $k = 0$ term survives in the sum, hence $F(x) = f(x)$. Clearly $F(x_0) = T_n f(x)$. Hence $F(x) - F(x_0) = f(x) - T_n f(x) = R_n f(x)$. It remains to verify (6.6).

By the product rule for derivatives (we set $n = 3$ to simplify this calculation)

$$\begin{aligned} F'(t) &= f'(t) + \left(f''(t)(x-t) - f'(t)\right) + \left(\tfrac{1}{2} f^{(3)}(t)(x-t)^2 - f''(t)(x-t)\right) \\ &\quad + \left(\tfrac{1}{6} f^{(4)}(t)(x-t)^3 - \tfrac{1}{2} f^{(3)}(t)(x-t)^2\right) \\ &= \tfrac{1}{6} f^{(4)}(t)(x-t)^3 = \tfrac{1}{n!} f^{(n+1)}(t)(x-t)^n. \end{aligned}$$

Let $G(t) := (x-t)^{n+1}$. Then $G(x) = 0$, $G(x_0) = (x-x_0)^{n+1}$, and $G'(t) = -(n+1)(x-t)^n$. By Cauchy's MVT there is a c between x and x_0 such that

$$F'(c)\left(G(x) - G(x_0)\right) = \left(F(x) - F(x_0)\right) G'(c).$$

Consequently,

$$\tfrac{1}{n!} f^{(n+1)}(c)(x-c)^n \left(0 - (x-x_0)^{n+1}\right) = R_n f(x)\left(-(n+1)(x-c)^n\right).$$

Rearranging the last equation gives the desired expression for $R_n f(x)$. ☺

Remark 6.6.6. Other choices of G leads to different formulas for the remainder. All that is required of G is that it satisfies the assumptions in the Cauchy MVT and that we do not divide by 0. Hence, any G that is continuous on the closed interval with endpoints x and x_0, is differentiable on the open interval with endpoints x and x_0, and whose derivative is nonzero on that interval gives a formula for the remainder.

Applications of Taylor's Formula

Example 6.6.7. If $f(x) := 1/\sqrt{1+x}$, then

$$f^{(n)}(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \sqrt{1+x}^{2n+1}},$$

for $n \geq 1$. Setting $x_0 = 0$ and $n = 3$ we get, for $0 < x$ that

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + R(x)$$

where

$$|R(x)| \leq \frac{M_4}{4!} x^4 = \frac{35}{128} x^4.$$

Since $\left| f^{(4)}(x) \right| \leq \left| f^{(4)}(0) \right|$, for all $0 \leq x$, we can use $M_4 := \left| f^{(4)}(0) \right| = \frac{105}{16}$ in Corollary 6.6.2.

As a direct consequence of Taylor's Formula we get a version of l'Hôpital's Rule.

Corollary 6.6.8 (l'Hôpital's Rule). *If f, g are $\mathscr{C}^{n+1}$, $f^{(j)}(x_0) = g^{(j)}(x_0) = 0$ for $0 \leq j < n$ and $g^{(n)}(x_0) \neq 0$, then*

$$\frac{f(x)}{g(x)} \to \frac{f^{(n)}(x_0)}{g^{(n)}(x_0)}$$

as $x \to x_0$.

Proof. For simplicity of notation, suppose $n = 1$. Since $f(x_0) = g(x_0) = 0$, Taylor's Formula applied to f and g gives

$$f(x) = f'(x_0) + \tfrac{1}{2} f''(c_x)(x - x_0)^2$$
$$g(x) = g'(x_0) + \tfrac{1}{2} g''(d_x)(x - x_0)^2$$

for some c_x and d_x between x and x_0.

Since c_x and d_x are between x and x_0, we see that $c_x \to x_0$ and $d_x \to x_0$ as $x \to x_0$. Hence continuity of f'' and g'' at x_0 guarantees $f''(c_x) \to f''(x_0)$ and $g''(d_x) \to g''(x_0)$ as $x \to x_0$. Consequently,

$$\frac{f(x)}{g(x)} = \frac{f'(x_0) + \frac{1}{2} f''(c_x)(x - x_0)^2}{g'(x_0) + \frac{1}{2} g''(d_x)(x - x_0)^2} \to \frac{f'(x_0) + \frac{1}{2} f''(x_0)(x_0 - x_0)^2}{g'(x_0) + \frac{1}{2} g''(x_0)(x_0 - x_0)^2} = \frac{f'(x_0)}{g'(x_0)}$$

as $x \to x_0$. ☺

A version of l'Hôpital's Rule that works under fewer assumptions on f and g can be found in Sect. 6.7. The version above is sufficient for most applications of l'Hôpital's Rule.

6.7 l'Hôpital's Rule★

The rule is named after Guillaume François Antoine, Marquis de l'Hôpital (Paris 1661, Paris to 2 February 1704, Paris). Johann Bernoulli (27 July 1667 Basel to 1 January 1748 Basel) was hired by l'Hôpital to teach him mathematics. One part of the arrangement allowed l'Hôpital to use Bernoulli's discoveries as he pleased. l'Hôpital published a calculus book largely based on Bernoulli's work, including what is commonly known as l'Hôpital's Rule.

Theorem 6.7.1 (l'Hôpital's Rule). *Suppose f and g are differentiable on $]a,b[$, $f(x) \to 0$ and $g(x) \to 0$ as $x \searrow a$, then*

$$\frac{f'(x)}{g'(x)} \to L \text{ as } x \searrow a \text{ implies } \frac{f(x)}{g(x)} \to L \text{ as } x \searrow a.$$

We assume $g(x) \neq 0$ and $g'(x) \neq 0$ for $x \in]a,b[$. L could be a real number or one of $\pm\infty$.

Proof. Since $f(x) \to 0$ and $g(x) \to 0$ as $x \searrow a$ we know that f and g have removable discontinuities at a. Removing the discontinuities f and g are continuous on $[a,b[$ and $f(a) = g(a) = 0$.

Using the Cauchy's MVT there is a $c = c(x)$ between a and x such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

Since $c(x)$ is between a and x, $c(x) \to a$ as $x \searrow a$. Hence, $f(c)/g(c) \to L$ as $x \searrow a$, by the Composition Theorem for Limits. Since $f(a) = g(a) = 0$, the previous equation can be rewritten as

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

letting $x \searrow a$ yields the desired result. ☺

If we replace the assumptions $f(x) \to 0$ and $g(x) \to 0$ as $x \searrow a$, by $f(x) \to 0$ and $g(x) \to 0$ as $x \nearrow b$, then a very similar argument shows that $\frac{f'(x)}{g'(x)} \to L$ as $x \nearrow b$ implies $\frac{f(x)}{g(x)} \to L$ as $x \nearrow b$.

A function f is $\mathscr{C}^1$ on a set D, in symbols $f \in \mathscr{C}^1(D)$, if f is differentiable on D and f' is continuous on D. This notation was also used Sect. 6.6. Section 6.6 also contains a version of l'Hôpital's Rule.

Exercise 6.7.2. If $f, g \in \mathscr{C}^1(]a,\infty[)$, $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$, then

$$\frac{f'(x)}{g'(x)} \to L \text{ as } x \to \infty \text{ implies } \frac{f(x)}{g(x)} \to L \text{ as } x \to \infty.$$

We assume $g(x) \neq 0$ and $g'(x) \neq 0$ for $x \in]a,\infty[$.

6.8 Convexity★

Many important inequalities in analysis can be proved using convexity. These inequalities are often consequences of Jensen's inequality, below we use Jensen's inequality to establish the Arithmetic–Geometric Mean inequality. The problems contain additional examples of inequalities that can be established using convexity.

This section has several subsections: Cords, Regularity, Calculus, and Applications. The Regularity subsection is not used in any of the other subsections.

Let I be an interval. A function $f : I \to \mathbb{R}$ is *convex* if it lies below its cords. See Fig. 6.4. That is, if for any $a < b$ in I, the cord

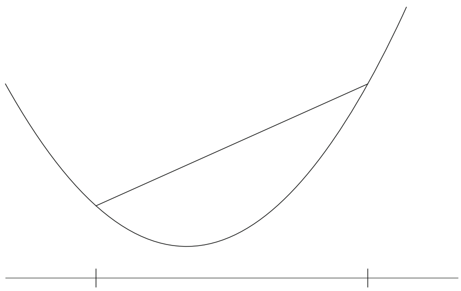

Fig. 6.4 A convex curve $y = f(x)$ and one a it's cords

$$L(x) = L_{a,b}(x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a),$$

satisfies

$$f(x) \le L(x) \text{ for all } a < x < b.$$

A function f is *concave*, if $-f$ is convex, that is if f lies above its cords. Looking at the graphs of $f(x) := x^2$ and of $g(x) := |x|$ suggests that these functions are convex on $\mathbb{R}$.

Slopes of Cords

Let $s_a(x)$ be the slope of the cord $L_{a,x}$ connecting $(a, f(a))$ to $(x, f(x))$, i.e.,

$$s_a(x) := \frac{f(x) - f(a)}{x - a}.$$

The following result shows that f is convex iff $x \to s_a(x)$ is an increasing function on $a < x$ for all a. See the left half of Fig. 6.5.

Lemma 6.8.1. *f is convex on I iff*

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(x)}{z - x} \textit{ for any } x < y < z \textit{ in } I.$$

Proof. By definition f is convex iff

$$f(y) \le f(x) + \frac{f(z)-f(x)}{z-x}(y-x) \text{ for all } x<y<z \text{ in } I.$$

Subtracting $f(x)$ and dividing by $y-x$ gives the desired conclusion. ☺

Exercise 6.8.2. (i) Prove $L(x) = M(x)$, where

$$M(x) := f(b) - \frac{f(b)-f(a)}{b-a}(b-x).$$

(ii) Prove f is convex iff

$$\frac{f(z)-f(x)}{z-x} \le \frac{f(z)-f(y)}{z-y} \text{ for all } x<y<z \text{ in } I.$$

The second part of this exercise says that f is convex iff the slope $s_b(x)$ of the cord connecting $(x, f(x))$ and $(b, f(b))$ is an increasing function of x, $x<b$, for all b. See the right half of Fig. 6.5.

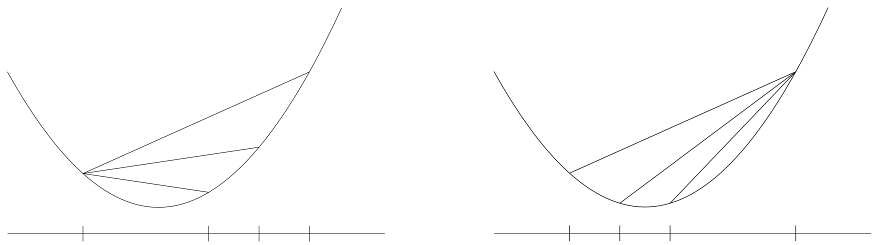

Fig. 6.5 A convex curve $y = f(x)$ and some of it's cords. **a** The figure on the *left* illustrates Lemma 6.8.1. In the notation of Lemma 6.8.1 $(x, f(x))$ is the point where the cords intersect. **b** The figure on the *right* illustrates Exercise 6.8.2. In the notation of Exercise 6.8.2 $(z, f(z))$ is the point where the cords intersect

Combining the two parts of Fig. 6.5 so all the cords have a point in common (see the left half of Fig. 6.6 for a the version that has one cord to the left of the common point and one cord to the right of the common point) leads to:

Corollary 6.8.3. *If f is convex on an interval I, then the difference quotient*

$$s_y(x) := \frac{f(x)-f(y)}{x-y}$$

is increasing as a function of x on the set $I \setminus \{y\}$. Since $s_x(y) = s_y(x)$ it is also increasing as a function of y on the set $I \setminus \{x\}$.

Proof. If $a<b<c$ are in I, then

$$s_a(b) \leq s_a(c) \tag{6.14}$$
$$s_c(a) \leq s_c(b) \tag{6.15}$$

by Lemma 6.8.1 and Exercise 6.8.2. Fix y in I. For $x < z$ with $x \neq y$ and $z \neq y$ we must show $s_y(x) \leq s_y(z)$. If $y < x < z$ this is (6.14) with $a = y$, $b = x$, and $c = z$. If $y < x < z$ it is (6.15) with $a = x$, $b = z$, and $c = y$. Finally, if $x < y < z$, then

$$s_y(x) = s_x(y) \leq s_x(z) = s_z(x) \leq s_z(y) = s_y(z),$$

by (6.14) and (6.15). ☺

If f is convex on an interval I, then $s_y(x) \leq s_y(z)$, for all $x < y < z$ in I by Corollary 6.8.3. Hence,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y} \text{ for all } x < y < z \text{ in } I. \tag{6.16}$$

See the left half of Fig. 6.6.

Exercise 6.8.4. Conversely, show that Eq. (6.16) implies $f(x) \leq L_{a,b}(x)$ for all $a < x < b$ in I.

We have proven:

Theorem 6.8.5. *f is convex on an interval I iff (6.16) holds for all $a < x < b$ in I.*

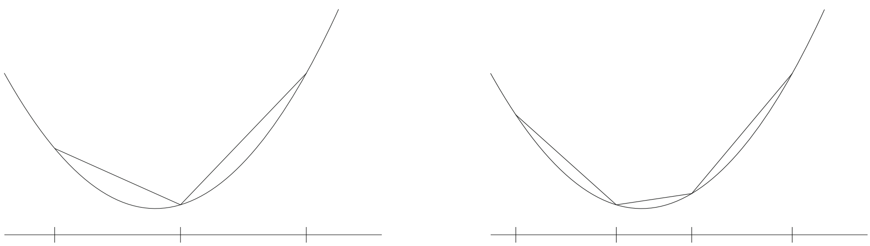

Fig. 6.6 A convex curve $y = f(x)$ and some of it's cords. The figure on the *left* illustrates (6.16). The figure on the *right* illustrates adding a third cord

Adding a third cord gives an inequality we can use to compare the slopes of cords that do not share an endpoint. See the right half of Fig. 6.6. Of course, this is corresponds to two applications of Corollary 6.8.3.

Regularity

In this subsection we show that geometric information: f lies below its cords, implies smoothness of f: f has one–sided derivatives.

A *Hölder condition* of order one, is also called a Lipschitz condition. Hence, f satisfies a Lipschitz condition on I, if there is a constant M, such that

$$|f(x)-f(y)| \leq M|x-y|$$

for all x and y in I. Clearly, any function satisfying a Lipschitz condition on an interval must he continuous on that interval. The Lipschitz condition is named after Rudolf Otto Sigismund Lipschitz (14 May 1832, Königsberg to 7 October 1903, Bonn), it is a strong form of uniform continuity.

Theorem 6.8.6. *Suppose I is an open interval and f is convex on I, then f is continuous on I. In fact, f satisfies a Lipschitz condition on any compact subinterval of I.*

Proof. Let $a<b$ be in I. Since I is open there is a $\delta>0$ such that $a-\delta$ and $b+\delta$ are in I. Fix $x<y$ in $[a,b]$. Applying Corollary 6.8.3 to

$$a-\delta<a \leq x<y \leq b<b+\delta$$

gives

$$s_a(a-\delta) \leq s_a(x) \leq s_y(x) \leq s_b(x) \leq s_b(b+\delta).$$

Taking the first, middle and last terms gives

$$s_a(a-\delta) \leq \frac{f(y)-f(x)}{y-x} \leq s_b(b+\delta).$$

Consequently, f satisfies the Lipschitz condition

$$|f(x)-f(y)| \leq \max\{|s_a(a-\delta)|,|s_b(b+\delta)|\}|x-y|$$

for all $x,y \in [a,b]$. In particular, f is continuous on $[a,b]$. Since $\bigcup[a,b]=I$, where the union is over all $a<b$ in I, f is continuous on I. ☺

Example 6.8.7. A convex function need not be continuous at an endpoint, for example, $f:[0,1] \to \mathbb{R}$ determined by $f(x):=\begin{cases} 0 & \text{when } x<1 \\ 2 & \text{when } x=1 \end{cases}$ is convex on $[0,1]$ and not continuous at 1.

Example 6.8.8. The function $f(x):=-\sqrt{1-x^2}$ is convex and continuous on the closed interval $[-1,1]$. However, it does not satisfy a Lipschitz condition on any subinterval containing at least one of the endpoints -1 or 1 of the interval $[-1,1]$.

We show that a convex function on an open interval has one-sided derivatives at every point in the interval and has a derivative at almost all points in the interval.

Theorem 6.8.9. *Let I be an open interval and f a convex function on I. The one sided derivatives f'^{+} and f'^{-} exist at every point in I. Furthermore, if $y<x$ are in I, then*

$$f'^{-}(y) \le f'^{+}(y) \le f'^{-}(x) \le f'^{+}(x). \tag{6.17}$$

In particular f'^{-} and f'^{+} are increasing functions on I.

Proof. Let I be an open interval and fix $x \in I$. By Corollary 6.8.3 $s_x(y)$ is an increasing function of y. Hence, $s_x(y)$ decreases as y decreases. Since I is open, there are a, b in I such that $a < x < b$. For $x < y < b$ we have $s_x(a) \le s_x(y)$. It follows from Exercise 5.1.2 that

$$\lim_{y \searrow x} s_x(y) = \inf\{s_x(y) \mid x < y\} \ge s_x(a).$$

Since $f'^{+}(x) = \lim_{y \searrow x} s_x(y)$, we have established the existence of the right hand derivative. Similarly, $f'^{-}(x) = \lim_{y \nearrow x} s_x(y)$ is a real number $\le s_x(b)$.

Fix $y < x$ in I. For u, v, w, z in I satisfying

$$u < y < z < w < x < v$$

Corollary 6.8.3 implies

$$s_y(u) \le s_y(z) \le s_x(w) \le s_x(v).$$

Letting $u \nearrow y$, we get

$$f'^{-}(y) \le s_y(z) \le s_x(w) \le s_x(v).$$

Letting $z \searrow y$, then $w \nearrow x$, and finally $v \searrow x$ we arrive at (6.17). ☺

Remark 6.8.10. Since f'^{+} exists at every point in the open interval, it follows that f is continuous on I. Hence, we have a second proof that a convex function defined on an open interval is continuous on that interval.

Corollary 6.8.11. *Let f be a convex function defined on an open interval I. There is a countable subset A of I, such that f is differentiable on $I \setminus A$.*

Proof. Since f'^{+} is increasing, it follows from Corollary 5.1.4 that there is a countable set A such that f'^{+} is continuous on $I \setminus A$. Fix $x \in I \setminus A$. For $y < x$ we have

$$f'^{+}(y) \le f'^{-}(x) \le f'^{+}(x).$$

Using f'^{+} is continuous at x, it follows that

$$f'^{+}(x) = \lim_{y \nearrow x} f'^{+}(y) \le f'^{-}(x) \le f'^{+}(x).$$

Hence $f'^{-}(x) = f'^{+}(x)$. Consequently, f is differentiable at x. ☺

Calculus

In this subsection we characterize convexity of differentiable functions in terms of properties of the first and second derivatives.

Theorem 6.8.12. *Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable. Then f' is increasing iff f is convex.*

Proof. Suppose f' is increasing. Let $x < y < z$ in I. By the MVT there are $c \in]x,y[$ and $d \in]y,z[$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} \text{ and } f'(d) = \frac{f(z) - f(y)}{z - y}.$$

Since $c < y < d$ and f' is increasing we have $f'(c) \leq f'(d)$, thus (6.16) holds, so f is convex.

Conversely, suppose f is convex. Let $x_0 < y_0$ in I. For any a, b in I with $a < x_0 < b < y_0$ we have

$$\frac{f(x_0) - f(a)}{x_0 - a} \leq \frac{f(y_0) - f(b)}{y_0 - b},$$

by Corollary 6.8.3. Letting $a \nearrow x_0$ shows

$$f'(x_0) \leq \frac{f(y_0) - f(b)}{y_0 - b} \text{ for any } x_0 < b < y_0 \text{ in } I.$$

Letting $b \nearrow y_0$, then yields $f'(x_0) \leq f'(y_0)$. ☺

The following is a simple criterion for convexity.

Corollary 6.8.13. *Let I be an open interval and let $f : I \to \mathbb{R}$. Suppose f'' exists on I. Then f is convex on I iff $f''(x) \geq 0$ for all $x \in I$.*

Proof. Let $g := f'$. Using g is increasing iff $g' \geq 0$, completes the proof ☺

The following exercise provides another geometrical interpretation of convexity.

Exercise 6.8.14. Let I be an open interval and $f : I \to \mathbb{R}$ be differentiable on I. Prove f is convex iff

$$f(x) \geq f(a) + f'(a)(x - a) \text{ for all } a, x \text{ in } I. \tag{6.18}$$

Thus, f is convex iff f lies above its tangents.

Suppose f has two derivatives. Recall, f is convex iff $f''(c) \geq 0$ for all c in I. The Taylor expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_1 f(x)$$

suggests this result, since $R_1 f(x) = \frac{f''(c)}{2}(x - x_0)^2$ and we need $R_1 f(x) \geq 0$ for f lie above its tangent at x_0.

Applications

The inequality $f(x) \leq L(x)$ for $a \leq x \leq b$, can be written as

$$f(\alpha a+(1-\alpha)b) \leq \alpha f(a)+(1-\alpha)f(b) \tag{6.19}$$

for $0 \leq \alpha \leq 1$.

Exercise 6.8.15. Verify this.

We establish the sum version of Jensen's Inequality. This inequality is due to Johan Ludwig William Valdemar Jensen (May 8, 1859, Nakskov to March 5, 1925, Copenhagen). Jensen worked for a telephone company as an engineer.

Theorem 6.8.16 (Jensen's Inequality). *Let I be an interval. If $f : I \to \mathbb{R}$ is convex, $x_k \in I$, $\alpha_k \geq 0$, and $\sum_{k=1}^{n} \alpha_k = 1$, then $\sum_{k=1}^{n} \alpha_k x_k$ is in I and*

$$f\left(\sum_{k=1}^{n} \alpha_k x_k\right) \leq \sum_{k=1}^{n} \alpha_k f(x_k).$$

Proof. If $x_{\min} := \min\{x_k \mid k=1,2,\ldots,n\}$ and $x_{\max} := \max\{x_k \mid k=1,2,\ldots,n\}$, then

$$\begin{aligned} x_{\min} = \sum_{k=1}^{n} \alpha_k x_{\min} &\leq \sum_{k=1}^{n} \alpha_k x_k \\ &\leq \sum_{k=1}^{n} \alpha_k x_{\max} = x_{\max} \end{aligned}$$

so $\sum_{k=1}^{n} \alpha_k x_k$ is a point in I.

The proof of the inequality is by induction. If $n=1$, the inequality is obvious. Suppose $n \geq 2$ and the inequality holds for $n-1$. If $\alpha_n = 1$ the inequality reduces to the $n=1$ case. If $\alpha_n \neq 1$, then

$$\sum_{k=1}^{n} \alpha_k x_k = (1-\alpha_n)\left(\sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} x_k\right) + \alpha_n x_n.$$

Hence, by convexity of f we have

$$\begin{aligned} f\left(\sum_{k=1}^{n} \alpha_k x_k\right) &= f\left((1-\alpha_n)\left(\sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} x_k\right) + \alpha_n x_n\right) \\ &\leq (1-\alpha_n) f\left(\sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} x_k\right) + \alpha_n f(x_n) \\ &\leq (1-\alpha_n) \sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} f(x_k) + \alpha_n f(x_n) \end{aligned}$$

$$= \sum_{k=1}^{n} \alpha_k f(x_k).$$

Where the first inequality follows from convexity of f (in the form (6.19)) and the second from the inductive hypothesis. ☺

Example 6.8.17. The triangle inequality is a special case of Jensen's inequality. Let $f(x) := |x|$. Clearly f is convex on $\mathbb{R}$. Applying Jensen's inequality with $\alpha_k := 1/n$ yields

$$\left| \sum_{k=1}^{n} \frac{1}{n} x_k \right| \leq \sum_{k=1}^{n} \frac{1}{n} |x_k|$$

for $x_k \in \mathbb{R}$. Multiplying by n, reduces this to the triangle inequality: $|\sum x_k| \leq \sum |x_k|$.

The *Arithmetic Mean* of $x_1, x_2, \ldots, x_n$ is $\frac{1}{n}\sum_{k=1}^{n} x_k$ and the *Geometric Mean* of $x_1, x_2, \ldots, x_n$ is $(x_1 x_2 \cdots x_n)^{1/n}$. A special case of the following result shows that the arithmetic mean is larger than the geometric mean when $x_k \geq 0$ for all k.

Theorem 6.8.18 (Arithmetic-Geometric Mean Inequality). *If $x_k \geq 0$, $\alpha_k \geq 0$, and $\sum_{k=1}^{n} \alpha_k = 1$, then*

$$\sum_{k=1}^{n} \alpha_k x_k \geq x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Proof. For the purpose of this proof we assume familiarity with the exponential function $\exp(x) = e^x$, and its inverse the logarithmic function $\log(x) = \ln(x)$. We study these functions in Chap. 8.

Since $\exp''(x) = \exp(x) \geq 0$ for all $x \in \mathbb{R}$, exp is convex on $\mathbb{R}$. Consequently,

$$\begin{aligned}
\sum_{k=1}^{n} \alpha_k x_k &= \sum_{k=1}^{n} \alpha_k \exp(\log(x_k)) \\
&\geq \exp\left(\sum_{k=1}^{n} \alpha_k \log(x_k) \right) \\
&= \exp\left(\log\left(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right)\right) \\
&= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.
\end{aligned}$$

The inequality is Jensen's. ☺

Corollary 6.8.19. *If $x_k \geq 0$ and $n \in \mathbb{N}$, then*

$$\frac{1}{n} \sum_{k=1}^{n} x_k \geq (x_1 x_2 \cdots x_n)^{1/n}.$$

Proof. Set $\alpha_k := 1/n$ for all k in the Arithmetic-Geometric Inequality. ☺

Problems

Problems for Sect. 6.1

1. Let $f(x) = x^2$ and let a be any real number. Show $f'(a) = 2a$.
2. Let $f(x) := |x|$. Show f is not differentiable at 0 and f is differentiable at any $a \neq 0$.
3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is even, i.e., $f(-x) = f(x)$ for all x. Assuming f is differentiable at 0, find $f'(0)$.
4. Show that the pseudo-sine function σ is differentiable at 1.

   The next two problems are concerned with the linearization of f near a. Suppose f is defined on some open interval containing a. If f is differentiable at a, then the line
   $$L(x) := f(a) + f'(a)(x-a)$$
   is called the *tangent* to f at a, and also the *linearization* of f at a.

5. Suppose $f'(a)$ exists. Let $L(x) := f(a) + f'(a)(x-a)$ and $E(x) := f(x) - L(x)$. Then
   $$f(x) = L(x) + E(x).$$
   Prove
   $$\frac{E(x)}{x-a} \to 0 \text{ as } x \to a.$$
   This if often written as $E(x) = o(x-a)$ as $x \to a$. Where $g(x) = o(\phi(x))$ as $x \to a$, means that $g(x)/\phi(x) \to 0$ as $x \to a$. This is known as “little oh” notation.

6. Let $k \in \mathbb{R}$. Let $L(x) := f(a) + k(x-a)$ and let $E(x) := f(x) - L(x)$. If
   $$\frac{E(x)}{x-a} \to 0 \text{ as } x \to a.$$
   Prove f is differentiable at a and $f'(a) = k$.
7. Let $\mathbb{R}^+ :=]0,\infty[$. Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is differentiable at $x = 1$ and
   $$f(xy) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}^+.$$
   a. Prove $f(1) = 0$.
   b. Prove $f(1/x) = -f(x)$.
   c. Prove f is differentiable on $\mathbb{R}^+$ and $f'(x) = f'(1)/x$ for all $x > 0$.

8. Suppose $f :]-1,1[\to \mathbb{R}$ is differentiable at 0, $f(x) < 0$ when $x < 0$, and $f(x) > 0$ when $x > 0$. Show $g(x) := |f(x)|$ is differentiable at 0 iff $f'(0) = 0$.

9. Suppose $f'(3) = 2$ and $f(3) = 4$. Prove there is a $\delta > 0$, such that
$$0 < x - 3 < \delta \implies x + 1 \le f(x).$$

10. We assume the familiar properties of $\sin(x)$. For example, (a) sin is differentiable and $\sin'$ is continuous on $\mathbb{R}$, (b) $-1 \le \sin(x) \le 1$ and $-1 \le \sin'(x) \le 1$ for all x. Let
$$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}.$$
Then $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ when $x \neq 0$. You need not verify this.
    a. Show f is differentiable at 0 and $f'(0) = 0$.
    b. $f'(x)$ is not continuous at $x = 0$.

Problems for Sect. 6.2

1. Let $f(x) := x^2$ on $\mathbb{R}$. Find $f'(x)$. Find the critical points. For each critical point decide if it is a local maximum, a local minimum, or neither.
2. Repeat the previous problem for $f(x) := x^3$.
3. If $f'^{+}(a)$ exists, then f is continuous from the right at a.
4. If $f'(a) \ge 0$ must f be increasing at a? [This is a proposed converse of Exercise 6.2.2.]
5. If f is strictly increasing at a and $f'(a)$ exists must $f'(a) > 0$? [This is a proposed converse of Exercise 6.2.3.]
6. If $f'(a) = 0$ must $f(a)$ be local extremum? [This is a proposed converse of Exercise 6.2.4.]
7. If $f(a)$ is a local extremum must $f'(a)$ exist?
8. How must the definition of increasing at a, strictly increasing at a, local maximum, etc. (in particular Exercises 6.2.2–6.2.4) be changed if we only assume a is an accumulation point of the domain D of f. (Instead of assuming f is defined near a.)

Problems for Sect. 6.3.

1. Calculate the derivate of
$$\frac{\sqrt{3x + x^2}}{(1 + x^2)\sqrt{x^3 + 7}}.$$

At each step in the calculation: Use at most one rule and state the name of that rule. You may use that constant functions have derivative equal to 0 and that $(x^n)' = nx^{n-1}$.

2. Let f be the restriction of the pseudo-sine function to the interval $[-1/2, 1/2]$. Prove f^{-1} is differentiable on the interval $]-1,1[$, but not at the endpoints of this interval.

Problems for Sect. 6.4.

1. Let $f(x) := \frac{1}{3}x^3$, $a := 0$ and $b := 2$. Then $f'(x) = x^2$. For $0 < k < 4$, show $x_{\min}$ in the proof of Darboux's Intermediate Value Theorem is $x_{\min} = \sqrt{k}$.
2. **The First Derivative Test**. Let f be continuous near c. Suppose f is differentiable on $]c-\gamma, c[$ and on $]c, c+\gamma[$ for some $\gamma > 0$. If $f'(x) > 0$ on $]c-\gamma, c[$ and $f'(x) < 0$ on $]c, c+\gamma[$, prove $f(c)$ is a local maximum. [*Hint*: Let $c-\gamma < x < c$. Applying the MVT to f on the interval $[x, c]$, yields $f(x) - f(c) < 0$.]
3. **The Second Derivative Test**. Suppose f is differentiable on $]c-\gamma, c+\gamma[$ for some $\gamma > 0$, $f'(c) = 0$, that f' is differentiable at c, and $f''(c) > 0$. (f'' denotes the derivative of f'.) Show that $f(c)$ is a local minimum, by completing the following steps:
   (*i*) Since, $\frac{f'(x)-f'(c)}{x-c} \to f''(c)$, there is a $\delta > 0$, such that $\frac{f'(x)-f'(c)}{x-c} > 0$ when $0 < |x-c| < \delta$.
   (*ii*) $f'(x) > 0$ for x in $]c-\delta, c[$.
   (*iii*) $f'(x) < 0$ for x in $]c, c+\delta[$.
   (*iv*) Use the First Derivative Test to show $f(c)$ is a local minimum of f.
4. Let

$$\phi(x) := \begin{cases} x^{3/2}\sigma(1/x) & \text{when } 0 < x \\ 0 & \text{when } x \le 0 \end{cases}.$$

Where σ is the pseudo-sine function. Prove ϕ is differentiable on $\mathbb{R}$ and $\limsup_{x \searrow 0} |\phi'(x)| = \infty$.

A function f satisfies a *global Lipschitz condition* on D, if there is a constant M, such that

$$|f(x) - f(y)| \le M|x-y| \text{ for all } x, y \in D.$$

5. If f is differentiable on $]a,b[$ and $|f'(x)| \le M$ for all x in $]a,b[$, then f satisfies a global Lipschitz condition on $]a,b[$. (In particular, if $M = 0$, then f is constant.)
6. If $f:]a,b[\to \mathbb{R}$ is $\alpha-$Hölder with $\alpha > 1$, then f is constant.
7. Let f be differentiable on $]0,1[$ and continuous on $[0,1]$. Suppose $f(0) = 0$ and f' is increasing on $]0,1[$. Let $g(x) := f(x)/x$. Prove g is increasing on $]0,1[$.
8. Let f be differentiable on $[a,b]$. Suppose $f'(x) \ge 0$ for all x in $[a,b]$ and f' is not identically 0 on any subinterval of $[a,b]$. Prove f is strictly increasing on $[a,b]$.

9. Suppose $f : [0,1] \to [0,1]$ is continuous on $[0,1]$ and differentiable on $]0,1[$. If $f'(x) \neq 1$ for all $x \in]0,1[$, prove there is at most one $c \in [0,1]$, such that $f(c) = c$.
   [If follows from the Intermediate Value Theorem that there is at least one such c, see the Problems for Sect. 5.2.]
10. Prove Bernoulli's inequality: $1 + \alpha x \leq (1+x)^\alpha$, for $x \geq -1$, when $\alpha > 1$ is rational.
    Your argument will probably work for irrational α also, but we have not yet shown that $f(t) := t^\alpha$ exists for $t \geq 0$, much less that is has the expected properties, e.g., that $f'(t) = \alpha t^{\alpha-1}$.
11. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose
    $$f(x+y) = f(x) + f(y) + 2xy \text{ for all } x, y \in \mathbb{R}.$$
    (i) Prove $f'(x) = f'(0) + 2x$, for all $x \in \mathbb{R}$.
    (ii) Prove $f(x) = x^2 + f'(0)x + f(0)$, for all $x \in \mathbb{R}$.
    In particular, f is a polynomial of degree two.
12. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be such that f''' exists. Suppose $f(a) = f'(a) = f(b) = f'(b) = 0$ for some $a < b$. Show $f'''(c) = 0$ for some c in $]a,b[$.
    (b) Part (a) applies to $f(x) := (x-a)^2 (x-b)^2$. In this case find an expression for c in terms of a and b.
13. Let I be an interval and suppose $f : I \to \mathbb{R}$ is differentiable. Let
    $$D := \left\{ f'(x) \mid x \in I \right\}$$
    and
    $$C := \left\{ \frac{f(b) - f(a)}{b - a} \mid a, b \in I, a < b \right\}.$$
    (a) Prove $C \subseteq D$.
    (b) Let k be a point in D. Show either k is in C or k is an accumulation point of C.
14. Let f be differentiable function defined on the open interval $]0,1[$.
    (a) Supposing f' is bounded on $]0,1[$, show f is bounded on $]0,1[$.
    (b) Give an example of a bounded and differentiable function f on $]0,1[$, such that f' is not bounded on $]0,1[$.

Problems for Sect. 6.5

1. Show that $\sum_{n=1}^{\infty} \frac{1}{2^{n!}}$ is transcendental.
2. For which $a \in \mathbb{R}$ is $\sum_{n=1}^{\infty} \frac{1}{a^{n!}}$ transcendental? [The series is only convergent for $a > 1$. See Chap. 10.]

3. Show $2^{1/3}$ is not algebraic of order 2.
4. Let R be the set of numbers of the form

$$\sum_{k=1}^{\infty} \frac{c_k}{10^{k!+1}}$$

where $c_k \in \{1, 10\}$ for all $k \in \mathbb{N}$.
(a) Show each number in R is transcendental.
(b) Show the set R is uncountable.

Problems for Sect. 6.6

1. Let $f(x) := x^{-1/3}$.

   a. (i) Verify $f^{(n)}(x) = (-1)^n 1 \cdot 4 \cdot 7 \cdots (3n-2) \cdot 3^{-n} \cdot x^{-(3n+1)/3}$ for $n \geq 1$.
   b. (ii) Show $\left|\frac{f^{(n)}(8)}{n!}\right| \leq \frac{1}{2^{3n+1}}$, when $n \geq 1$.
   c. (iii) Let $x_0 := 8$ and $x := 9$, how large must n be for $|R_n f(9)| \leq 1/60000$?
   d. (iv) For this n, evaluate $T_n f(9)$.
      You have now found $9^{-1/3}$ with error $< 1/60000$. Multiplying by 3 gives $3^{1/3}$ with an error $< 1/20000 = 0.00005$.

2. Let $f(x) := \frac{1}{1-x}$ and $x_0 = 0$.

   a. (i) Obtain an expression for $R_1 f$.
   b. (ii) Find the value of c when $x = \frac{3}{4}$.

3. Let $f(x) = \sqrt[3]{1+3x}$.

   a. Find the second Taylor polynomial $T_2 f(x)$ for f at $x_0 = 0$.
   b. Use $T_2 f(x)$ to approximate $\sqrt[3]{1.006}$.
   c. Use the formula for the Lagrange formula for the remainder $R_2 f(x)$ to estimate the error in the approximation in part (b).

Problems for Sect. 6.7

1. If, in the proof of l'Hopital's Rule we use the MVT in the numerator and denominator separately we get

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c(x))}{g'(d(x))},$$

where $c(x)$ and $d(x)$ are between a and x. Does this imply $f(x)/g(x) \to L$? More precisely, if $\phi(x)/\psi(x) \to L$ as $x \to a$, and $\alpha(x) \to a$, $\beta(x) \to a$ as $x \to a$, must $\phi(\alpha(x))/\psi(\beta(x)) \to L$ as $x \to a$?

2. Formulate and prove l'Hopital's Rule for the case where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$. [*Hint*: A proof can be based on the formula

$$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f(x)}{g(x)} \frac{1-\frac{f(b)}{f(x)}}{1-\frac{g(b)}{g(x)}}$$

by making a suitable choice for b.]

Problems for Sect. 6.8

1. Prove Theorem 6.8.6 when I is unbounded. Note, I may be unbounded in one or both directions.
2. Let $I :=]0,1[$. Give an example of a function $f : I \to \mathbb{R}$, such that f is convex on I and f not uniformly continuous on I.
3. Verify the claims in Example 6.8.8.
4. Construct a convex and continuous $f : [0,1] \to \mathbb{R}$ such that f is not differentiable at any point in $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

   Suppose f is convex on an open interval I and y is a point in I. A real number m is called a *sub-derivative* of f at y, if

   $$f(x) \geq f(y) + m(x-y)$$

   for all $x \in I$., i.e., if f lies below the line $L_{m,y}(x) := f(y) + m(x-y)$. Compare to Exercise 6.8.14.
5. If f is convex on an open interval I, y is a point in I, and m is a real number. Then m is a sub-derivative of f at y iff $f'^{-}(y) \leq m \leq f'^{+}(y)$.
6. Young's inequality [William Henry Young, London, 20 October 1863 to Lausanne, 7 July 1942)] Let $x, y \geq 0$. Suppose $p, q > 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then

   $$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

   [*Hint*: Set $n = 2$, $\alpha_1 = 1/p$, and $\alpha_2 := 1/q$ in the algebraic-geometric mean inequality.]

7. Hölder's inequality [Otto Ludwig Hölder, Stuttgart, December 22, 1859 to Leipzig, August 29, 1937] Let $x_k, y_k \in \mathbb{C}$. Suppose $p, q > 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}.$$

[*Hint*: It is sufficient to consider $x_k, y_k \geq 0$. Apply Young's inequality to

$$x = \frac{x_k}{\left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}} \text{ and } y = \frac{y_k}{\left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}},$$

then sum over $k = 1, 2, \ldots, n$ and simplify.]
When $p = q = 2$ Hölder's inequality is usually called the Cauchy–Schwarz inequality. It is usually named after Augustin-Louis Cauchy and Karl Hermann Amandus Schwarz (25 January 1843, Hermsdorf to 30 November 1921, Berlin). Cauchy gave the first proof for sums, Viktor Yakovych Bunyakovsky (December 16 1804, Bar, to December 12 1889, St. Petersburg) first proved it for integrals, and Schwarz later gave a simpler proof for integrals.
For $p \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$, $x_k \in \mathbb{C}$, the $p-$norm of x is $\|(x_k)\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$. The next problem establishes the triangle inequality for the $p-$norm.
8. Minkowski's inequality [Hermann Minkowski, Aleksotas, June 22, 1864 to Göttingen, January 12, 1909] Let $x_k, y_k \in \mathbb{C}$. Suppose $p, q > 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

[*Hint*: It is sufficient to consider $x_k, y_k \geq 0$. If $p = 1$ there is nothing to prove. Assume $p > 1$ and let $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Since $(p-1)q = p$

$$\begin{aligned}\sum_{k=1}^{n} (x_k + y_k)^p &= \sum_{k=1}^{n} x_k (x_k + y_k)^{p-1} + \sum_{k=1}^{n} y_k (x_k + y_k)^{p-1} \\ &\leq \left(\sum_{k=1}^{n} (x_k)^p\right)^{1/p} \left(\sum_{k=1}^{n} (x_k + y_k)^p\right)^{1/q} \\ &\quad + \left(\sum_{k=1}^{n} (y_k)^p\right)^{1/p} \left(\sum_{k=1}^{n} (x_k + y_k)^p\right)^{1/q}\end{aligned}$$

by Hölder's inequality. Simplifying gives us Minkowski's inequality.]

There are versions of Jensen's, Hölder's, and Minkowski's inequalities for integrals.

Solutions and Hints for the Exercises

Exercise 6.2.2 If f is increasing at a, then $(f(x)-f(a))/(x-a) \geq 0$ near a.

Exercise 6.2.3 If $\lim_{x\to a}(f(x)-f(a))/(x-a) > 0$, then $(f(x)-f(a))/(x-a) > 0$ near a.

Exercise 6.2.4 If $f'(a) > 0$, Exercise 6.2.3 leads to a contradiction.

Exercise 6.3.9. Use the examples, the product rule and induction.

Exercise 6.3.10. One way is to use the constant, sum and product rules as well as Exercise 6.3.9

Exercise 6.3.11. By Exercise 6.3.10 and the quotient rule.

Exercise 6.3.12. Let $f(x) := x^n$ and use the inverse function rule.

Exercise 6.4.1. Look at the discussion prior to this exercise.

Exercise 6.4.4. If $x \neq 0$, then $g(x) = x^2\sigma(1/x)$. So, $g'(x) = 2x\sigma(1/x) - \sigma'(1/x)$ when $x \neq 0$. Also $g(x)/x = x\sigma(1/x) \to 0$ as $x \to 0$, hence, $g'(0) = 0$.

Exercise 6.4.8. If $h := f - g$, then $h' = 0$ on I.

Exercise 6.4.9. Apply the MVT to $h = f - g$ on the interval with endpoints x and c.

Exercise 6.4.10. If f is increasing then the definition of f' shows that $f'(x) \geq 0$. On the other hand suppose $f' \geq 0$ on I. Let $x < y$ be points in I. Apply MVT on the interval $[x,y]$.

Exercise 6.7.2. Let $t := 1/x$, then $x \to \infty$ iff $t \searrow 0$.

Exercise 6.8.2. (*i*) Rearrange the expression for $L(x)$. (*ii*) Rearrange $f(x) \leq M(x)$.

Exercise 6.8.4. Solve (6.16) for $f(x)$.

Exercise 6.8.14. If f is convex the inequality in (6.18) follows from Lemma 6.8.1 by taking an appropriate limit.

If (6.18) holds for all a, x in I. Let $y < z$ be points in I. Applying (6.18) with $x = z$ and $a = y$ gives, $f(z) \geq f(y) + f'(y)(z-y)$. Similarly, applying (6.18) with $x = y$ and $a = z$ gives, $f(y) \geq f(z) + f'(z)(y-z)$. Consequently,

$$f'(y) \leq \frac{f(z)-f(y)}{z-y} \leq f'(z).$$

Exercise 6.8.15. Verify $x : [0,1] \to [a,b]$ determined by $x(\alpha) := \alpha a + (1-\alpha)b$ is a bijection. Then replace x in $f(x) \leq L(x)$ by $\alpha a + (1-\alpha)b$ and rearrange.

Chapter 7
The Riemann Integral

The Riemann integral is defined in terms of lower and upper step functions. The major theorems are concerned with characterizations of integrability, the integrability of monotone and continuous functions, the algebra of integrable functions, and the two versions of the fundamental theorem of calculus.

We will mostly be concerned with integrals of functions $f : [a,b] \to \mathbb{R}$. However, the reader can without difficulty extend the integral to functions $f : [a,b] \times [c,d] \to \mathbb{R}$, and is asked to do so in some of the exercises. It is essential that we have an order on the values of f – not that we have an order on the domain of f. While most of the discussing is for integrals of real valued functions, the last section considers integrals of complex valued functions. Arc length is discussed in Sect. 11.3 it could be covered at any time after Sect. 7.3.

7.1 Definition of the Integral

The idea of our approach to the integral of a function $f : [a,b] \to \mathbb{R}$ is to approximate the graph from below by the areas of a finite number of rectangles (see Fig. 7.1), then take the best such approximation, that is the supremum of the lower approximations. We also approximate from above by a finite number of rectangles (see Fig. 7.2), then take the best such approximation, the infimum of the upper approximations. When the two best approximations agree we say f is integrable. To have rectangles of *finite* height we need f to be bounded. To have a *finite* number of rectangles of *finite* width we need the interval $[a,b]$ to be bounded.

Step Functions

A finite collection of points $a = x_0 < x_1 < \cdots < x_n = b$ is called a *partition* of the closed and bounded interval $[a,b]$. Given a partition $a = x_0 < x_1 < \cdots < x_n = b$

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a function $s : [a,b] \to \mathbb{R}$ that is constant on the open intervals $]x_{k-1}, x_k[$ is a *step function*. So, a step function is determined by a partition and real numbers A_k by setting

$$s(x) := \sum_{k=1}^{n} A_k \mathbb{1}_{]x_{k-1},x_k[}(x)$$

for x in $[a,b]$. Recall,

$$\mathbb{1}_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

The values of a step function at the partition points $x_0, x_1, \ldots, x_n$ are not important in what follows. For a step function $s := \sum_{k=1}^{n} A_k \mathbb{1}_{]x_{k-1},x_k[}$ we write

$$\sum s = \sum_P s = \sum_{i=1}^{n} A_i (x_i - x_{i-1}),$$

where $P = \{x_0, x_1, \ldots, x_n\}$ is the set of partition points.

If $f : [a,b] \to \mathbb{R}$ be a bounded function, then $m = m(f)$ is some lower bound for f and $M = M(f)$ is some upper bound for f. That is $m \le f(x) \le M$ for all x.

If $s : [a,b] \to \mathbb{R}$ is a step function such that $s(x) \le f(x)$ for all $x \in [a,b]$, except possibly at the partition points, then s is a *lower step function for* f and $\sum s$ is a *lower sum for* f. The shaded areas in Fig. 7.1 illustrates lower sums.

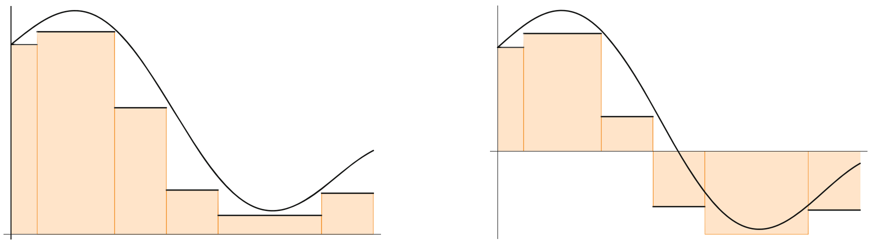

Fig. 7.1 Lower step functions—in one case the function is positive. The two parts of this figure are essentially identical except for a vertical shift

Lemma 7.1.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The set of lower sums for f is nonempty and bounded above.*

Proof. The set of lower sums is nonempty, because any bounded function has at least one lower step function. For example, $s(x) = m$ for $x \in [a,b]$ is a lower step function for f. If $s = \sum_{k=1}^{n} a_k \mathbb{1}_{]x_{k-1},x_k[}$ is any lower step function for f, then $a_i \le f(x) \le M$ for any $x_{i-1} < x < x_i$, so

$$\sum s = \sum_{i=1}^{n} a_i (x_i - x_{i-1}) \le \sum_{i=1}^{n} M (x_i - x_{i-1}) = M(b-a).$$

Hence, $M(b-a)$ is an upper bound for the set of lower sums. ☺

The *lower integral* $\underline{\int_a^b} f$ *of* f is the supremum of the set of lower sums for f. That is

$$\underline{\int_a^b} f := \sup\left\{\sum s \mid s \text{ is a lower step function for } f\right\}.$$

Similarly, if $S : [a,b] \to \mathbb{R}$ is a step function such that $f(x) \leq S(x)$ for all $x \in [a,b]$, except possibly for the partition points, then S is an *upper step function for* f and $\sum S$ is an *upper sum for* f. The shaded areas in Fig. 7.2 illustrates upper sums.

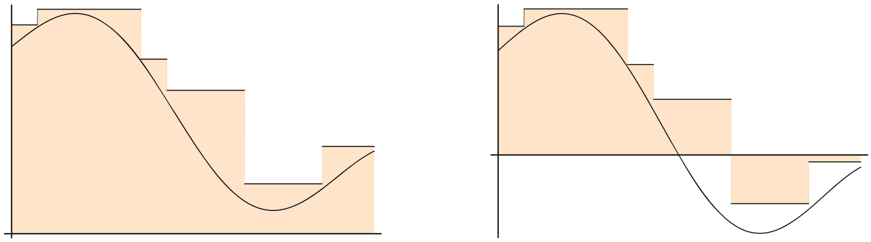

Fig. 7.2 Upper step functions—in one case the function is positive. The two parts of this figure are essentially identical except for a vertical shift

Lemma 7.1.2. *The set of upper sums is nonempty and bounded below.*

Exercise 7.1.3. Prove this Lemma.

The *upper integral* $\overline{\int_a^b} f$ *of* f is the infimum of the set of upper sums for f. That is

$$\overline{\int_a^b} f := \inf\left\{\sum S \mid S \text{ is an upper step function for } f\right\}.$$

Definition 7.1.4. A bounded function $f : [a,b] \to \mathbb{R}$ is *(Riemann) integrable* if $\underline{\int_a^b} f = \overline{\int_a^b} f$. We write $\int_a^b f$ for this common value and call this number the *integral* of f.

Remark 7.1.5. Consider a positive function $f(x) \geq 0$ for $x \in [a,b]$. Consider the region R below the graph of f and above the x-axis. A lower sum for f is a lower bound for the area of R, see Fig. 7.1. Similarly, an upper sum for f gives an upper bound for the area of R, see Fig. 7.2. Hence, it is natural to say that R has an *area* when f is integrable and that this area equals $\int_a^b f$. Calculating areas by approximating both from the outside and from the inside was also done by Archimedes (c. 287 BC Syracuse–c. 212 BC Syracuse).

In the next section, we will obtain a useful characterization of (Riemann) integrability. In particular, we will show that a step function s is integrable and $\int_a^b s = \sum s$.

Example 7.1.6. The function $f : [0,1] \to \mathbb{R}$, determined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise is not integrable.

Proof. By density of the rationals any upper sum is ≥ 1. Similarly, it follows from density of the irrationals that any lower sum is ≤ 0. Consequently, $\underline{\int_0^1} f \leq 0 < 1 \leq \overline{\int_0^1} f$. ☺

The integral is monotone in the sense:

Exercise 7.1.7. If $f, g : [a,b] \to \mathbb{R}$ are integrable and $f \leq g$, then $\int_a^b f \leq \int_a^b g$.

More on Step Functions

We establish some technical results related to step functions. The main result shows that if we have a step function s and construct a new step function s' by inserting additional partition points, then the two step functions s and s' have the same sum. Hence, when considering two step function s and S we can assume they have the same partition points.

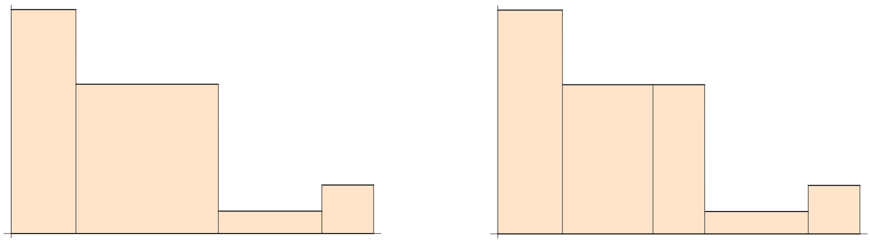

Fig. 7.3 Inserting one additional partition point. The figure on the *right* has one additional partition point. Clearly, the two step functions have the same sum

Let $P : a = x_0 < x_1 < \cdots < x_m = b$ and $Q : a = y_0 < y_1 < \cdots < y_n = b$ be partitions of $[a,b]$. If $\{x_j \mid j = 1,2,\ldots,m\}$ is a subset of $\{y_k \mid k = 0,1,\ldots,n\}$, then Q is a *refinement* of P. If g is a step function with respect to some partition P and that Q is a *refinement* of P, then we can also view g as a step function with respect to Q, the next lemma shows that $\sum_P g = \sum_Q g$. See Fig. 7.3.

Lemma 7.1.8. *Let g be a step function. If g' is obtained from g by inserting a finite number of additional partition points, then $\sum g' = \sum g$.*

Proof. Consider a step function $g = \sum_{k=1}^{n} A_k \mathbb{1}_{]x_{k-1} x_k[}$. Let t a point in $[a,b]$ which is not one of the x_i's. Suppose $x_{j-1} < t < x_j$. Let $y_k := x_k$ and $B_k := A_k$, when $k < j$; $y_j := t$, $B_j := A_j$; and $y_k := x_{k-1}$ and $B_k := A_{k-1}$, when $k > j$. Let $g_1 := \sum_{k=1}^{n+1} B_k \mathbb{1}_{]y_{k-1}, y_k[}$. Then $g_1(x) = A_i$ if $x_{i-1} < x < x_i$ and $i \neq j$, and $g(x) = A_j$ if $x_{j-1} <$

$x < t$ and if $t < x < x_j$. So g_1 has the same values as g, but there is one more division point. Hence,

$$\begin{aligned}\sum_{P_1} g_1 &= \sum_{i\neq j} A_i(x_i - x_{i-1}) + A_j(t - x_{j-1}) + A_j(x_j - t)\\ &= \sum_{i=1}^{n} A_i(x_i - x_{i-1}) = \sum_{P} g,\end{aligned}$$

because $(t - x_{j-1}) + (x_j - t) = x_j - x_{j-1}$. Suppose g_2 is similarly obtained from g_1 by inserting an additional division point, then we get $\sum g_2 = \sum g_1$. But g_2 is obtained from g by inserting two additional division points and $\sum g_2 = \sum g$. By induction we can insert any finite number of additional partition points. ☺

The following consequence of this lemma is very useful.

Corollary 7.1.9. *If s,t are step functions with respect to the partitions P,Q of $[a,b]$, then we can view s,t as step functions with respect to the common partition $R = P\cup Q$ and $\sum_R s = \sum_P s$ and $\sum_R t = \sum_Q t$.*

If $s = \sum_{k=1}^{n} A_k \mathbb{1}_{]x_{k-1}x_k[}$, then $as = \sum_{k=1}^{n} aA_k \mathbb{1}_{]x_{k-1}x_k[}$, hence a multiple of a step function is a step function.

Exercise 7.1.10. The sum of two step functions is a step function.

If s is a lower step function for f and S is an upper step function for f then $s \le S$. But, since s and S may correspond to different subdivisions of $[a,b]$ it is not easy to compare their sums $\sum s$ and $\sum S$. The next lemma shows that, in fact, the upper sum is larger than the lower sum.

Lemma 7.1.11. *Let f be a bounded function. If s is a lower step function for f and S is an upper step function for f, then $\sum s \le \sum S$.*

Proof. Suppose s corresponds to the partition $P: y_0 < \cdots < y_l$ and S corresponds to the partition $Q: z_0 < \cdots < z_m$. Let $\{x_0,\ldots,x_n\}$ be the union of $\{y_0,\ldots,y_l\}$ and $\{z_0,\ldots,z_m\}$. So $R: x_0 < \cdots < x_n$ is a refinement of P and of Q. By Lemma 7.1.8 $\sum_P s = \sum_R s$. Similarly, for $\sum_Q S = \sum_R S$. With respect to the partition $x_0 < \cdots < x_n$ we have $s(x) = a_i$ and $S(x) = A_i$ for $x_{i-1} < x < x_i$. Also, $a_i = s(x) \le f(x) \le S(x) = A_i$, when $x_{i-1} < x < x_i$, hence $a_i \le A_i$. It follows that

$$\sum_R s = \sum_{i=1}^{n} a_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} A_i(x_i - x_{i-1}) = \sum_R S$$

as we needed to establish. ☺

This result seems obvious, if we think of the upper and lower sums in terms of areas. See Figs. 7.1 and 7.2.

Exercise 7.1.12. Write down a definition of step functions and of the integral for functions defined on a rectangle.

7.2 Characterizations of Riemann Integrability

In this section, we establish techniques that simplify verifying that a given function is integrable and in the affirmative case simplifies calculating its integral.

Preliminary Versions

We begin by observing that Lemma 7.1.11 implies that the lower integral is always smaller than the upper integral.

Lemma 7.2.1. *If $f : [a,b] \to \mathbb{R}$ is a bounded function, then $\underline{\int_a^b} f \le \overline{\int_a^b} f$.*

Proof. If s is any lower step function and S is any upper step function, then $\sum s \le \sum S$ by Lemma 7.1.11. Thinking of S as fixed, we see that $\sum S$ is an upper bound for the set of all lower sums, and therefore

$$\underline{\int_a^b} f = \sup\left\{\sum s \mid s\right\} \le \sum S. \tag{7.1}$$

But, since S is an arbitrary upper step function, then $\underline{\int_a^b} f$ is a lower bound for the set of upper sums. So we get

$$\underline{\int_a^b} f \le \inf\left\{\sum S \mid S\right\} = \overline{\int_a^b} f. \tag{7.2}$$

As we needed to show. ☺

We could have referred to Proposition 3.3.1, instead we repeated the proof.

For any bounded function f any lower step function s and any upper step function S we have

$$\sum s \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \sum S \tag{7.3}$$

by Lemma 7.2.1. In particular, we have an analogue of the Nested Interval Theorem.

Corollary 7.2.2. *Suppose f is integrable. If I is a real number such that*

$$\sum s \le I \le \sum S$$

for all lower step functions s for f and all upper step functions S for f, then $\int_a^b f = I$.

Our next aim it to make these observations easier to use by showing it is not necessary to consider <u>all</u> step functions.

Improved Versions

The results in the previous subsection require us to consider all lower sums and all upper sums. The results in this subsection allow us to use conveniently chosen lower and upper sums (Fig. 7.4).

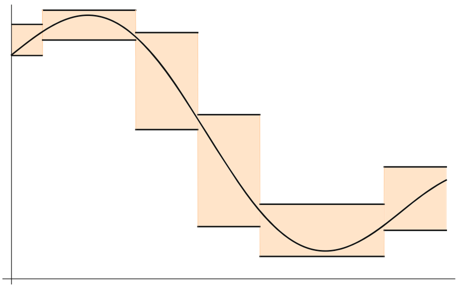

Fig. 7.4 The *shaded area* illustrates $\sum S - \sum s$. Compare to Figs. 7.1 and 7.2

Theorem 7.2.3 (Existence Theorem for Integrals). *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if given any $\varepsilon > 0$ there exists a lower step function s_ε for f and an upper step function S_ε for f such that $\sum S_\varepsilon - \sum s_\varepsilon < \varepsilon$.*

Proof. Suppose f is integrable. Let $\varepsilon > 0$ be given. Let s be a lower step function such that $\underline{\int_a^b} f < \sum s + \frac{\varepsilon}{2}$ and let S be an upper step function such that $\sum S - \frac{\varepsilon}{2} < \overline{\int_a^b} f$. Since $\underline{\int_a^b} f = \overline{\int_a^b} f$ we get $\sum S - \frac{\varepsilon}{2} < \sum s + \frac{\varepsilon}{2}$. Rearranging, leads to $\sum S - \sum s < \varepsilon$.

Conversely, suppose f is not integrable. Let $\varepsilon := \overline{\int_a^b} f - \underline{\int_a^b} f$, then $\varepsilon > 0$. If s is any lower step function and S is any upper step function, then

$$\sum s \leq \underline{\int_a^b} f = \overline{\int_a^b} f - \varepsilon \leq \sum S - \varepsilon.$$

Hence, $\sum S - \sum s \geq \varepsilon$ as we needed to show. ☺

The condition that given any $\varepsilon > 0$ we can find a lower step function s for f and an upper step function S for f such that $\sum S - \sum s < \varepsilon$, is satisfied if we can find a sequence (s_n) of lower step functions for f and a sequence (S_n) upper step functions for f such that the sequence $(\sum S_n - \sum s_n)$ is null, hence:

Corollary 7.2.4. *Let $f : [a,b] \to \mathbb{R}$ be bounded. If there are lower step functions s_n and upper step functions S_n for f, such that the sequence $(\sum S_n - \sum s_n)$ is null, then f is integrable.*

The next result allows us to evaluate $\int_a^b f$ at the same time as we show that f is integrable.

Theorem 7.2.5 (Evaluation Theorem for Integrals). *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let I be a real number. Suppose for any $\varepsilon > 0$ we can find a lower step function s_ε for f and an upper step function S_ε for f, such that $\sum S_\varepsilon - \sum s_\varepsilon < \varepsilon$ and $\sum s_\varepsilon \leq I \leq \sum S_\varepsilon$, then f is integrable and $\int_a^b f = I$.*

Proof. It follows from the Existence Theorem (Theorem 7.2.3) that f is integrable. Suppose $\int_a^b f \neq I$. Let $\varepsilon := \left| I - \int_a^b f \right| /2$. Then I and $\int_a^b f$ both are in the interval $[s_\varepsilon, S_\varepsilon]$. Since this interval has length ε, $\left| I - \int_a^b f \right| \leq \varepsilon$. That is

$$\left| I - \int_a^b f \right| \leq \left| I - \int_a^b f \right| /2.$$

This contradiction completes the proof. ☺

As above, we can formulate this in terms of sequences:

Corollary 7.2.6. *Let $f : [a,b] \to \mathbb{R}$ be bounded. If there are lower step functions s_n and upper step functions S_n for f, such that the sequence $(\sum S_n - \sum s_n)$ is null and I is a real number such that $s_n \leq I \leq S_n$ for all n, then f is integrable and $\int_a^b f = I$.*

Example 7.2.7. Let $f(x) = 2x$. Then f is integrable on $[0,3]$ and $\int_0^3 f = 9$.

Proof. Consider the partition $x_k := \frac{3k}{n}$, $k = 0, 1, \ldots, n$ of the interval $[0,3]$. Since f is increasing

$$s_n := \sum_{k=1}^{n} f(x_{k-1}) \mathbb{1}_{]x_{k-1}, x_k[} = \sum_{k=1}^{n} \frac{6}{n} \cdot (k-1) \cdot \mathbb{1}_{]\frac{3k-3}{n}, \frac{3k}{n}[}$$

is a lower step function for f and

$$S_n := \sum_{k=1}^{n} f(x_k) \mathbb{1}_{]x_{k-1}, x_k[} = \sum_{k=1}^{n} \frac{6}{n} \cdot k \cdot \mathbb{1}_{]\frac{3k-3}{n}, \frac{3k}{n}[}$$

is an upper step function for f. Using $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$ it follows that

$$\sum s_n = \frac{6}{n} \left(\sum_{k=1}^{n-1} k \right) \frac{3}{n} = \frac{6}{n} \cdot \frac{(n-1)n}{2} \cdot \frac{3}{n} = 9 \left(1 - \frac{1}{n} \right).$$

Similarly,

$$\sum S_n = 9 \left(1 + \frac{1}{n} \right).$$

Hence, the sequence $\sum S_n - \sum s_n = \frac{18}{n}$ is null and

$$\sum s_n \leq 9 \leq \sum S_n$$

for all n. ☺

7.3 Examples of Integrable Functions

In this section, we show that step functions, monotone functions, and continuous functions defined on compact intervals are integrable.

Proposition 7.3.1. *If $f:[a,b]\to\mathbb{R}$ is a step function, then f is integrable and $\int_a^b f = \sum f$.*

Proof. Let $s=f$ and $S=f$ then s is a lower step function for f and S is an upper step function for f. Clearly, $\sum S-\sum s=0$ and $\sum s\le\sum f\le\sum S$. Hence, the result follows from Theorem 7.2.5. ☺

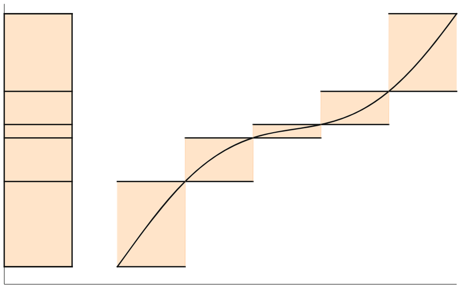

Fig. 7.5 Illustrating $\sum S-\sum s=(f(b)-f(a))\frac{b-a}{n}$ in the proof of Theorem 7.3.2. The rectangle on the left has height $f(b)-f(a)$ and width $\frac{b-a}{n}$

Theorem 7.3.2. *Monotone functions are integrable.*

Proof. Let $f:[a,b]\to\mathbb{R}$ be increasing. f is bounded since $f(a)\le f(x)\le f(b)$. Let $\varepsilon>0$ be given. Let n be an integer such that

$$(f(b)-f(a))\frac{b-a}{n}<\varepsilon.$$

Let $x_i=a+\frac{b-a}{n}i$ for $i=0,\dots,n$; and let $s(x)=f(x_{i-1})$ for $x_{i-1}<x<x_i$. Since f is increasing s is a lower step function for f. Similarly, $S(x)=f(x_i)$ for $x_{i-1}<x<x_i$ determines an upper step function for f.

$$\sum S-\sum s=\sum f(x_i)(x_i-x_{i-1})-\sum f(x_{i-1})(x_i-x_{i-1})$$

since $x_i-x_{i-1}=\frac{b-a}{n}$ we get

$$\sum S-\sum s=\sum(f(x_i)-f(x_{i-1}))\frac{b-a}{n}=(f(b)-f(a))\frac{b-a}{n}<\varepsilon.$$

See Fig. 7.5. So f is integrable by Theorem 7.2.3. ☺

Recall, a monotone function can be discontinuous at an infinite number of points, in fact, it can be discontinuous at a dense set of points, see Example 5.1.5.

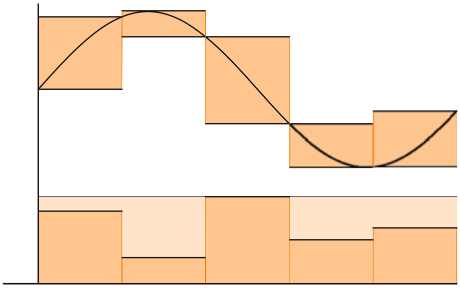

Fig. 7.6 Illustrating $\sum S - \sum s < \varepsilon$ in the proof of Theorem 7.3.3. Since $M_i - m_i < \frac{\varepsilon}{b-a}$, the rectangle at the bottom has height $< \frac{\varepsilon}{b-a}$. Clearly, it has width $b-a$

Theorem 7.3.3. *Continuous functions are integrable.*

Proof. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is bounded, since a continuous function on a compact intervals has a largest and a smallest value. Let $\varepsilon > 0$ be given. Since f is defined on a compact interval f is uniformly continuous. Hence, there exists a $\delta > 0$, such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Let n be an integer such that $\frac{b-a}{n} < \delta$. Let $x_i = a + \frac{b-a}{n} i$ for $i = 0, \ldots, n$. Let $m_i = f(x_i^*)$ be the smallest value of f on the interval $[x_{i-1}, x_i]$. Let $M_i = f(y_i^*)$ be the largest value of f on the interval $[x_{i-1}, x_i]$. Since x_i^* and y_i^* are in $[x_{i-1}, x_i]$ we have

$$|x_i^* - y_i^*| \leq |x_i - x_{i-1}| = \frac{b-a}{n} < \delta,$$

hence $M_i - m_i = f(y_i^*) - f(x_i^*) < \frac{\varepsilon}{b-a}$. Let $s(x) = m_i$ on $x_{i-1} < x < x_i$. Then s is a lower step function for f by the construction of m_i. Similarly, $S(x) = M_i$ on $x_{i-1} < x < x_i$ is an upper step function for f. By construction of s and S we have (Fig. 7.6)

$$\begin{aligned} \sum S - \sum s &= \sum (M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum (x_i - x_{i-1}) \\ &= \varepsilon. \end{aligned}$$

Hence, f is integrable by Theorem 7.2.3. ☺

Below is an outline that shows how the preceding two proofs are similar.

Proof. [Unified proof of the two previous theorems] Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $\varepsilon > 0$ be given. Let s and S be a lower and an upper step function for f. By Corollary 7.1.9 we may assume that s and S corresponds to the same partition. If

$$\sum s = \sum m_i(x_i - x_{i-1})$$
$$\sum S = \sum M_i(x_i - x_{i-1})$$

then

$$\sum S - \sum s = \sum (M_i - m_i)(x_i - x_{i-1}).$$

If f is continuous, then f is uniformly continuously, hence we can choose the partition such that

$$M_i - m_i < \widetilde{\varepsilon} := \frac{\varepsilon}{b-a}.$$

Consequently,

$$\sum S - \sum s \le \widetilde{\varepsilon} \sum (x_i - x_{i-1}) = \widetilde{\varepsilon}(b-a) = \varepsilon.$$

If f is increasing, then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. So, we choose the partition such that

$$x_i - x_{i-1} \le \delta := \frac{\varepsilon}{f(b) - f(a)}.$$

Consequently,

$$\sum S - \sum s \le \sum (f(x_i) - f(x_{i-1}))\delta = (f(b) - f(a))\delta = \varepsilon.$$

In either case, f is integrable by Theorem 7.2.3. ☺

Exercise 7.3.4. If $f : [a,b] \to \mathbb{R}$ is continuous, $f \ge 0$, and $\int_a^b f = 0$, then $f = 0$.

Exercise 7.3.5. If $f : [a,b] :\to \mathbb{R}$ is continuous, $f \ge 0$, and $f(c) > 0$ for some $c \in [a,b]$, then $\int_a^b f > 0$.

7.4 Algebra of Integrable Functions

In order not to have to refer to the lower/upper step functions every time we consider an integral we need to develop some properties of the integral. Doing so is the purpose of this section and of Sect. 7.5.

The transformation $f \to \int_a^b f$ is linear:

Proposition 7.4.1. *Let f and g be integrable functions on $[a,b]$ and let c and d be constants, then $cf + dg$ is integrable and $\int_a^b (cf + dg) = c\int_a^b f + d\int_a^b g$.*

Proof. Suppose $k > 0$. Let $\varepsilon > 0$ be given. Suppose s (resp. S) is a lower (resp. upper) step function for f such that $\sum S - \sum s < \frac{\varepsilon}{k}$. Since $k > 0$, ks (resp. kS) is a lower (resp. upper) step function for kf such that $\sum kS - \sum ks = k(\sum S - \sum s) < \varepsilon$. Furthermore,

$$\sum ks = k \sum s \le k \int_a^b f \le k \sum S = \sum kS.$$

Hence, it follows from Theorem 7.2.5 that kf is integrable and $\int_a^b kf = k\int_a^b f$.

It remains to show that $f+g$ is integrable and that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. Let $\varepsilon > 0$ be given. Let s_f, s_g, S_f, S_g be step functions such that $s_f(x) \le f(x) \le S_f(x)$, $s_g(x) \le g(x) \le S_g(x)$ for all x in $[a,b]$, (except possibly at partition points, as usual,) and such that $\sum S_f - \sum s_f < \frac{\varepsilon}{2}$ and $\sum S_g - \sum s_g < \frac{\varepsilon}{2}$. Then

$$s_f(x) + s_g(x) \le (f+g)(x) \le S_f(x) + S_g(x),$$

so $s_f + s_g$ is a lower step function for $f+g$ and $S_f + S_g$ is an upper step function for $f+g$. Clearly,

$$\begin{aligned}\sum(S_f+S_g) - \sum(s_f+s_g) &= \sum S_f - \sum s_f + \sum S_g - \sum s_g \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and

$$\begin{aligned}\sum(s_f+s_g) = \sum s_f + \sum s_g &\le \int_a^b f + \int_a^b g \\ &\le \sum S_f + \sum S_g = \sum(S_f+S_g).\end{aligned}$$

So $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. ☺

Exercise 7.4.2. Complete the proof when $k \le 0$.

Remark 7.4.3. Many proof using step functions follow the pattern used in the proof of linearity: Assume f and g are integrable and you want to show that some "transform" of f and/or g is integrable. Pick lower/upper step functions s_f/S_f and s_g/S_g for f and g, such that $\sum S_f - \sum s_f$ and $\sum S_g - \sum s_g$ are "small." Apply the transform to the step functions. Check that the transformed step functions give lower/upper step functions for the transformed functions and that the difference of the sums of the transformed step functions is small.

Given a function f the function $f^+(x) := \max\{f(x), 0\}$ is the *positive part* of f and $f^-(x) := \max\{-f(x), 0\}$ is the *negative part* of f. Note $f = f^+ - f^-$ and $f^+f^- = 0$. (Fig. 7.7)

Before dismissing the following theorem as trivial the reader may want to recall that, if σ is the pseudosine function, then $x\sigma(1/x)$ is continuous on $[0,1]$ and changes sign infinitely many times on the interval $[0,1]$.

Proposition 7.4.4. *If f is integrable, then so are the positive and negative parts $f^{\pm}$ of f.*

Proof. Let $\varepsilon > 0$ be given. Let $s = \sum_{k=1}^n a_k \mathbb{1}_{]x_{k-1},x_k[}$ be a lower step function for f and let $S = \sum_{k=1}^n A_k \mathbb{1}_{]x_{k-1},x_k[}$ be an upper step function for f, such that $\sum S - \sum s < \varepsilon$. Let

$$a_k^+ := \begin{cases} a_k & \text{if } a_k \ge 0 \\ 0 & \text{if } a_k < 0 \end{cases}$$

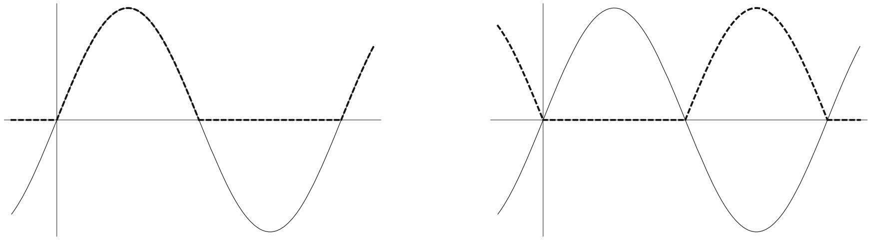

Fig. 7.7 Illustrating the positive and negative parts $f^{\pm}$ of a function f. In this example, $f = \sigma$ is the pseudosine function. In both graphs f is the *thin curve*. In the graph on the left f^+ is the *thicker dashed curve* and in the graph on the right f^- is the *thicker dashes curve*

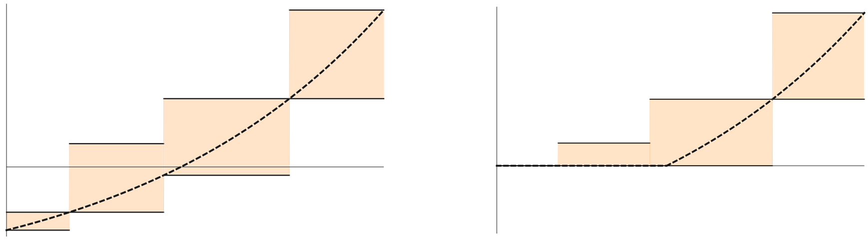

Fig. 7.8 The figure on the *left* shows f, s, and S. The figure on *right* shows f^+, s^+, and S^+. Illustrating s^+ is a lower step function for f^+, S^+ is an upper step function for f^+, and $A_k^+ - a_k^+ \le A_k - a_k$

and similarly for A_k^+. Then $s^+ = \sum_{k=1}^n a_k^+ \mathbb{1}_{]x_{k-1},x_k[}$ and $S^+ = \sum_{k=1}^n A_k^+ \mathbb{1}_{]x_{k-1},x_k[}$. It follows, see Fig. 7.8, that s^+ is a lower step function for f, S^+ is an upper step function for f, and $A_k^+ - a_k^+ \le A_k - a_k$. The inequality yields

$$\begin{aligned}\sum S^+ - \sum s^+ &= \sum_{k=1}^n \left(A_k^+ - a_k^+\right)(x_k - x_{k-1}) \\ &\le \sum_{k=1}^n (A_k - a_k)(x_k - x_{k-1}) \\ &= \sum S - \sum s < \varepsilon.\end{aligned}$$

Thus f^+ is integrable. The claim for f^- can be established in a similar fashion or by showing that $f^- = (-f)^+$ and then using the result for f^+. ☺

Exercise 7.4.5. Complete the proof by showing that f^- is integrable.

The next two results show that, if f is integrable so is the absolute value $|f|(x) := |f(x)|$ and establishes a triangle inequality for integrals.

Exercise 7.4.6. If f is integrable, then so is $|f|$.

Exercise 7.4.7. If f is integrable, then $\left|\int_a^b f\right| \le \int_a^b |f|$.

The next result shows that the restriction of an integrable function on $[a,b]$ to some subinterval $[c,d] \subset [a,b]$ is integrable on $[c,d]$. The result will be useful when we discuss the Fundamental Theorem of Calculus.

Theorem 7.4.8. *If f in integrable on $[a,b]$ and $[c,d] \subseteq [a,b]$, then f is integrable on $[c,d]$.*

Proof. Let g be the restriction of f to $[c,d]$. We must show that g is integrable on $[c,d]$.

Let $\varepsilon > 0$ be given. Let s,S be upper and lower step functions for f, such that $\sum S - \sum s < \varepsilon$. We may assume that s and S have the same partition points $x_0 < x_1 < \cdots < x_n$, and that c,d are among these partition points. Suppose $c = x_m$ and $d = x_{m+k}$. If t is the restriction of s to $[c,d]$ and T is the restriction of S to $[c,d]$, then

$$\begin{aligned}\sum T - \sum t &= \sum_{i=m+1}^{m+k} (M_i - m_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum S - \sum s \leq \varepsilon.\end{aligned}$$

The inequality uses all the terms in both sums are positive and that all the terms in the first sum are also terms in the second sum. See Fig. 7.9. ☺

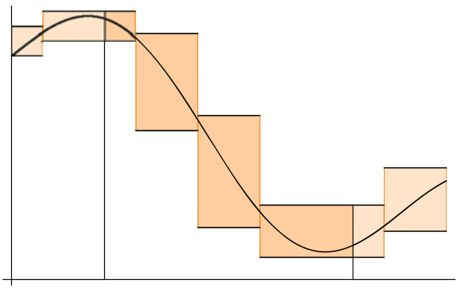

Fig. 7.9 Illustrating the proof of Theorem 7.4.8. In particular, the inequality $\sum T - \sum t \leq \sum S - \sum s$

The next two related results (Exercises 7.4.9 and 7.4.10) can be established using similar arguments.

Exercise 7.4.9. If $a < c < b$, then $\int_a^b f = \int_a^c f + \int_c^b f$.

We define $\int_a^a f := 0$ and if $a < b$ we define $\int_b^a f := -\int_a^b f$. With this notation Exercise 7.4.9 implies $\int_a^b f = \int_a^c f + \int_c^b f$ for all a,b,c provided the three integrals exists.

The previous two results involved splitting up the interval, the next exercise puts two intervals together.

Exercise 7.4.10. Suppose $a < b < c$. Let $g : [a,b] \to \mathbb{R}$ and $h : [b,c] \to \mathbb{R}$ be integrable. Show that

$$f(x) := \begin{cases} g(x) & \text{when } a \le x \le b \\ h(x) & \text{when } b < x \le c \end{cases}$$

determines an integrable function f, and

$$\int_a^c f = \int_a^b g + \int_b^c h.$$

Clearly, this can be extended to a finite number of terms by induction. Our next goal is to show that the product of two integrable functions is an integrable function.

Exercise 7.4.11. If f is integrable on $[a,b]$ and $f \ge 0$, then f^2 is integrable on $[a,b]$.

Exercise 7.4.12. If f is integrable on $[a,b]$, then f^2 is integrable on $[a,b]$.

Theorem 7.4.13. *If f and g are integrable on $[a,b]$, then the product fg is integrable on $[a,b]$.*

Proof. $f+g$ and $f-g$ are integrable, since sums and constant multiples of integrable functions are integrable. So $(f+g)^2$ and $(f-g)^2$ are integrable by the previous exercise. Since

$$fg = \tfrac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

we conclude fg is integrable. ☺

The composition of two integrable functions need not be integrable, see the problems for an example illustrating this.

7.5 The Fundamental Theorem of Calculus

In this section, we relate the integral to the derivative, that is, we prove the two versions of the Fundamental Theorem of Calculus. Isaac Newton (4 January 1643 Woolsthorpe-by-Colsterworth to 31 March 1727 Kensington) established the Fundamental Theorem as we know it. We also derive some consequences of the Fundamental Theorem.

Theorem 7.5.1. *[The Fundamental Theorem of Calculus, Part I, FTC-Derivative] Suppose $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and continuous at x_0. Let $g(x) := \int_a^x f$. Then g is differentiable at x_0 and $g'(x_0) = f(x_0)$.*

Proof. Fix $x_0 \in [a,b]$. Let $\varepsilon > 0$ be given. Since f is continuous at x_0 there is a $\delta > 0$ such that $|t - x_0| < \delta$ implies $|f(t) - f(x_0)| < \varepsilon$. Clearly,

$$\frac{g(x) - g(x_0)}{x - x_0} = \frac{1}{x - x_0}\left(\int_a^x f - \int_a^{x_0}\right) = \frac{1}{x - x_0}\int_{x_0}^x f.$$

Hence, when $x_0 < x < x_0 + \delta$

$$\begin{aligned}\left|\frac{g(x)-g(x_0)}{x-x_0} - f(x_0)\right| &= \left|\left(\frac{1}{x-x_0}\int_{x_0}^{x} f\right) - f(x_0)\right| \\ &= \left|\frac{1}{x-x_0}\int_{x_0}^{x} (f - f(x_0))\right| \\ &\leq \frac{1}{x-x_0}\int_{x_0}^{x} |f - f(x_0)| \\ &\leq \frac{1}{x-x_0}\int_{x_0}^{x} \varepsilon = \varepsilon.\end{aligned}$$

Thus, $g'(x_0)$ exists and $g'(x_0) = f(x_0)$. ☺

Exercise 7.5.2. Complete the proof by considering $x_0 - \delta < x < x_0$.

Theorem 7.5.3. *[The Fundamental Theorem of Calculus, Part II, FTC-Evaluation] If $f : [a,b] \to \mathbb{R}$ is integrable and $F : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on $]a,b[$, and $F' = f$ on $]a,b[$, then*

$$\int_a^b f = F(b) - F(a).$$

Proof. Let $S = \sum A_i \mathbb{1}_{]x_{i-1},x_i[}$ be an upper step function for f. By the Mean Value Theorem there are $x_{i-1} < c_i < x_i$ such that

$$\begin{aligned}F(b) - F(a) &= \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right) \\ &= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} A_i (x_i - x_{i-1}) \\ &= \sum S.\end{aligned}$$

The inequality used that $f(x) \leq A_i$ for all $x_{i-1} < c_i < x_i$. Similarly, $\sum s \leq F(b) - F(a)$ for any lower step function for f. An application of the Evaluation Theorem for Integrals completes the proof. ☺

Exercise 7.5.4. Verify, the claim, that $\sum s \leq F(b) - F(a)$ for any lower step function for f.

Example 7.5.5. If F is not differentiable at all points in the interval, then the conclusion of part II of the fundamental theorem may fail. To see this let $F(x) := \begin{cases} 0 & \text{when } -1 \leq x < 0 \\ 1 & \text{when } 0 < x \leq 1 \end{cases}$, then $f(x) := F'(x) = 0$, $x \neq 0$, determines an integrable function, but $\int_{-1}^{1} f = 0 \neq 1 = F(1) - F(-1)$. Hence, F must be differentiable at all points in the interval.

The reader might have noticed that the F is the previous problem is not even continuous, much less differentiable.

Remark 7.5.6. (For those familiar with the Devil's Staircase.) Let F be the Devil's Staircase. Then $F' = 0$ on the complement $[0,1] \setminus C$ of the Cantor set C. Hence, $\int_0^1 f = 0 \neq 1 = F(1) - F(0)$, where f is any bounded function that equals F' on the complement of the Cantor set.

Remark 7.5.7. A weaker version of the second part of the fundamental theorem can be derived from the first part. To apply the first part of the fundamental theorem we must assume f is continuous not just integrable. This proof of the weaker version of part II of the fundamental theorem is outlined below.

Proof. Suppose f is continuous. Let $G(x) = \int_a^x f$. By part I of the Fundamental Theorem of Calculus, $G' = f = F'$. So $(F-G)' = 0$ and therefore $F - G$ is constant. In particular,

$$F(b) - G(b) = F(a) - G(a).$$

Consequently,

$$F(b) - F(a) = G(b) = \int_a^b f$$

since $G(a) = 0$. ☺

The basic computational rules: Integration by Parts and the Change of Variables Formula are consequences of FTC-Evaluation.

Theorem 7.5.8 (Integration by Parts). *Suppose f and g are differentiable and that the derivatives f' and g' are integrable on $[a,b]$, then*

$$\int_a^b f g' = f(b)g(b) - f(a)g(a) - \int_a^b f' g.$$

Proof. It follows from the product rule that

$$(fg)' = f'g + fg' \text{ hence } fg' = (fg)' - f'g.$$

All the terms in the last equality are integrable. For example, f is differentiable, hence continuous and therefore integrable and g' is integrable by assumption. Since the product of two integrable functions is an integrable function we conclude the product fg' is integrable. Similarly $f'g$ is integrable. Finally, $(fg)'$ is integrable, since it equals the sum of the integrable functions $f'g$ and fg'. So by linearity of the integral

$$\int_a^b fg' = \int_a^b (fg)' - \int_a^b f'g.$$

Since $(fg)'$ is integrable we can apply FTC-Evaluation to conclude $\int_a^b (fg)' = (fg)(b) - (fg)(a)$. ☺

Example 7.5.9. If f has a continuous second derivative f'', that is $f \in \mathscr{C}^2$, then

$$f(b) = f(a) + (b-a)f'(a) + \int_a^b (b-x)f''(x)\,dx.$$

Proof. Using integration by parts and FTC-Evaluation we get

$$\begin{aligned}\int_a^b (b-x)f''(x)\,dx &= (b-b)f'(b) - (b-a)f'(a) - \int_a^b (0-1)f'(x)\,dx \\ &= -(b-a)f'(a) + f(b) - f(a).\end{aligned}$$

Rearranging leads to the desired formula. ☺

By induction this argument leads to Taylor's Formula, with the remainder in integral form. See the problems.

The following is sometimes called Integration by Substitution.

Theorem 7.5.10 (Change of Variables). *Suppose F and g are differentiable. Assume $f := F'$ is integrable on the closed interval with endpoints $g(a)$ and $g(b)$ and that $f \circ g$ and g' are integrable on $[a,b]$, then*

$$\int_a^b (f \circ g)\, g' = \int_{g(a)}^{g(b)} f. \tag{7.4}$$

Proof. The functions f, F satisfies the assumptions of FTC-Evaluation on the interval with endpoints $g(a)$ and $g(b)$, hence $\int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a))$.

The product $(f \circ g)g'$ is integrable, since $f \circ g$ and g' are integrable and the product of integrable functions is integrable. Using

$$(F(g(x)))' = f(g(x))g'(x)$$

and FTC-Evaluation we conclude $\int_a^b (f \circ g)\, g' = F(g(b)) - F(g(a))$.

Since both integrals equals $F(g(b)) - F(g(a))$, they are equal. Consequently, we have established (7.4). ☺

Sometimes it is convenient to write

$$\int_a^b f(x)\,dx := \int_a^b f.$$

With this notation (7.4) is sometimes written as

$$\int_a^b f(g(x))\, g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

where the change of variables is $u = g(x)$.

Example 7.5.11. Consider the integral

$$I = \int_0^1 x\sqrt{1+x^2}\,dx.$$

If $f(u) := \sqrt{u}$ and $g(x) := 1+x^2$, then $f \circ g(x) = f(g(x)) = \sqrt{1+x^2}$ and $g'(x) = 2x$. Hence,

$$I = \frac{1}{2}\int_0^1 f(g(x))g'(x)\,dx.$$

Now $g(0) = 1$ and $g(1) = 2$, so by (7.4) and FTC-Evaluation

$$I = \frac{1}{2}\int_1^2 f(u)\,du = \frac{1}{2}\int_1^2 \sqrt{u}\,du = \frac{1}{3}\left(2^{3/2} - 1^{3/2}\right).$$

Linear Change of Variables

In the change of variables theorem we assumed that $f \circ g$ is integrable. The reason for this is that there are integrable functions f and g such that $f \circ g$ makes sense, yet $f \circ g$ is not integrable. In some cases $f \circ g$ is integrable, for example, if f and g both are continuous, then $f \circ g$ is continuous and therefore integrable. We will have use for the case where f is integrable and g is linear.

Theorem 7.5.12. *If $f : [a,b] \to \mathbb{R}$ is integrable, $\alpha \neq 0$ and β are real numbers, and $g(x) := \alpha x + \beta$, then $f \circ g$ is integrable on the interval with endpoints $g^{-1}(a) = \frac{a-\beta}{\alpha}$ and $g^{-1}(b) = \frac{b-\beta}{\alpha}$ and*

$$\alpha \int_{g^{-1}(a)}^{g^{-1}(b)} f(\alpha x + \beta)\,dx = \int_a^b f(u)\,du.$$

Proof. Suppose $\alpha > 0$. Let $\varepsilon > 0$ be given. Pick a lower step function $s = \sum m_k \mathbb{1}_{]x_{k-1},x_k[}$ for f and an upper step function $S = \sum M_k \mathbb{1}_{]x_{k-1},x_k[}$ for f such that $\sum S - \sum s < \alpha\varepsilon$. Then $t := \sum m_k \mathbb{1}_{]g^{-1}(x_{k-1}),g^{-1}(x_k)[}$ is a lower step function for $f \circ g$ and $T := \sum M_k \mathbb{1}_{]g^{-1}(x_{k-1}),g^{-1}(x_k)[}$ is an upper step function for $f \circ g$. Since, $g^{-1}(x) = \frac{x-\beta}{\alpha}$, we see that

$$\sum t = \sum m_k \left(g^{-1}(x_k) - g^{-1}(x_{k-1})\right) = \frac{1}{\alpha}\sum m_k\left(x_k - x_{k-1}\right) = \frac{1}{\alpha}\sum s$$

and similarly $\sum T = \frac{1}{\alpha}\sum S$. Consequently, $\sum T - \sum t = \frac{1}{\alpha}\left(\sum S - \sum s\right) < \varepsilon$, so $f \circ g$ is integrable, by the Existence Theorem for Integrals. Also,

$$\sum s = \alpha \sum t \leq \alpha \int_{g^{-1}(a)}^{g^{-1}(b)} f \circ g \leq \alpha \sum T = \sum S$$

hence $\int_a^b f = \alpha \int_{g^{-1}(a)}^{g^{-1}(b)} f \circ g$, by the Evaluation Theorem for Integrals.

The case where $\alpha < 0$ is similar. ☺

7.6 Improper Integrals

Unbounded Intervals

We defined the integral on closed and bounded intervals. We extend this to unbounded intervals in the following way.

We say the integral $\int_a^\infty f$ exists, if f is integrable on $[a,b]$ for all $b > a$ and the limit

$$\int_a^\infty f := \lim_{b\to\infty} \int_a^b f$$

exists. Similarly, we can consider $\int_{-\infty}^b f$. We let

$$\int_{-\infty}^\infty f := \int_{-\infty}^0 f + \int_0^\infty f$$

provided both limits exists.

Exercise 7.6.1. Let $f(x) := 1/x^p$ for some $p > 1$. Show $\int_1^\infty f$ exists.

Unbounded Functions

We will write

$$\int_a^b f := \lim_{c\nearrow b} \int_a^c f$$

if f is integrable on $[a,c]$ for all $c \in]a,b[$ and the limit exists. Similarly,

$$\int_a^b f := \lim_{c\searrow a} \int_c^b f,$$

if f is integrable on $[c,b]$ for all $c \in]a,b[$ and the limit exists. This sometimes allows us to consider the integral in cases, where f may not be bounded on the interval of interest.

Exercise 7.6.2. Let $f(x) := 1/x^p$ for some $0 < p < 1$. Show $\int_0^1 f$ exists.

7.7 Complex Valued Functions

Complex values functions are integrated by integrating their real and imaginary parts. Let $f : [a,b] \to \mathbb{C}$. We say f is *integrable* on $[a,b]$, if the real and imaginary parts, $\operatorname{Re} f$ and $\operatorname{Im} f$, of f both are integrable on $[a,b]$. If f is integrable on

$[a,b]$ we set

$$\int_a^b f = \int_a^b (\operatorname{Re} f + i \operatorname{Im} f) := \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.$$

Since the integral of a complex valued function is determined by its real and imaginary parts, we get many properties of the integral of complex valued functions by considering the real and imaginary parts of such functions. For example,

Exercise 7.7.1. If $f,g : [a,b] \to \mathbb{C}$ are integrable, then

1. $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
2. fg is integrable.

The main result in this section is:

Theorem 7.7.2. *If $f : [a,b] \to \mathbb{C}$ is integrable, then $|f|$ is integrable on $[a,b]$.*

Proof. Since products and sums of real valued integrable functions are integrable and $|f| = \sqrt{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2}$, it is sufficient to show, if $f \geq 0$ is integrable, then $\sqrt{f}$ is integrable. This is established in Proposition 7.7.4. ☺

Exercise 7.7.3. Show $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|}$ for all $x,y \geq 0$. [That is $x \to \sqrt{x}$ is $\alpha-$Hölder on $[0,\infty[$ with $\alpha = 1/2$.]

Proposition 7.7.4. *If $f \geq 0$ is integrable on $[a,b]$, then $\sqrt{f}$ is integrable on $[a,b]$.*

Proof. Suppose $f \geq 0$ is integrable on $[a,b]$. Let $\varepsilon > 0$ be given. Let

$$\delta := \min\left\{\varepsilon, \left(\frac{\varepsilon}{2(b-a)}\right)^2\right\}.$$

In particular, $0 < \delta \leq \varepsilon$.

Since f is integrable, f is bounded. Let M be an upper bound for f. Since f is integrable, there is a lower step function $\widetilde{s} = \sum_{i=1}^n \widetilde{a}_i \mathbb{1}_{]x_{i-1},x_i[}$ for f and an upper step function $\widetilde{S} = \sum_{i=1}^n \widetilde{A}_i \mathbb{1}_{]x_{i-1},x_i[}$ for f such that $\sum \widetilde{S} - \sum \widetilde{s} < \frac{\delta^2}{2\sqrt{M}}$. Let $a_i := \max\{0, \widetilde{a}_i\}$ and $A_i := \min\left\{M, \widetilde{A}_i\right\}$, then $0 \leq a_i \leq A_i \leq M$, $s = \sum_{i=1}^n a_i \mathbb{1}_{]x_{i-1},x_i[}$ is a lower step function for f and $S = \sum_{i=1}^n A_i \mathbb{1}_{]x_{i-1},x_i[}$ in an upper step function for f. Since $0 \leq A_i - a_i \leq \widetilde{A}_i - \widetilde{a}_i$ we have

$$\begin{aligned}\sum S - \sum s &= \sum_{i=1}^n (A_i - a_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n \left(\widetilde{A}_i - \widetilde{a}_i\right)(x_i - x_{i-1}) \\ &= \sum \widetilde{S} - \sum \widetilde{s} < \frac{\delta^2}{2\sqrt{M}}.\end{aligned} \tag{7.5}$$

Since $\sqrt{s} = \sum_{i=1}^n \sqrt{a_i} \mathbb{1}_{]x_{i-1},x_i[}$ is a lower step function for $\sqrt{f}$ and $\sqrt{S} = \sum_{i=1}^n \sqrt{A_i} \mathbb{1}_{]x_{i-1},x_i[}$ is an upper step function for $\sqrt{f}$, it remains to check that $\sum \sqrt{S} - \sum \sqrt{s} < \varepsilon$.

Clearly,

$$\sum\sqrt{S}-\sum\sqrt{s}=\sum_{i=1}^{n}\left(\sqrt{A_i}-\sqrt{a_i}\right)(x_i-x_{i-1})\,.$$

We split the set $\{1,2,\dots,n\}$ into two disjoint subsets and estimate the sum over each of those subsets and show that each of those sums is $<\varepsilon/2$. Let

$$\begin{aligned}I_1 &:= \{i \mid A_i-a_i<\delta\} \text{ and}\\ I_2 &:= \{i \mid A_i-a_i\geq\delta\}\,.\end{aligned}$$

We complete the proof by showing that

$$\sum_{i\in I_k}\left(\sqrt{A_i}-\sqrt{a_i}\right)(x_i-x_{i-1})<\frac{\varepsilon}{2}$$

for $k=1,2$.

$\underline{k=1:}$ Let $i\in I_1$. Then $A_i-a_i<\delta$, hence Exercise 7.7.3 implies

$$\left|\sqrt{A_i}-\sqrt{a_i}\right|<\sqrt{\delta}\leq\frac{\varepsilon}{2(b-a)}. \tag{7.6}$$

Using (7.6) we get

$$\sum_{i\in I_1}\left(\sqrt{A_i}-\sqrt{a_i}\right)(x_i-x_{i-1})<\sum_{i\in I_1}\frac{\varepsilon}{2(b-a)}(x_i-x_{i-1})\leq\frac{\varepsilon}{2}.$$

$\underline{k=2:}$ For $i\in I_2$ we have

$$\begin{aligned}\delta\sum_{i\in I_2}(x_i-x_{i-1}) &\leq \sum_{i\in I_2}(A_i-a_i)(x_i-x_{i-1})\\ &\leq\sum_{\text{all } i}(A_i-a_i)(x_i-x_{i-1})\\ &=\sum S-\sum s\\ &<\frac{\delta^2}{2\sqrt{M}}.\end{aligned}$$

The first inequality is the definition of I_2, the last inequality is (7.5). Dividing by δ leads to

$$\sum_{i\in I_2}(x_i-x_{i-1})<\frac{\delta}{2\sqrt{M}}.$$

Consequently, $0\leq\sqrt{A_i}-\sqrt{a_i}\leq\sqrt{A_i}\leq\sqrt{M}$ implies

$$\begin{aligned}\sum_{i\in I_2}\left(\sqrt{A_i}-\sqrt{a_i}\right)(x_i-x_{i-1}) &\leq\sum_{i\in I_2}\sqrt{M}(x_i-x_{i-1})\\ &<\sqrt{M}\frac{\delta}{2\sqrt{M}}=\frac{\delta}{2}\leq\frac{\varepsilon}{2}.\end{aligned}$$

The last inequality is $\delta\leq\varepsilon$. ☺

We can now establish a triangle inequality for the integral of complex valued functions.

Theorem 7.7.5. *If $h : [a,b] \to \mathbb{C}$ is integrable, then* $\left|\int_a^b h\right| \leq \int_a^b |h|$.

Proof. Let $f := \operatorname{Re} h$ and $g := \operatorname{Im} h$, then f and g are real valued integrable functions and we can write the desired conclusion as

$$\sqrt{\left(\int_a^b f\right)^2 + \left(\int_a^b g\right)^2} \leq \int_a^b \sqrt{f^2 + g^2}. \tag{7.7}$$

Since, $\left|\int_a^b f\right| \leq \int_a^b |f|$ and $\left|\int_a^b g\right| \leq \int_a^b |g|$ (by Exercise 7.4.7) and $f^2 + g^2 = |f|^2 + |g|^2$, the inequality (7.7) follows from

$$\sqrt{\left(\int_a^b |f|\right)^2 + \left(\int_a^b |g|\right)^2} \leq \int_a^b \sqrt{|f|^2 + |g|^2}.$$

But, this is just (7.7) for $|f|$ and $|g|$. Thus, we will establish (7.7) for $f \geq 0$ and $g \geq 0$.

Suppose $f \geq 0$ and $g \geq 0$ are integrable. Let $0 \leq s_n = \sum_{i=1}^n a_i \mathbb{1}_{]x_{i-1},x_i[}$ be a sequence of lower step functions for f such that $\sum s_n \to \int_a^b f$, and similarly, let $0 \leq t_n = \sum_{i=1}^n b_i \mathbb{1}_{]x_{i-1},x_i[}$ be a sequence of lower step functions for g such that $\sum t_n \to \int_a^b g$. In particular,

$$\sqrt{\left(\sum s_n\right)^2 + \left(\sum t_n\right)^2} \to \sqrt{\left(\int_a^b f\right)^2 + \left(\int_a^b g\right)^2}. \tag{7.8}$$

Using $\left(\mathbb{1}_{]x_{i-1},x_i[}\right)^2 = \mathbb{1}_{]x_{i-1},x_i[}$, we see that for $x_{k-1} < x < x_k$ we have

$$\begin{aligned} u_n(x) := \sqrt{(s_n)^2 + (t_n)^2}(x) &= \sqrt{\left(\sum_{i=1}^n a_i \mathbb{1}_{]x_{i-1},x_i[}(x)\right)^2 + \left(\sum_{i=1}^n b_i \mathbb{1}_{]x_{i-1},x_i[}(x)\right)^2} \\ &= \sqrt{(a_k)^2 + (b_k)^2} = \sum_{i=1}^n \sqrt{(a_i^2) + (b_i)^2}\mathbb{1}_{]x_{i-1},x_i[}(x). \end{aligned}$$

In particular, u_n is a lower step function for $\sqrt{|f|^2 + |g|^2}$. Consequently,

$$\sum_{i=1}^n \sqrt{(a_i^2) + (b_i)^2}\,(x_i - x_{i-1}) \sum u_n \leq \int_a^b \sqrt{f^2 + g^2}. \tag{7.9}$$

If

$$\sqrt{\left(\sum s_n\right)^2 + \left(\sum t_n\right)^2} \leq \sum u_n, \tag{7.10}$$

then (7.8) and (7.9) gives (7.7) and the proof is complete. Write (7.10) as

$$\begin{aligned}
&\sqrt{\left(\sum_{i=1}^{n} a_i\left(x_i - x_{i-1}\right)\right)^2 + \left(\sum_{i=1}^{n} b_i\left(x_i - x_{i-1}\right)\right)^2} \\
&= \sqrt{\left(\sum s_n\right)^2 + \left(\sum t_n\right)^2} \\
&\leq \sum u_n = \sum_{i=1}^{n} \sqrt{(a_i)^2 + (b_i)^2}\,(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sqrt{\left(a_i\left(x_i - x_{i-1}\right)\right)^2 + \left(b_i\left(x_i - x_{i-1}\right)\right)^2}.
\end{aligned}$$

Setting $\alpha_i := a_i\left(x_i - x_{i-1}\right)$ and $\beta_i := b_i\left(x_i - x_{i-1}\right)$, reduces the previous inequality to

$$\sqrt{\left(\sum_{i=1}^{n} \alpha_i\right)^2 + \left(\sum_{i=1}^{n} \beta_i\right)^2} \leq \sum_{i=1}^{n} \sqrt{(\alpha_i)^2 + (\beta_i)^2} \tag{7.11}$$

for $\alpha_i \geq 0$ and $\beta_i \geq 0$. ☺

Exercise 7.7.6. Verify (7.11) for all real α_i and β_i.

Problems

Problems for Sect. 7.1

1. If

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases},$$

then f is integrable on $[-1,1]$ and $\int_{-1}^{1} f = 0$. [*Hint*: Any lower sum is ≤ 0 and some lower sum is $= 0$. Hence, $\underline{\int_{-1}^{1}} f = 0$.]

2. Let C be the Cantor set and let $f : [0,1] \to \mathbb{R}$ be determined by

$$f(x) := \begin{cases} 1 & \text{when } x \in C \\ 0 & \text{when } x \notin C \end{cases}.$$

Find an upper step function S for f such that $\sum S < \frac{1}{2}$.

Problems for Sect. 7.2

1. Let $f(x)=2x$. Use Corollary 7.2.6 to show f is integrable on $[0,2]$ and $\int_0^2 f=4$.
2. Let

$$f(x)=\begin{cases}0 & \text{if } x\notin\{1/n\mid n\in\mathbb{N}\}\\ 1 & \text{if } x\in\{1/n\mid n\in\mathbb{N}\}.\end{cases}$$

   Prove that f is integrable on $[0,1]$ and $\int_0^1 f=0$.

3. Prove a characterization of integrability in the spirit of this section for functions defined on a rectangle.

   The following method for evaluating the integral of x^k is due to Pierre de Fermat (17 August 1601 Beaumont-de-Lomagne to 12 January 1665 Castres).

4. Fix $k\in\mathbb{N}$. Let $f(x):=x^k$ and $a>1$. Let $r:=a^{1/n}$. Consider the partition $1<r<r^2<\cdots<r^{n-1}<a$ of $[1,a]$.

   a. Write the corresponding upper and lower sums for $\int_1^a f$.
   b. Find the limit of these sums as $n\to\infty$.
   c. Evaluate $\int_1^a f$.

5. In this problem, we assume familiarity with trigonometric functions. Evaluate $\int_0^{\pi/2}\sin$. You may want to prove

$$\sum_{k=0}^{n-1}\sin(a+kb)=\frac{\sin(a+(n-1)b/2)\sin)nb/2)}{\sin(b/2)}$$

   and use partitions determined by $x_i-x_{i-1}=\frac{\pi}{2n}$. [*Hint*: One way to prove the summation formula is to use $e^{it}=\cos(t)+i\sin(t)$.]

Problems for Sect. 7.3

1. Suppose $a\le x_n\le b$ and (x_n-c) is null, for some $c\in[a,b]$. Show the characteristic function $\mathbb{1}_A$ of $A:=\{x_n\mid n\in\mathbb{N}\}$ is integrable on $[a,b]$ and and $\int_a^b \mathbb{1}_A=0$.

2. Give an example of a sequence x_n such that $0\le x_n\le 1$ and the characteristic function $\mathbb{1}_A$ of $A:=\{x_n\mid n\in\mathbb{N}\}$ is not integrable on $[0,1]$.

3. Let C be the Cantor set. Show the characteristic function $\mathbb{1}_C$ of the Cantor set is integrable on $[0,1]$ and $\int_0^1 \mathbb{1}_C=0$. [That is, the "length" of the Cantor set is 0.]

4. Let $f:[0,1]\to\mathbb{R}$ be determined by

$$f(x):=\begin{cases}\frac{(-1)^p}{q} & \text{when } x=p/q\\ 0 & \text{when } x\notin\mathbb{Q}\end{cases}.$$

   Show that f is Riemann integrable on $[0,1]$ and $\int_0^1 f=0$.

5. Show that a continuous function defined on a rectangle is integrable on that rectangle.

6. Consider the points $a_1=\frac{1}{2}$, $a_2=\frac{1}{4}$, $a_3=\frac{3}{4}$, $a_4=\frac{1}{8}$, $a_5=\frac{3}{8},\ldots$. Let $f(x)=\sum_{k,a_k<x}\frac{1}{2^k}$ as in Example 5.1.5. Calculate $\int_0^1 f$.

7. Let f and g be continuous real valued functions defined on the compact interval $[a,b]$. Suppose $g\geq 0$, show there is a point c in $[a,b]$ such that

$$\int_a^b fg=f(c)\int_a^b g.$$

Problems for Sect. 7.4

The purpose of the first three problems below is to show that f,g integrable $\not\Rightarrow g\circ f$ integrable.

1. Let $f:[0,1]\to\mathbb{R}$ be determined by

$$f(x):=\begin{cases}\frac{1}{q} & \text{when } x=p/q\\ 0 & \text{when } x\notin\mathbb{Q}\end{cases}.$$

   Show that f is Riemann integrable on $[0,1]$ and $\int_0^1 f=0$.

2. Let $g:[0,1]\to\mathbb{R}$ be determined by

$$g(x):=\begin{cases}1 & \text{when } x>0\\ 0 & \text{when } x=0\end{cases}.$$

   Show that g is Riemann integrable on $[0,1]$ and $\int_0^1 g=0$.

3. Let f and g be as in the preceding two exercises. Note both f and g are functions $[0,1]\to[0,1]$. Let $h:=g\circ f$, show that h is not integrable on $[0,1]$.

4. If f is continuous on $[c,d]$, g is integrable on $[a,b]$, and $g([a,b])\subseteq[c,d]$, then $f\circ g$ is integrable on $[a,b]$. [*Hint*: f is uniformly continuous.]

   It can be show that if f is integrable and g is continuous, then $f\circ g$ need not be integrable. However,

5. If f is integrable, g is $\mathscr{C}^1$ and $g'(x) \neq 0$ for all x, then $f \circ g$ is integrable. [*Hint*: g is monotone.]

Problems for Sect. 7.5

The converse of FTC-Derivative is false, in the sense that differentiability of $g(x) = \int_a^x f$ does not imply continuity of f.

1. Give an example of an integrable function f and a point x_0 such that $g(x) = \int_a^x f$ is differentiable at x_0 and f is not continuous at x_0.

   What can we say about g if f is only integrable? One answer is:

2. Suppose f is integrable, prove that $g(x) = \int_a^x f$ is continuous.
3. [Taylor's formula with integral remainder] If f is $\mathscr{C}^{n+1}$, then

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b-a)^j + \frac{1}{n!}\int_a^b (b-x)^n f^{(n+1)}(x)\,dx.$$

Problems for Sect. 7.6

1. Suppose $0 \leq g(x) \leq f(x)$ for all x and the improper integral $\int_a^\infty f$ exists. Prove the improper integral $\int_a^\infty g$ exists.
2. Suppose $0 < a < 1$.
   a. Show that
   $$0 \leq \int_0^a \frac{dx}{\sqrt{1-x^2}} \leq \int_0^a \frac{dx}{\sqrt{1-x}} \leq 2.$$
   b. Show that $I(a) := \int_0^a \frac{dx}{\sqrt{1-x^2}}$ is increasing and bounded by 2.
   c. Deduce that $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ exists as an improper integral.

3. If f is integrable on $[a,b]$, then

$$\int_a^b f = \lim_{c \nearrow b} \int_a^c f.$$

The point of the last problem is that limit used to define the improper integral agrees with the usual integral, when f is integrable.

Solutions and Hints for the Exercises

Exercise 7.1.3. Similar to the proof of Lemma 7.1.1.

Exercise 7.1.7. Any lower step function for f is a lower step function for g, consequently $\underline{\int_a^b} f \leq \underline{\int_a^b} g$.

Exercise 7.1.10. If it is not clear how to proceed: examine the proof of Lemma 7.1.11.

Exercise 7.1.12. If $[a,b] \times [c,d]$ is a rectangle, $a = x_0 < \cdots < x_m = b$ is a partition of $[a,b]$, $c < y_0 \cdots < y_n = d$ is a partition of $[c,d]$, and $A_{j,k}$ are real numbers, then $f(x,y) := \sum_{j=1}^m \sum_{k=1}^n A_{j,k} \mathbb{1}_{]x_{j-1}x_j[\times]y_{k-1}y_k[}(x,y)$ is a step function.

Exercise 7.3.4. If $f \neq 0$, then $f(c) > 0$ for some c. By Local Positivity for Limits f is $\geq f(c)/2$ on some open interval I containing c. Let L be the length of I. Then any upper sum for f is $\geq Lf(c)/2$. Hence, the upper integral of f is $\geq Lf(c)/2$. Thus, $\int_a^b f \geq Lf(c)/2$.

Exercise 7.3.5. Equivalent to the Exercise 7.3.4.

Exercise 7.4.2. If $k = 0$, kf is the zero function which has integral zero. The case where $k < 0$ is similar to the $0 < k$ case, the only changes are kS is a lower step function and ks is an upper step function for kf.

Exercise 7.4.5. One way is to mimic the proof that f^+ is integrable. Another is to use that $f^- = f^+ - f$.

Exercise 7.4.6. Prove and use $|f| = f^+ + f^-$.

Exercise 7.4.7. Integrate the inequalities $-|f| \leq f \leq |f|$.

Exercise 7.4.9. By the theorem both $\int_a^c f$ and $\int_c^b f$ exists. Let $\varepsilon > 0$ be given. Since f is integrable on $[a,c]$ there is a lower step function s_1 for f on $[a,c]$ and an upper step function S_1 for f on $[a,c]$, such that $\sum S_1 - \sum s_1 < \varepsilon/2$. Similarly, there is a lower step function s_2 for f on $[c,b]$ and an upper step function S_2 for f on $[c,b]$, such that $\sum S_2 - \sum s_2 < \varepsilon/2$. If $s_1 = \sum_{k=1}^m m_k \mathbb{1}_{]x_{k-1},x_k[}$, $S_1 = \sum_{k=1}^m M_k \mathbb{1}_{]x_{k-1},x_k[}$, $s_2 = \sum_{k=m+1}^n m_k \mathbb{1}_{]x_{k-1},x_k[}$, and $S_2 = \sum_{k=m+1}^n M_k \mathbb{1}_{]x_{k-1},x_k[}$, then $s := \sum_{k=1}^n m_k \mathbb{1}_{]x_{k-1},x_k[}$ is a lower step function for f on $[a,b]$ and $S := \sum_{k=1}^n M_k \mathbb{1}_{]x_{k-1},x_k[}$ is an upper step function for f on $[a,b]$.

The rest is similar to the last part of the proof that the sum of two integrable functions is integrable. We include some of the details. $\sum S - \sum s = (\sum S_1 + \sum S_2) - (\sum s_1 + \sum s_2) = (\sum S_1 - \sum s_1) + (\sum S_2 - \sum s_2) < \varepsilon$. It follows that $\int_a^b f$ and $\int_a^c f + \int_c^b f$ both are in the interval $[\sum s, \sum S]$. Since this interval has length $< \varepsilon$ we have $\left| \int_a^b f - \left(\int_a^c f + \int_c^b f \right) \right| < \varepsilon$. But $\varepsilon > 0$ is arbitrary, consequently $\int_a^b f - \left(\int_a^c f + \int_c^b f \right)$.

Exercise 7.4.10. Similar to Exercise 7.4.9.

Exercise 7.4.11. Let M be an upper bound for f. Let s a lower step function for f with sum $\sum s = \sum m_i (x_i - x_{i-1})$ and S be an upper step function for f with sum

$\sum S = \sum M_i(x_i - x_{i-1})$, such that $\sum S - \sum s < \frac{\varepsilon}{2M}$. We may assume $0 \le m_i$ and $M_i \le M$ for all i. Then s^2 is a lower step function for f^2, S^2 is an upper step function for f^2, and $\sum S^2 - \sum s^2 < \varepsilon$.

Exercise 7.4.12. For some K, $f + K \ge 0$ and $f^2 = (f+K)^2 - 2Kf - K^2$.

Exercise 7.5.2. Similar to the proof in the text, except we need to pay attention to signs.

Exercise 7.5.4. Similar to the proof in the text.

Exercise 7.6.1. Use FTC-Evaluation to calculate $\int_1^b f$, then take the limit as $b \to \infty$.

Exercise 7.6.2. Similar to Exercise 7.6.1.

Exercise 7.7.1. Consider the real and imaginary parts.

Exercise 7.7.3. We may assume $y < x$. Then squaring $\sqrt{x} - \sqrt{y} \le \sqrt{x-y}$ and simplifying leads to $\sqrt{y} < \sqrt{x}$. Rewriting this as $y < x$ implies $\sqrt{y} < \sqrt{x}$, etc., ending with $\sqrt{x} - \sqrt{y} \le \sqrt{x-y}$ proves the inequality.

Exercise 7.7.6. This is a simple proof by induction. When $n = 1$, (7.11) is an equality. Suppose

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2^2)} \le \sqrt{(x_1)^2 + (y_1)^2} + \sqrt{(x_2)^2 + (y_2)^2},$$

that is (7.11) holds with $n = 2$. Then

$$\begin{aligned}
&\sqrt{\left(\left(\sum_{i=1}^{n} \alpha_i\right) + \alpha_{n+1}\right)^2 + \left(\left(\sum_{i=1}^{n} \beta_i\right) + \beta_{n+1}\right)^2} \\
&= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2^2)} \\
&\le \sqrt{(x_1)^2 + (y_1)^2} + \sqrt{(x_2)^2 + (y_2)^2} \\
&= \sqrt{\left(\sum_{i=1}^{n} \alpha_i\right)^2 + \left(\sum_{i=1}^{n} \beta_i\right)^2} + \sqrt{(\alpha_{n+1})^2 + (\beta_{n+1})^2}.
\end{aligned}$$

Hence, (7.11) follows by induction, if it holds for $n = 2$.

Chapter 8
The Logarithm and the Exponential Function

The natural logarithmic and exponential functions are constructed in this chapter. In addition to establishing the standard properties of these functions we show the number e is transcendental, construct a smooth compactly supported function (a "bump" function), and define the Euler constant γ.

8.1 Logarithms

For a real number $x > 0$, let

$$\log(x) := \int_1^x \frac{1}{t}\,dt.$$

Note $\log(1) = 0$, $\log(x) < 0$ when $0 < x < 1$, and $\log(x) > 0$ when $1 < x$.

The FTC-Derivative shows that log is differentiable and

$$\log'(x) = \frac{1}{x}.$$

Since we can differentiate $1/x$ as many times as we please $\log \in \mathscr{C}^n(]0,\infty[)$ for any n. Consequently, $\log \in \mathscr{C}^\infty(]0,\infty[)$. The basic computational property of a logarithm is:

Theorem 8.1.1. *For all $x, y > 0$ we have*

$$\log(xy) = \log + \log(y).$$

Proof. This is a simple calculation. If $x, y > 0$, then

$$\log(xy) = \int_1^{xy} \frac{1}{t}\,dt$$

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$$\begin{aligned}
&= \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt \\
&= \int_1^x \frac{1}{t}\,dt + \int_1^{y} \frac{1}{t}\,dt \\
&= \log(x) + \log(y)
\end{aligned}$$

establishing the claim. ☺

Exercise 8.1.2. Explain the steps in the calculation above.

Exercise 8.1.3. $\log(x^n) = n\log(x)$ for $n \in \mathbb{N}$ and $x > 0$.

Lemma 8.1.4. *For all $x > 0$ and all $n \in \mathbb{Z}$ we have $\log(x^n) = n\log(x)$. Recall the convention $x^0 = 1$.*

Proof. Using the previous exercise: $\log(x^{-n}) + \log(x^n) = \log(x^{-n}x^n) = \log(1) = 0$. Hence, $\log(x^{-n}) = -n\log(x)$. ☺

Theorem 8.1.5. *The function $\log :]0,\infty[\to \mathbb{R}$ is strictly increasing and onto.*

Proof. If $x < y$, then $\int_x^y \frac{1}{t}\,dt > 0$, because $t \to 1/t$ is continuous (Exercise 7.3.5). Consequently, $\log(y) = \int_1^y \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^y \frac{1}{t}\,dt > \log(x)$. So log is strictly increasing. (This also follows from $\log'(x) = \frac{1}{x} > 0$.)

Since

$$\frac{1}{t} \geq g(t) := \begin{cases} 1/2 & \text{when } 1 \leq t < 2 \\ 1/3 & \text{when } 2 \leq t < 3 \\ 1/4 & \text{when } 3 \leq t \leq 4 \end{cases},$$

it follows from monotonicity of the integral (i.e. $f \geq g \implies \int_a^b f \geq \int_a^b g$) that

$$\log(4) = \int_1^4 \frac{1}{t}\,dt \geq \int_1^4 g = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1.$$

Let y be a real number, then $-n < y < n$ for some positive integer n, hence

$$\log\left(4^{-n}\right) = -n\log(4) < -n < y.$$

Similarly, $y < \log(4^n)$. Consequently, $\log(4^{-n}) < y < \log(4^n)$. By the Intermediate Value Theorem $y = \log(x)$ for some x between 4^{-n} and 4^n. ☺

Remark 8.1.6. It follows from the proof that $\log(x) \to \infty$ as $x \to \infty$. For example, $x > 4^n$ implies $\log(x) > n$. Similarly, $\log(x) \to -\infty$ as $x \searrow 0$.

The Euler constant is the number

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1)\right) = \lim_{n\to\infty} \int_1^{n+1} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx.$$

It is not known if γ is rational or irrational. Godfrey Harold Hardy (7 February 1877, Cranleigh to 1 December 1947, Cambridge) offered to give up his Savilian Chair at Oxford to anyone who proved γ to be irrational. Hilbert mentioned the irrationality of gamma as an unsolved problem.

8.2 Exponentials

Since $\log :]0,\infty[\to \mathbb{R}$ is strictly increasing and surjective it has a strictly increasing inverse function exp mapping $\mathbb{R}$ onto the interval $]0,\infty[$. In symbols:

$$\exp(x) := \log^{-1}(x).$$

In particular, $\exp(\log(x)) = x$ for all $x > 0$ and $\log(\exp(y)) = y$ for all y in $\mathbb{R}$.

Remark 8.2.1. Using that exp is strictly increasing we see that $\exp(x) \to \infty$ as $x \to \infty$. In fact, $x > \log(N)$ implies $\exp(x) > \exp(\log(N)) = N$. Similarly, $\exp(x) \to 0$ as $x \to -\infty$.

Since log is differentiable, we infer from the Inverse Function Rule for Derivatives that exp is differentiable and

$$\exp'(x) = \frac{1}{\log'(\exp(x))} = \frac{1}{1/\exp(x)} = \exp(x).$$

Since $\exp' = \exp$, the right hand side is differentiable, so we can differentiate exp as many times as we please: $\exp \in \mathscr{C}^{\infty}(\mathbb{R})$.

The basic computational property of an exponential function is:

Exercise 8.2.2. $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$.

The exponential function as the solution to a differential equation:

Theorem 8.2.3. *Let $f : \mathbb{R} \to \mathbb{C}$ be differentiable. Fix $a, b \in \mathbb{C}$. If $f' = bf$ and $f(0) = a$, then $f(x) = a\exp(bx)$ for all $x \in \mathbb{R}$.*

Proof. Let $g(x) = f(x)/\exp(bx)$. By the quotient rule for derivatives

$$g'(x) = \frac{bf'(x)\exp(x) - f(x)b\exp'(bx)}{\exp^2(bx)} = 0$$

for all x. Consequently, g is a constant function. In particular, $g(x) = g(0) = a$ for all x. ☺

The exponential function as a limit:

Theorem 8.2.4. *For any real number x,*

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Part of the claim is that the limit exists.

Proof. As $n \to \infty$ we have $x/n \to 0$, hence, since log is differentiable at 1 :

$$\begin{aligned} n\log\left(1+\frac{x}{n}\right) &= x\,\frac{\log(1+x/n)-\log(1)}{x/n} \\ &\underset{n\to\infty}{\longrightarrow} x\log'(1) = x. \end{aligned}$$

Hence, since exp is continuous

$$\begin{aligned} \left(1+\frac{x}{n}\right)^n &= \exp\left(\log\left(\left(1+\frac{x}{n}\right)^n\right)\right) \\ &= \exp\left(n\log\left(1+\frac{x}{n}\right)\right) \\ &\underset{n\to\infty}{\longrightarrow} \exp(x) \end{aligned}$$

by the previous calculation. ☺

Let

$$e := \exp(1) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Then $\log(1) = 0$, $\log(e) = \log(\exp(1)) = 1$, and $1 < \log(4)$, so $\log(1) < \log(e) < \log(4)$, and therefore $1 < e < 4$, since log is an increasing function.

Exercise 8.2.5. Prove $\exp(p/q) = e^{p/q}$ for any rational number p/q.

Exercise 8.2.5 shows that $\exp(r) = e^r$ when r is rational. Since exp is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$, it is natural to define

$$e^x := \exp(x) \text{ for all } x \text{ in } \mathbb{R}.$$

More generally, for $a > 0$ define

$$a^x := e^{x\log(a)} = \exp(x\log(a)).$$

Prior to this definition we could only consider x^r for rational r. The usual computational properties hold:

$$\begin{aligned} a^{x+y} &= \exp((x+y)\log(a)) \\ &= \exp(x\log(a)+y\log(a)) \\ &= \exp(x\log(a))\,\exp(y\log(a)) \\ &= a^x a^y \end{aligned}$$

for all $a > 0$ and all $x, y \in \mathbb{R}$ and

$$\begin{aligned} (xy)^a &= \exp(a\log(xy)) \\ &= \exp(a(\log(x)+\log(y))) \end{aligned}$$

$$
\begin{aligned}
&= \exp(a\log(x) + a\log(y)) \\
&= \exp(a\log(x))\exp(a\log(y)) \\
&= x^a y^a
\end{aligned}
$$

for all $x, y > 0$ and all $a \in \mathbb{R}$. We leave the statement and verification of other familiar properties of the functions $x \to a^x$ and of $x \to x^a$ to the interested reader.

Example 8.2.6. We will show that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for all } x \in \mathbb{R}.$$

In particular, setting $x = 1$ gives

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Proof. Fix x. Since $\exp' = \exp$ it follows that $\exp^{(k)} = \exp$ for all $k \in \mathbb{N}_0$. In particular, $\exp^{(k)}(0) = 1$ for all k. Hence, the nth Taylor polynomial for exp at 0 is

$$T_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}$$

and the corresponding Lagrange Remainder is

$$R_n(x) := e^c \frac{x^{n+1}}{(n+1)!}$$

for some $c = c(n,x)$ between 0 and x. Since $\exp(x) - T_n(x) = R_n(x)$ we must show $R_n(x) \to 0$ as $n \to \infty$. Note,

$$e^c = e^{c(n,x)} \leq \max\left\{e^0, e^x\right\}$$

for all n. Hence, we must show $\frac{x^{n+1}}{(n+1)!} \to 0$, as $n \to \infty$.

Let M be an integer such that $|x| \leq M$. Then for $M \leq n$

$$
\begin{aligned}
\left|\frac{x^{n+1}}{(n+1)!}\right| &\leq \frac{M}{1}\cdot\frac{M}{2}\cdots\frac{M}{M}\cdot\frac{M}{M+1}\cdots\frac{M}{n}\cdot\frac{M}{n+1} \\
&\leq \frac{M}{1}\cdot\frac{M}{2}\cdots\frac{M}{M}\cdot\quad 1 \quad\cdots\quad 1 \quad\cdot\frac{M}{n+1} \\
&= \frac{M^{M+1}}{M!}\cdot\frac{1}{n+1}
\end{aligned}
$$

Hence, $\left|\frac{x^{n+1}}{(n+1)!}\right| \to 0$ as $n \to \infty$. ☺

8.3 The Napier Constant $e^\star$

Irrationality of e was first proven by Leonhard Euler (15 April 1707 Basel to 18 September 1783 St. Petersburg). Charles Hermite (24 December 1822 Dieuze, Moselle to 14 January 1901 Paris), was the first to prove that e is transcendental.

Euler named the Napier constant (John Napier 1550 Merchiston Tower to 4 April 1617 Edinburgh) after himself, calling it e. The first reference to the constant was published in 1618 in work on logarithms by Napier in the form of a table of logarithms based on e. The discovery of the constant $e := \exp(1)$ itself is credited to Jacob Bernoulli, who attempted to find the value of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

By Example 8.2.6 $e = \sum_{k=0}^{\infty}\frac{1}{k!}$. Observe that when $k \geq 3$, then $k! > 4^{k-2}$. Hence

$$2 < \frac{5}{2} = \sum_{k=0}^{2}\frac{1}{k!} < e < \sum_{k=0}^{2}\frac{1}{k!} + \sum_{k=3}^{\infty}\left(\frac{1}{4}\right)^{k-2} = \frac{5}{2}+\frac{1}{3} < 3.$$

In particular, e is not an integer.

Theorem 8.3.1 (Euler). *The number e is irrational.*

There are many proofs of this fact. The one I have chosen uses the idea used to show $e < 3$ above. The same method was used in our proof that $\sqrt{2}$ is irrational.

Proof. Suppose e is rational. Let a, b be natural numbers, such that $e = a/b$. Then $b > 1$ since e is not an integer and

$$\frac{a}{b} = \sum_{n=0}^{b}\frac{1}{n!} + \sum_{n=b+1}^{\infty}\frac{1}{n!}$$

since both sides equals e. Multiplying by $b!$ and rearranging gives

$$b!\sum_{n=b+1}^{\infty}\frac{1}{n!} = b!\frac{a}{b} - b!\sum_{n=0}^{b}\frac{1}{n!} = (b-1)!a - \sum_{n=0}^{b} b(b-1)\cdots(b-(n+1)) \quad (8.1)$$

(the empty product is interpreted as being $= 1$). The right hand side is a sum/difference of products of integers, hence an integer. The left hand side is > 0, hence the right hand side is a natural number. But estimating the left hand side leads to a contradiction, in fact

$$0 < \sum_{n=b+1}^{\infty}\frac{b!}{n!} < \sum_{n=b+1}^{\infty}\left(\frac{1}{b+1}\right)^{n-b} = \frac{1/(b+1)}{1-1/(b+1)} = \frac{1}{b} < 1. \quad (8.2)$$

We used $b > 1$ to establish the last inequality. So $\sum_{n=b+1}^{\infty}\frac{b!}{n!}$ is an integer in the open interval $]0,1[$. ☺

Remark 8.3.2. Comparing to Remark 3.5.5 the equation $x = y$ is Eq. (8.1).

Charles Hermite is perhaps best known for proving that e is transcendental.

Theorem 8.3.3 (Hermite 1873). *The number e is transcendental.*

Proof. Suppose e is algebraic. Let a_j be integers such that

$$a_n e^n + a_{n-1} e^{n-1} + \cdots + a_1 e + a_0 = 0. \tag{8.3}$$

We may assume $a_0 \neq 0$. If not we can reduce the degree of the polynomial by dividing by e. Let

$$f_p(x) := \frac{x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p}{(p-1)!}$$

where p is an integer > 1. Then f_p is a polynomial of degree $(n+1)p-1$, sometimes called the *Hermite polynomial*. Let

$$F_p(x) := \sum_{j=0}^{(n+1)p-1} f_p^{(j)}(x), \tag{8.4}$$

where $f_p^{(j)}$ is f_p differentiated j times. Then

$$F_p'(x) - F_p(x) = \sum_{j=1}^{(n+1)p-1} f_p^{(j)}(x) - \sum_{j=0}^{(n+1)p-1} f_p^{(j)}(x) = -f_p(x),$$

since $f_p^{(np+p)}(x) = 0$. It follows that

$$\left(F_p(x)e^{-x}\right)' = \left(F_p'(x) - F_p(x)\right) e^{-x} = -f_p(x)e^{-x}.$$

Using the fundamental theorem of calculus we get

$$\int_0^m f_p(x)e^{-x}dx = F_p(0) - F_p(m)e^{-m}.$$

Multiply by $a_m e^m$ and sum over $m = 0, 1, \ldots, n$ we get

$$\begin{aligned}\sum_{m=0}^n a_m e^m \int_0^m f_p(x)e^{-x}dx &= \sum_{m=0}^n a_m e^m F_p(0) - \sum_{m=0}^n a_m F_p(m) \\ &= -\sum_{m=0}^n a_m F_p(m).\end{aligned} \tag{8.5}$$

Where we used the sum $\sum_{m=0}^n a_m e^m F_p(0)$ is $= 0$ by (8.3).

To complete the proof we will show (a) that the left hand side of (8.2) converges to zero as $p \to \infty$ and (b) that the right hand side of (8.2) is a non-zero integer for an infinite number of integers p. This is a contradiction, hence verifying the claims (a) and (b) completes the proof.

(a): If $0 \le x \le n$, then

$$\left|f_p(x)\right| \le \frac{n^{(n+1)p-1}}{(p-1)!} \text{ for } 0 \le x \le n, \tag{8.6}$$

since $|x-k| \le n$ for $k = 0, 1, \ldots, n$ and $0 \le x \le n$. Now, if $n^{n+1} < p$, then (as in the verification of Example 8.2.6)

$$\begin{aligned}\left|\frac{n^{(n+1)p-1}}{(p-1)!}\right| &= n^n \cdot \frac{n^{n+1}}{1} \cdot \frac{n^{n+1}}{2} \cdots \frac{n^{n+1}}{n^{n+1}} \cdot \frac{n^{n+1}}{n^{n+1}+1} \cdots \frac{n^{n+1}}{p-2} \cdot \frac{n^{n+1}}{p-1} \\ &\le n^n \cdot \frac{n^{n+1}}{1} \cdot \frac{n^{n+1}}{2} \cdots \frac{n^{n+1}}{n^{n+1}} \cdot \quad 1 \quad \cdots \quad 1 \quad \cdot \frac{n^{n+1}}{p-1} \\ &= n^n \cdot \frac{M^{M+1}}{M!} \cdot \frac{1}{p-1},\end{aligned}$$

where $M := n^{n+1}$. Hence,

$$\left|\int_0^m f_p(x)e^{-x}dx\right| \le m \cdot n^n \cdot \frac{M^{M+1}}{M!} \cdot \frac{1}{p-1} \underset{p\to\infty}{\longrightarrow} 0.$$

Consequently, the left hand side of (8.2) converges to zero as $p \to \infty$.

(b) *Conceptual Version*: By definition of $f_p(x)$, each $f_p^{(i)}(m)$ is an integer, divisible by p except when $m = 0$ and $j = p-1$.

In fact, when $m \ne 0$ the only non-zero terms in $f_p^{(j)}(m)$ are from terms where the factor $(x-m)^p$ has been differentiated exactly p times, and then $p!$ cancels $(p-1)!$ leaving a factor of p.

When $m = 0$ the only non-zero terms in $f_p^{(j)}(m) = f_p^{(j)}(0)$ are from terms where the factor x^{p-1} has been differentiated exactly $p-1$ times giving a factor of $(p-1)!$. If $j \ge p$, then one of the terms $(x-m)^p$ with $m \ne 0$ has been differentiated at least once giving an additional factor of p, yielding a factor of $p!$ as before. Finally, when $j = p-1$ and $m = 0$ we get $f_p^{(p-1)}(0) = (-1)^p \ldots (-n)^p$.

So

$$\sum_{m=0}^{n} a_m F_p(m) = a_0(-1)^p \ldots (-n)^p + pM$$

where M is an integer derived from all the non-zero terms, where $m \ne 0$ or where $m = 0$ and $j \ge p$. If p is a prime $> n$ and $> |a_0|$, then the term $a_0(-1)^p \ldots (-n)^p$ is not divisible by p. Thus $\sum_{m=0}^n a_m F_p(m)$ is a non-zero integer. This completes the proof.

(b) *Computational Version*: Write

$$(p-1)! \cdot f_p(x) = g_0(x) g_1(x) \cdots g_n(x),$$

where $g_0(x) := x^{p-1}$ and $g_i(x) := (x-i)^p$ for $1 \le i \le n$. It follows from the product rule that

$$(p-1)! \cdot f_p^{(j)}(x) = \sum_{j_0+j_1+\cdots+j_n=j} g_0^{(j_0)}(x) g_1^{(j_1)}(x) \cdots g_n^{(j_n)}(x), \tag{8.7}$$

where the sum is over all vectors $(j_0, j_1, \ldots, j_n)$ of integers $j_i \ge 0$ such that $j_0 + j_1 + \cdots + j_n = j$.

Repeatedly differentiating g_0 we see that

$$g_0^{(k)}(x) = (p-1)(p-2)\cdots(p-k)x^{p-k-1} \tag{8.8}$$

when $1 \le k \le p-1$ and $g_0^{(k)}(x) = 0$ when $p-1 < k$. By (8.8)

$$g_0^{(k)}(m) = \begin{cases} (p-1)(p-2)\cdots(p-k))m^{p-k-1} & \text{if } 0 \le k \le p-1 \\ 0 & \text{if } k > p-1 \end{cases}. \tag{8.9}$$

The boundary cases $k=0$ and $k=p-1$ are interpreted as $g_0^{(0)}(m) = m^{p-1}$ and $g_0^{(p-1)}(m) = (p-1)!$.

Let $1 \le i \le n$. Repeatedly differentiating g_i we see that

$$g_i^{(k)}(x) = p(p-1)(p-2)\cdots(p-k+1)(x-i)^{p-k} \tag{8.10}$$

when $1 \le k \le p$ and $g_i^{(k)}(x) = 0$ when $p < k$. By (8.10), if $1 \le i \le n$, then

$$g_i^{(k)}(m) = \begin{cases} p(p-1)\cdots(p-k+1))m^{p-k} & \text{if } 0 \le k \le p \\ 0 & \text{if } k > p \end{cases}. \tag{8.11}$$

The boundary cases $k=0$ and $k=p$ are interpreted as $g_i^{(0)}(m) = (m-i)^p$ and $g_i^{(p)}(m) = p!$. Note, $g_i^{(k)}(m)$ is an integer for all $0 \le i \le n$, all $0 \le k$, and all $0 \le m \le n$. Setting $x = m$ in (8.7) gives

$$(p-1)! \cdot f_p^{(j)}(m) = \sum_{j_0+j_1+\cdots+j_n=j} g_0^{(j_0)}(m) g_1^{(j_1)}(m) \cdots g_n^{(j_n)}(m). \tag{8.12}$$

For $m=0$, it follows from (8.9) that $g_0^{(k)}(0) = 0$ if $k \ne p-1$ and $g_0^{(p-1)}(0) = (p-1)!$. So using (8.12) it follows that $f_p^{(j)}(0) = 0$ when $j < p-1$ and when $p-1 \le j$ only the terms with $j_0 = p-1$ in the sum (8.12) can be non-zero. Since $g_0^{(p-1)}(0) = (p-1)!$ all the terms in the sum (8.12) with $j_0 = p-1$ have a factor of $(p-1)!$. So for $p-1 \le j$, (8.12) can be written as

$$f_p^{(j)}(0) = \sum_{j_1+\cdots+j_n=j-p+1} g_1^{(j_1)}(0) g_2^{(j_2)}(0) \cdots g_n^{(j_n)}(0). \tag{8.13}$$

Hence, in either case $j < p-1$ or $p-1 \leq j$, $f_p^{(j)}(0)$ is an integer. By (8.4) $F_p(0)$ is an integer.

Fix $1 \leq m \leq n$. It follows from (8.11) that $g_m^{(k)}(m) = 0$ if $k \neq p$ and $g_m^{(p)}(m) = p!$. So using (8.12) it follows that $f_p^{(j)}(m) = 0 = p \cdot 0$ when $j < p$ and when $p \leq j$ only the terms with $j_m = p$ in the sum (8.12) can be non-zero. Since $g_m^{(p)}(m) = p!$ all the terms in the sum (8.12) with $j_m = p$ have a factor of $p!$. In either case $j < p$ or $p \leq j$, $f_p^{(j)}(m)$ is an integer multiple of p. By (8.4)

$$F_p(m) \text{ is an integer multiple of } p \text{ when } 1 \leq m \leq n. \tag{8.14}$$

Since $f_p^{(j)}(m)$ is an integer for all $0 \leq j$ and all $0 \leq m \leq n$, it follows from (8.4) that $F_p(m)$ is an integer. Hence the right hand side $-\sum_{m=0}^{n} a_m F_p(m)$ of (8.2) is an integer. To complete the proof we must show that the integer $\sum_{m=0}^{n} a_m F_p(m)$ is non-zero for an infinite number of p. We will complete the proof by showing that $\sum_{m=0}^{n} a_m F_p(m)$ is not a multiple of p for infinitely many p.

If $m \neq 0$, then $F_p(m)$ is a multiple of p by (8.14). Hence we must show that $a_0 F_p(0)$ is not a multiple of p for infinitely many p. Since $f_p^{j)}(0) = 0$ when $j < p-1$, it follows from (8.4) and (8.13) that

$$F_p(0) = \sum_{j=p-1}^{(n+1)p-1} f_p^{(j)}(0) = \sum_{j=p-1}^{(n+1)p-1} \sum_{j_1+\cdots+j_n=j-p+1} g_1^{(j_1)}(0) \cdots g_n^{(j_n)}(0).$$

If $j_i > 0$ for some $1 \leq i \leq n$, then it follows from (8.11) that $g_i^{(j_i)}(0)$ is a multiple of p. Hence,

$$F_p(0) = g_1(0) \cdots g_n(0) + pM$$

for some integer M. So

$$a_0 F_p(0) = a_0 (-1)^p (-2)^p \cdots (-n)^p + p a_0 M.$$

Consequently, if p is a prime and $p > \max\{|a_0|, n\}$, then $a_0 F_p(0)$ is not a multiple of p. ☺

8.4 Bump Functions★

The purpose of this section is to construct a function $\phi : \mathbb{R} \to \mathbb{R}$, such that

- $0 \leq \phi(x) \leq 1$ for all x in $\mathbb{R}$
- $\phi(0) = 1$
- $\phi(x) = 0$ for all $|x| \geq 1$
- ϕ is $\mathscr{C}^\infty$ on $\mathbb{R}$

Such a function is called a *bump* function. See Fig. 8.1.

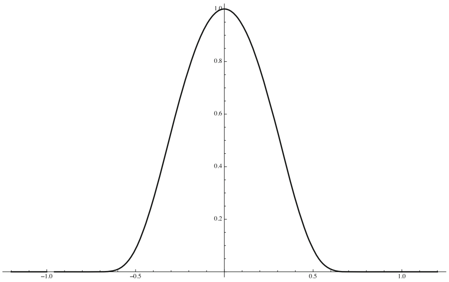

Fig. 8.1 The bump function ϕ constructed in this section

Step 1: If $x > 0$, then Taylor's Theorem with Lagrange remainder tells us

$$\frac{x^n}{e^x} = \frac{x^n}{\sum_{k=0}^{n+1} \frac{x^k}{k!} + e^c \frac{x^{n+2}}{(n+2)!}} \leq \frac{x^n}{\frac{x^{n+1}}{(n+1)!}} = \frac{(n+1)!}{x}$$

we used all terms in the first denominator are ≥ 0 and retained the term with $k = n+1$. Consequently,

$$\frac{x^n}{e^x} \to 0 \text{ as } x \to \infty.$$

Hence, if p is a polynomial, then

$$\frac{p(x)}{e^x} \to 0 \text{ as } x \to \infty.$$

If $1 < x$, then $x < x^2$, so $e^x < e^{x^2}$. Consequently,

$$0 \leq \frac{|p(x)|}{e^{x^2}} \leq \frac{|p(x)|}{e^x} \to 0 \text{ as } x \to \infty.$$

Since, $1/t \to \infty$ as $t \searrow 0$ it follows that for any polynomial p

$$p(1/t)e^{-1/t^2} \to 0 \text{ as } t \searrow 0.$$

A similar argument yields the same conclusion for $t \nearrow 0$.

Step 2: Let $f(x) := e^{-1/x^2}$ for $x \neq 0$. Then $f'(x) = \frac{2}{x^3} f(x)$, $f''(x) = \left(\frac{-6}{x^4} + \frac{2}{x^3}\right) f(x)$, by induction $f^{(n)}(x) = q_n(1/x) f(x)$, where q_n is a polynomial.

Exercise 8.4.1. Carry out the induction suggested above.

Step 3: (Euler) Let

$$\psi(x) := \begin{cases} e^{-1/x^2} & \text{when } 0 < x \\ 0 & \text{when } x \leq 0 \end{cases}.$$

Clearly, ψ is $\mathscr{C}^\infty$ on $\mathbb{R} \setminus \{0\}$. See Fig. 8.2.

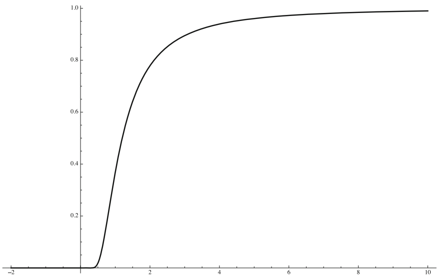

Fig. 8.2 The function ψ

- ψ is continuous at 0, by Step 1 with $p(1/x) := 1$, i.e., $p(t) = 1$.
- ψ is differentiable at 0 and $\psi'(0) = 0$, because

$$\frac{\psi(x) - \psi(0)}{x - 0} = \frac{1}{x}\psi(x) \to 0 \text{ as } x \to 0$$

  by Step 1 with $p(1/x) = 1/x$, i.e., $p(t) = t$.
- ψ' is differentiable at 0 and $\psi''(0) = 0$, because

$$\frac{\psi'(x) - \psi'(0)}{x - 0} = \frac{q_1(1/x)\psi(x)}{x} \to 0 \text{ as } x \to 0$$

  by Step 1 with $p(1/x) = q_1(1/x)/x$, i.e., with $p(t) = tq_1(t)$.

Exercise 8.4.2. Show that for any $n \in \mathbb{N}_0$, the nth derivative of ψ exists at 0 and $\psi^{(n)}(0) = 0$.

Remark 8.4.3. In particular, for any n, the Taylor polynomial $T_n\psi$ associated with ψ at 0 is $T_n\psi(x) = 0$.

Proposition 8.4.4. ψ *maps* $]-\infty, 0]$ *onto* $\{0\}$ *and* $]0, \infty[$ *onto* $]0, 1[$.

Proof. Note $-1/x^2$ is a strictly increasing function mapping $]0, \infty[$ onto $]-\infty, 0[$. Since exp is a strictly increasing function mapping $]-\infty, 0[$ onto $]0, 1[$, the composition $e^{-1/x^2} = \exp(-1/x^2)$ is a strictly increasing function mapping $]0, \infty[$ onto $]0, 1[$.

Step 4: Let $\phi : \mathbb{R} \to \mathbb{R}$ be determined by

$$\phi(x) := e^2\psi(1+x)\psi(1-x).$$

See Fig. 8.1.

- ϕ is $\mathscr{C}^\infty$ on $\mathbb{R}$, since ψ is.
- If $x \geq 1$, $1 - x \leq 0$, so $\phi(x) = 0$, since $\psi(1-x) = 0$. Similarly, if $x \leq -1$, then $\phi(x) = 0$, since $\psi(1+x) = 0$.
- $\phi(0) = e^2\psi(1)\psi(1) = e^2e^{-1}e^{-1} = 1$.
- $0 \leq \phi(x)$ for all x, since $0 \leq \psi(x)$ for all x.

It remains to verify that $\phi(x) \le 1$ for all x between -1 and 1.

Exercise 8.4.5. Prove $\phi'(x) \ge 0$ when $-1 < x < 0$ and $\phi'(x) \le 0$ when $0 < x < 1$.

It follows from the exercise above that $\phi(0)$ is the global maximum of ϕ on $]-1,1[$. Thus ϕ is a bump function.

Problems

Problems for Sect. 8.1

1. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$.
2. $\log(x^{1/n}) = \frac{1}{n}\log(x)$ for any $n \in \mathbb{N}$.
3. $\log\left(x^{p/q}\right) = \frac{p}{q}\log(x)$, for any rational number p/q. (We cannot replace p/q by a real number r, since we have not yet defined x^r for irrational r's.)
4. Let $f(x) := \log(1+x)$. Note f is defined for $x > -1$.
   (i) Show by induction that

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n} \text{ for } n \in \mathbb{N}.$$

   (ii) Show that the n^{th} Taylor polynomial for f at 0 is

$$T_n f(x) = \sum_{k=1}^{n} (-1)^{k+1}\frac{x^k}{k}$$

and the Lagrange Remainder is

$$R_n f(x) = \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}$$

for some c between 0 and x.

Problems for Sect. 8.2

1. Prove $\log(x^r) = r\log(x)$ for any real number r and any $x > 0$.
2. If $f : [0,1] \to \mathbb{R}$ is differentiable and $f(0) = f(1) = 0$, then

$$\forall a \in \mathbb{R}, \exists c \in]0,1[, f'(c) = af(c).$$

   [*Hint:* Apply Rolle's Theorem to $f(x)e^{ax}$.]

3. There are irrational numbers a and b such that a^b is rational. [*Hint*: $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$.]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathscr{C}^1$. Suppose $f(s+t) = f(s)f(t)$ for all s,t in $\mathbb{R}$. Use Theorem 8.2.3 to show there exists A, B in $\mathbb{R}$, such that $f(x) = Ae^{Bx}$ for all x in $\mathbb{R}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(s+t) = f(s)f(t)$ for all s,t in $\mathbb{R}$. Show there exists A, B in $\mathbb{R}$, such that $f(x) = Ae^{Bx}$ for all x in $\mathbb{R}$.

6. Use upper and lower sums to evaluate $\int_0^1 e^x dx$.

7. Let $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$. The first two parts of the problem shows that $\Gamma(x)$ exists for $x > 0$. The last two parts establish a connection to the factorial $n!$.

   a. Prove $\int_0^1 t^{x-1}e^{-t}dt = \lim_{a \searrow 0} \int_a^1 t^{x-1}e^{-t}dt$ exists for all $x > 0$.
   b. Prove $\int_1^\infty t^{x-1}e^{-t}dt = \lim_{b \to \infty} \int_1^b t^{x-1}e^{-t}dt$ exists for all $x > 0$.
   c. Prove $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.
   d. Prove $\Gamma(n+1) = n!$ for all integers $n \geq 0$.

8. Let $f(x) = \log(x)$. Recall, Taylor's formula with integral remainder $f(x) = T_n f(x) + R_n f(x)$ where

$$T_n f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j, \quad R_n f(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

   Let $a = 1$. Suppose $1 < x$.

   (a) Find the second Taylor polynomial $T_2 f(x)$.
   (b) When $1 \leq x \leq t$ we have $(x-t)^n \leq (x-1)^n$. Use this to find an upper bound for the integral determining $R_2 f(x)$.[This upper bound will depend on x.]
   (c) Find $T_2(2)$.
   (d) Use part (b) to find a number $\varepsilon > 0$ such that $|\log(2) - T_2 f(2)| < \varepsilon$.
   (e) Use parts (c) and (d) to find numbers α and β such that $\alpha \leq \log(2) \leq \beta$.

9. Suppose $f :]0,\infty[\to \mathbb{R}$ is continuous and $f(st) = f(s) + f(t)$ for all $s,t > 0$. Let $g(x) = f(e^x)$ for all x in $\mathbb{R}$. Prove the following claims:

   (a) g is continuous on $\mathbb{R}$.
   (b) $g(x+y) = g(x) + g(y)$ for all real numbers x and y.
   (c) Prove there is a real number k, such that $g(x) = kx$ for all real numbers x.
   (d) Prove $f(t) = k\log(t)$ for all $t > 0$.

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then $f(x) = kx$ for some real number k. See the Problems for Sect. 2.1. The next problem constructs a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, yet $f(x) \neq kx$. Hence, the continuity assumption cannot be omitted.

First Recall a Bit of Linear Algebra

Let V be a vector space over the field F. A subset B of V is a basis for V if B is linearly independent and spans V.

Linear independent:

$$a_1, a_2, \ldots, a_n \in F, b_1, b_2, \ldots, b_n \in B, \sum_{k=1}^{n} a_k b_k = 0 \implies b_1 = b_2 = \cdots = b_n = 0$$

Spans:

$$\forall v \in V, \exists a_1, a_2, \ldots, a_n \in F, b_1, b_2, \ldots, b_n \in B, \sum_{k=1}^{n} a_k b_k = v$$

By linear independence $\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a'_k b_k \implies a_k = a'_k$ for $k = 1, 2, \ldots, n$.

A basic fact of linear algebra is that any vector space has a basis.

$\mathbb{R}$ as a Vector Space Over $\mathbb{Q}$

Suppose B is a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. If B is finite, then it follows, that $\mathbb{R}$ is countable, hence B is not finite. The same argument shows that B is not countable. See Chap. 4.

Let $p_1, p_2, \ldots$ be the primes. The numbers (vectors) $\log(p_1)$, $\log(p_2), \ldots$ are linearly independent over the rationals.

Proof. Let $c_1, c_2, \ldots, c_n$ be rationals such that

$$c_1 \log(p_1) + c_2 \log(p_2) + \cdots + c_n \log(p_n) = 0.$$

Multiplying by the common denominator of $c_1, c_2, \ldots, c_n$ we may assume the $c'_k s$ all are integers. But

$$\log\left(p_1^{c_1} p_2^{c_2} \cdots p_n^{c_n}\right) = 0 \implies p_1^{c_1} p_2^{c_2} \cdots p_n^{c_n} = 1 \implies c_1 = c_2 = \cdots = c_n = 0.$$

The last implication is the fundamental theorem of arithmetic. ☺

One can show that the sets

$$\sqrt{p_1}, \sqrt{p_2}, \ldots$$

and

$$\sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}}, \ldots$$

both are linearly independent over the rationals. Any decent Modern Algebra book contains the details.

10. Let B be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Without Loss Of Generality we may assume $1 \in B$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(a_1 + a_2 b_2 + \cdots + a_n b_n) = a_1$$

where $a_k \in \mathbb{Q}$ and $b_1 = 1, b_2, \ldots, b_n$ are in B. Prove:

(a) f is well-defined, i.e, prove f is a function.
(b) For all $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$.
(c) f is not continuous.
(d) There is no constant $k \in \mathbb{R}$, such that $f(x) = kx$ for all $x \in \mathbb{R}$.

Problems for Sect. 8.3

The problems for this section should be worked in order, or the results of the previous problems taken on faith before attempting one of the later problems.

The first problem below is an example implementing the strategy in Remark 3.5.5.

1. Let $p > 0$ be an integer. Suppose $e^p = \frac{a}{b}$ for some positive integers a and b. For positive integers n consider the polynomials

$$f_n(x) := \frac{x^n (1-x)^n}{n!}$$

and

$$F_n(x) := \sum_{j=0}^{2n} p^{2n-j} f_n^{(j)}(x)$$

where $f_n^{(j)}(x)$ is $f_n(x)$ differentiated j times. Some hints are provided in the brackets.

(i) Show $pF_n(x) - F_n'(x) = p^{2n+1} f_n(x)$. [First show $F_n'(x) = \sum_{j=1}^{2n} p^{2n-j+1} f_n^{(j)}(x)$.]
(ii) Show $\left(F_n(x) e^{p(1-x)}\right)' = -p^{2n+1} f_n(x) e^{p(1-x)}$. [Use ($i$).]
(iii) Show $F_n(0)e^p - F_n(1) = \int_0^1 p^{2n+1} f_n(x) e^{p(1-x)} dx$. [Use ($ii$).]
(iv) Show $0 < aF_n(0) - bF_n(1)$. [Use $e^p = a/b$ and that $p^{2n+1} f_n(x) e^{p(1-x)}$ is a positive continuous function and not $= 0$ for all x in $[0,1]$.]
(v) Show $\int_0^1 p^{2n+1} f_n(x) e^{p(1-x)} dx \le \frac{p^{2n+1} e^p}{n!}$. [Use $f_n(x) e^{-x} \le 1/n!$.]
(vi) Show $\frac{p^{2n+1}}{n!} \le \frac{p^{2p^2+3}}{p^2!} \cdot \frac{1}{n}$ when $p^2 < n$. [Similar to an argument from the verification of Example 8.2.6.]
(vii) Show $0 < aF_n(0) - bF_n(1) < 1$ for large n. [By (iii) $aF_n(0) - bF_n(1) = b\int_0^1 p^{2n+1} f_n(x) e^{p(1-x)} dx$. Use ($iv$), ($v$), and ($vi$).]

(*viii*) Show $n!f_n^{(j)}(x) = \sum_{k=0}^{j} \binom{j}{k} g_n^{(k)}(x) h_n^{(j-k)}(x)$, where $g_n(x) = x^n$ and $h_n(x) = (1-x)^n$. [Apply the product rule for derivatives.]

(*ix*) Show $g_n^{(k)}(0) = 0$ when $k < n$, $g_n^{(n)}(x) = n!$, and $g_n^{(k)}(x) = 0$ when $n < k$. [Differentiating $g_n(x) = x^n$ gives $g_n^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k}$ for $k \le n$. As usual, $x^0 = 1$.]

(*x*) Show $h_n^{(k)}(0)$ is an integer for all k. [Differentiating $h_n(x) = (1-x)^n$ gives $h_n^{(k)}(x) = (-1)^k n(n-1)\cdots(n-k+1)(1-x)^{n-k}$ for $k \le n$.]

(*xi*) Show $f_n^{(j)}(0)$ is an integer. [Combine (*viii*), (*ix*), and (*x*).]

(*xii*) Show $F_n(0)$ is an integer. [Use (*xi*) and the definition of $F_n(x)$.]

(*xiii*) Show $F_n(1)$ is an integer. [Repeat (*ix*), (*x*), (*xi*), (*xii*) interchanging the roles of $g_n(x)$ and $h_n(x)$ and of 0 and 1, as appropriate.]

(*xiv*) Show $aF_n(0) - bF_n(1)$ is an integer. [Combine (*xii*) and (*xiii*).]

(*xv*) Conclude e^p is irrational. [Use (*vii*) and (*xiv*).]

In the language of Remark 3.5.5 $e^p \approx \frac{F_n(1)}{F_n(0)}$ with error $\frac{1}{F_n(0)} \int_0^1 p^{2n+1} f_n(x) e^{p(1-x)} dx$. The key parts of the remark are provided by (*vii*) and (*xiv*).

2. If p and q are positive integers, then $e^{p/q}$ is irrational.

3. If $r \neq 0$ is rational, then e^r is irrational.

Problems for Sect. 8.4

Solutions and Hint for the Exercises

Exercise 8.1.2. Splitting the interval and using a change of variables does the job.

Exercise 8.1.3. This is clear if $n = 1$. Suppose $\log(x^n) = n\log(x)$ for some $n \in \mathbb{N}$. Then $\log(x^{n+1}) = \log(x \cdot x^n) = \cdots$.

Exercise 8.2.2. For example, $\log(\exp(x)\exp(y)) = \log(\exp(x)) + \log(\exp(y)) = x + y$. Applying exp to both sides gives the desired equality.

Exercise 8.2.5. $\exp(3) = \exp(1+1+1) = \exp(1)\exp(1)\exp(1) = e^3$ and $e = \exp(1) = \exp\left(\frac{1}{2} + \frac{1}{2}\right) = \exp\left(\frac{1}{2}\right)\exp\left(\frac{1}{2}\right)$, so $\exp\left(\frac{1}{2}\right) = e^{1/2}$.

Exercise 8.4.1. If $q(t) = \sum a_k t^k$ then $\frac{q(1/x)}{x^2} = p(1/x)$, where $p(t) = \sum a_k t^{k+2}$.

Exercise 8.4.2. A simple proof by induction works.

Exercise 8.4.5. If $-1 < x < 1$, then $0 < 1 - x$ and $0 < 1 + x$.

Part II

Analysis

Sequences and series of numbers and functions, including power series and Fourier series, form the core of this part. The last chapter is an introduction to point set topology. Among the applications are proofs of the fundamental theorem of algebra, of the Weierstrass approximation theorem, of Weyl's uniform distribution theorem, a proof of the irrationality of the number π. Constructions of a space filling curve, of the trigonometric functions, of the number π, and of nowhere differentiable continuous functions,

Much of the material in this part involves the interaction of two limits. In particular, much of the material in this part can be interpreted as studying when two limit can be interchanged.

Chapter 9
Convergence of Sequences

This chapter covers convergence of numerical sequences in some detail. This is followed by a discussion of convergence of sequences of functions, in particular, of uniform convergence of a sequence of functions. Applications include, Leibnitz Integral Rule, Fubini's Theorem, the Approximate Identity Lemma, and a proof of the Fundamental Theorem of Algebra. The later completes the discussion of polynomials begun in Proposition 1.4.11.

9.1 Real and Complex Sequences

We discussed basic ideas related to sequences in Sects. 1.6 and 1.7. In this section we will expand on these considerations. In the first part of this section we discuss sequences of complex numbers. Then we characterize continuity of a function in terms convergence of sequences. Finally, we discuss properties of sequences that require an order, hence we restrict attention to sequences of real numbers.

A sequence in a set X is a mapping from $\mathbb{N}$ to X. Instead of using function notation it is customary to use subscripts for sequences. So, a *sequence* is a mapping $a : \mathbb{N} \to X$ and the value of the mapping at the positive integer n is $a(n) = a_n$. We will usually write (a_n) or $a_1, a_2, \ldots$ in place of $a : \mathbb{N} \to X$. Two sequences are equal if they are equal as functions, that is, $(a_n) = (b_n)$, if $a_n = b_n$ for all $n \in \mathbb{N}$.

Exercise 9.1.1. The sequences

$$1,2,3,2,3,2,3,\ldots \text{ and } 1,2,3,1,2,3,\ldots$$

are not equal, because ______________. However, the sets

$$\{1,2,3,2,3,2,3,\ldots\} \text{ and } \{1,2,3,1,2,3,\ldots\}$$

are equal, both are equal to ______________. Fill in the two blanks.

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Convergence

We discussed elementary properties of convergence of sequences in Sect. 1.6. For emphasis we restate the definition here.

Definition 9.1.2. Let (a_n) be a sequence of points in $\mathbb{C}$ and let $a \in \mathbb{C}$. We say that (a_n) is *converges* to a if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n > N \implies |a_n - a| < \varepsilon.$$

If (a_n) converges to a, then we say that a is the *limit* of (a_n) and we write $a_n \to a$, $a_n \underset{n\to\infty}{\longrightarrow} a$, $\lim a_n = a$, $\lim_n a_n = a$, or $\lim_{n\to\infty} a_n = a$. We say (a_n) is *convergent,* if (a_n) converges to some point in $\mathbb{C}$. We say (a_n) is *divergent* or *diverges* if (a_n) is not convergent.

Note $|a_n - a| < \varepsilon$ is equivalent to $a_n \in B_\varepsilon(a)$. So we can rewrite the definition of convergence as

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n > N \implies a_n \in B_\varepsilon(a).$$

Recall, from Sect. 1.6, as sequence converging to zero is called a *null sequence*. Clearly,

Proposition 9.1.3. *$a_n \to a$ iff $(a_n - a)$ is null.*

Recall, from Sect. 1.2, that a point p is an accumulation point of the set A, if there are points in $A \setminus \{p\}$ arbitrarily close to p.

Exercise 9.1.4. Let A be a subset of $\mathbb{C}$ and let $p \in \mathbb{C}$. Then p is an accumulation point of A if and only if there is a sequence of points $a_n \in A$ such that, $a_n \neq p$ for all n and $a_n \to p$.

Subsequences

If $b_1, b_2, b_3, \ldots$ is obtained from $a_1, a_2, a_3, \ldots$ by striking out some of the terms of $a_1, a_2, a_3, \ldots$, then (b_n) is a *subsequence* of (a_n).

Example 9.1.5. The sequence 2, 3, 2, 3, 2, 3, ... is a subsequence of 1, 2, 3, 2, 3, 2, 3, ... obtained by striking out the first term. The sequence 2, 3, 2, 3, 2, 3, ... is also a subsequence of 1, 2, 3, 1, 2, 3, ... obtained by striking out all the 1's, an infinite number of term, but still leaving an infinite number of terms.

More formally, a sequence (b_n) is a subsequence of the sequence (a_n), is there is a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, such that $b_n = a_{\varphi(n)}$. It is customary to write $b_n = a_{i_n}$, that is, a subsequence of (a_n) is usually denoted by (a_{i_n}).

Exercise 9.1.6. $i_n = \varphi(n) \geq n$ for all n.

Exercise 9.1.7. If $a_n \to L$ and $(b_n) = (a_{i_n})$, then $b_n = a_{i_n} \to L$.

Exercise 9.1.8. Basic facts about subsequences.

1. Any sequence is a subsequence of itself. What is φ?
2. A subsequence of a subsequence is a subsequence of the original sequence. That is, if (c_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n), then (c_n) is a subsequence of (a_n). How are the φ's related?
3. If (b_n) is the subsequence of (a_n) obtained by omitting the first m terms, write a formula for φ.

The following is one of the "big" theorems about sequences, so the proof must be non-trivial.

Theorem 9.1.9 (Sequential Compactness, Bolzano–Weierstrass). *Any bounded sequence of complex numbers has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence of real numbers. Since the sequence (x_n) is bounded it has a lower bound, i.e., there is a r such that $r \leq x_n$ for all n. The sequence $y_n := x_n - r$ is ≥ 0. If $y_{i_n} \to b$ then $x_{i_n} = y_{i_n} + r \to b + r$. Hence, we may assume $x_n \geq 0$ for all n.

Exercise 9.1.10. Why is there a positive integer m, such that the set $A_0 := \{n \in \mathbb{N} \mid m \leq x_n < m+1\}$ is infinite?

Let $d_0 := m$, then

$$A_0 = \{n \in \mathbb{N} \mid d_0 \leq x_n < d_0 + 1\}.$$

Exercise 9.1.11. Why is there an $m \in \{0, 1, \ldots, 9\}$, such that the set $A_1 := \{n \in \mathbb{N} \mid d_0.m \leq x_{0,n} < d_0.m + 1/10\}$ is infinite?

Let $d_1 := m$. Then

$$A_1 = \{n \in \mathbb{N} \mid d_0.d_1 \leq x_n < d_0.d_1 + 1/10\}.$$

Since $d_0 \leq d_0.d_1 \leq x_n < d_0.d_1 + 1/10 \leq d_0 + 1$ we conclude $A_1 \subseteq A_0$.

Inductively, suppose $k \in \mathbb{N}$, and we have constructed $d_j \in \{0, 1, \ldots, 9\}$, such that

$$A_k = \{n \in \mathbb{N} \mid d_0.d_1 \cdots d_k \leq x_n < d_0.d_1 \cdots d_k + 1/10^k\}$$

is infinite. Then there is $m \in \{0, 1, \ldots, 9\}$, such that the set $A_{k+1} := \{n \in \mathbb{N} \mid d_0.d_1 \cdots d_k m \leq x_n < d_0.d_1 \cdots d_k m + 1/10^{k+1}\}$ is infinite. Let $d_{k+1} := m$, then

$$A_{k+1} = \{n \in \mathbb{N} \mid d_0.d_1 \cdots d_k d_{k+1} \leq x_n < d_0.d_1 \cdots d_k d_{k+1} + 1/10^{k+1}$$

and $A_{k+1} \subseteq A_k$.

Let i_1 be the smallest element of A_1, i_2 the smallest element of $A_2 \setminus \{i_1\}$, i_3 the smallest element of $A_3 \setminus \{i_1, i_2\}$, and so on. Let $y_n := x_{i_n}$.

Exercise 9.1.12. Why is (y_n) a subsequence of (x_n)?

Let $b := d_0.d_1 d_2 \cdots$.

Exercise 9.1.13. Why does $y_n \to b$?

We have established Bolzano–Weierstrass for real sequences.

Exercise 9.1.14. Any bounded sequence of complex numbers has a convergent subsequence.

This completes our proof of the Bolzano–Weierstrass Theorem. ☺

Corollary 9.1.15 (Limit Point Compactness). *A bounded infinite set of complex numbers has an accumulation point.*

Proof. Let D be a bounded infinite set. Let $x_1 \in D$, let $x_2 \in D \setminus \{x_1\}$, let $x_3 \in D \setminus \{x_1, x_2\}$, and so on. Since D is infinite the process does not stop after a finite number of step, hence we get a sequence (x_n). The sequence (x_n) is bounded, since D is bounded. Hence (x_n) has a convergent subsequence (x_{i_n}). The limit of (x_{i_n}) is an accumulation point of D. ☺

Exercise 9.1.16. Prove the last claim in the proof of Limit Point Compactness.

Cauchy Sequences

A sequence (x_n) of complex numbers is *Cauchy*, if given any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $\forall m, n \in \mathbb{N}$,

$$m, n \geq N \implies |x_m - x_n| < \varepsilon.$$

Exercise 9.1.17. Any convergent sequence is a Cauchy sequence.

The main result in Cauchy sequences is that the converse is true, this is Theorem 9.1.21. The proof is an application of Limit Point Compactness.

Lemma 9.1.18. *Let (x_n) be a Cauchy sequence of complex numbers. If some subsequence (x_{i_n}) of (x_n) is convergent, then (x_n) is convergent.*

Exercise 9.1.19. Prove the Lemma.

Exercise 9.1.20. Any Cauchy sequence is bounded.

As a consequence of sequential compactness we have:

Theorem 9.1.21 (Cauchy Completeness). *Any Cauchy sequence of complex numbers is convergent to a complex number.*

Exercise 9.1.22. Prove the Theorem.

Verifying that a sequence is convergent using the definition of convergence requires us to "guess" the limit. Verifying that a sequence is Cauchy does not require us to know the value of the limit. Consequently, it is sometimes simpler to show that a sequence is convergent by showing it is Cauchy.

Example 9.1.23. Let $x_1 := 1$ and inductively $x_{n+1} := \frac{1}{1+x_n}$ when $n \geq 1$. Then (x_n) is converges to $\frac{\sqrt{5}-1}{2}$. The first few terms of this sequence are $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \cdots$ The Fibonacci sequence make an (unexpected?) appearance in analysis. The sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ is named after Leonardo Pisano Bigollo (c. 1170, Pisa to c. 1250, Pisa) who was known as Fibonacci.

Proof. We begin by showing (x_n) Cauchy and consequently convergent. Note $x_1 \geq \frac{1}{2}$. If $x_n \geq \frac{1}{2}$, then $x_{n+1} \geq \frac{1}{1+1/2} = \frac{2}{3} \geq \frac{1}{2}$. By induction $x_n \geq \frac{1}{2}$ for all n. Hence,

$$\begin{aligned}|x_{n+1} - x_n| &= \left| \frac{1}{1+x_n} - \frac{1}{1+x_{n-1}} \right| \\ &= \frac{|x_{n-1} - x_n|}{(1+x_n)(1+x_{n+1})} \\ &\leq \frac{|x_{n-1} - x_n|}{(3/2)(3/2)} \leq \frac{1}{2} |x_n - x_{n-1}|,\end{aligned}$$

since $\frac{4}{9} < \frac{1}{2}$. It follows that $|x_3 - x_2| \leq \frac{1}{2} |x_2 - x_1|$, $|x_4 - x_3| \leq \frac{1}{2} |x_3 - x_2| \leq \frac{1}{2^2} |x_2 - x_1|$, continuing in this manner we see

$$|x_{n+1} - x_n| \leq \frac{1}{2^{n-1}} |x_2 - x_1| = \frac{1}{2^n}.$$

for all $n \geq 1$. Where the equality used $x_2 = 1/2$, so that $x_1 - x_2 = 1/2$. Hence, if $m > n$, then

$$\begin{aligned}|x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \\ &= \frac{1}{2^n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}} \right) < \frac{1}{2^{n-1}}.\end{aligned}$$

Consequently, (x_n) is Cauchy and therefore convergent.

It remains to determine the limit. Suppose $x_n \to L$. Since $x_n \geq 0$ we have $L \geq 0$. Also, $x_{n+1} \to L$ and by the algebra of limits $\frac{1}{1+x_n} \to \frac{1}{1+L}$, using $1 + L \neq 0$. Hence $x_{n+1} = \frac{1}{1+x_n}$ implies $L = \frac{1}{1+L}$. Solving this equation for its positive root gives $L = \frac{\sqrt{5}-1}{2}$. ☺

Remark 9.1.24. The argument, used to show that if $|x_{n+1} - x_n| \leq \frac{1}{2} |x_n - x_{n-1}|$ for all $n \geq 2$, then (x_n) is Cauchy, is a standard technique.

Sequences and Continuity

Definition 9.1.25. Let M and N be subsets of the complex plane. Let $f : M \to N$ be a function and let $a \in M$. We say that f is *sequentially continuous* at a, if $(f(a_n))$ converges to $f(a)$, whenever (a_n) converges to a.

We will show that a function is continuous if and only if it is sequentially continuous. As an application, we show that the sequence determined by $x_1 := 1$ and $x_{n+1} := \sqrt{1+x_n}$ is Cauchy and use sequential continuity of the function $f(y) = \sqrt{1+y}$ to find the limit of (x_n).

Theorem 9.1.26 (Sequential Continuity). *Let D be a subset of the complex plane $f : D \to \mathbb{C}$ and $a \in D$. Then f is continuous at a iff $a_n \in D$ and $a_n \to a$ implies $f(a_n) \to f(a)$.*

Proof. Suppose f is continuous at a and $a_n \to a$. Let $\varepsilon > 0$ be given. Pick $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. Pick N such that $n \geq N$ implies $|a_n - a| < \delta$. Then $n \geq N \implies |a_n - a| < \delta \implies |f(a_n) - f(a)| < \varepsilon$.

Conversely, suppose f is not continuous at a. Since f is not continuous at a there is an $\varepsilon > 0$ so that for any $\delta > 0$ there is at least one $x \in B_M(a,\delta)$ such that $f(x) \notin B_N(f(a),\varepsilon)$. Let $\delta = \frac{1}{n}$, there is an $a_n \in D$ such that $|a - a_n| < \frac{1}{n}$ and $|f(a) - f(a_n)| \geq \varepsilon$. Consequently, $a_n \to a$ and $f(a_n) \not\to f(a)$. ☺

Example 9.1.27. Let $x_1 := 1$ and inductively $x_{n+1} := \sqrt{1+x_n}$. Then $x_n \to \frac{\sqrt{5}+1}{2}$. The first few terms of this sequence are

$$1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, \cdots .$$

Proof. As in Example 9.1.23 we begin by showing that (x_n) is Cauchy. By induction $x_n \geq 1$ for all n. Let $f(y) := \sqrt{1+y}$. Then $x_{n+1} = f(x_n)$ and $f'(y) = \frac{1}{2\sqrt{1+y}} \leq \frac{1}{2}$ for all $y \geq 0$. By the Mean Value Theorem

$$\begin{aligned} |x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\ &= \left|f'(c)\right| |x_n - x_{n-1}| \\ &\leq \frac{1}{2} |x_n - x_{n-1}| . \end{aligned}$$

Repeating an argument from Example 9.1.23 it follows that (x_n) is Cauchy.

Let $L := \lim_{n\to\infty} x_n$. Since f is continuous $f(x_n) \to f(L)$ by sequential continuity. Hence $x_{n+1} = f(x_n)$ implies $L = f(L)$, that is $L = \sqrt{1+L}$. Solving this equation for its positive root gives $L = \frac{\sqrt{5}+1}{2}$. ☺

Monotone Sequences in $\mathbb{R}$

A sequence (x_n) of real numbers is *increasing*, if $x_n \leq x_{n+1}$ for all n, it is *strictly increasing*, if $x_n < x_{n+1}$ for all n. If $x_n \geq x_{n+1}$ (resp. $x_n > x_{n+1}$) for all n, then (x_n) is *decreasing* (resp. *strictly decreasing*). A sequence is *monotone*, if it is increasing or decreasing. Similarly, a sequence is *strictly monotone* if it is either strictly increasing or strictly decreasing.

The following result is similar to Exercise 5.1.2.

Theorem 9.1.28 (Monotone Convergence). *If* (a_n) *is increasing, then* $a_n \to \sup\{a_n \mid n \in \mathbb{N}\}$.

Proof. There are two cases. $\sup\{a_n \mid n \in \mathbb{N}\} < \infty$ and $\sup\{a_n \mid n \in \mathbb{N}\} = \infty$.

Suppose $\sup\{a_n \mid n \in \mathbb{N}\} < \infty$. In this case $\{a_n \mid n \in \mathbb{N}\}$ is bounded and $b := \sup\{a_n \mid n \in \mathbb{N}\} < \infty$ is the least upper bound for $\{a_n \mid n \in \mathbb{N}\}$. For any $\varepsilon > 0$, $b - \varepsilon < b$, so $b - \varepsilon$ is not an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. Hence, for some N, $b - \varepsilon < a_N$. Since (a_n) is increasing, $b - \varepsilon < a_N \leq a_{n,}$ for all $n \geq N$. Consequently,

$$0 \leq b - a_n < \varepsilon \text{ for all } n \geq N.$$

Thus $a_n \to b$.

Suppose $\sup\{a_n \mid n \in \mathbb{N}\} = \infty$. In this case $\{a_n \mid n \in \mathbb{N}\}$ does not have an upper bound. So for any K, there is an N, such that $K < a_N$. Since (a_n) is increasing

$$K < a_N \leq a_n \text{ for all } n \geq N.$$

Thus $a_n \to \infty$. ☺

Corollary 9.1.29. *If* (a_n) *is decreasing, then* $a_n \to \inf\{a_n \mid n \in \mathbb{N}\}$.

Proof. Applying the theorem to $(-a_n)$, gives

$$-a_n \to \sup\{-a_n \mid n \in \mathbb{N}\} = -\inf\{a_n \mid n \in \mathbb{N}\}.$$

Hence an application of the theorem completes the proof. ☺

Example 9.1.30. It follows from monotone convergence that $0 < r < 1$ implies $r^n \to 0$. We already established this in Sect. 1.7 using a different argument.

Proof. Let $a_n := r^n$. Multiplying $r < 1$ by r^n gives $a_{n+1} = r^{n+1} < r^n = a_n$, so (a_n) is decreasing and bounded below by 0. Hence, by monotone convergence, $a_n = r^n$ is convergent to $L := \inf\{r^n \mid n \in \mathbb{N}\}$ as $n \to \infty$. It remains to show that $L = 0$.

Since $r > 0$, $a_n = r^n > 0$, in particular, $L \geq 0$. Now $b_n := a_{n+1} = r^{n+1}$ is a subsequence of (a_n), hence $b_n \to L$. But $b_n = r a_n \to rL$. Since a sequence has at most one limit $L = rL$. So $(1 - r)L = L - rL = 0$. Consequently, $1 - r \neq 0$ implies $L = 0$. ☺

Limit Superior and Limit Inferior

Recall, from Sect. 1.2 and the beginning of this section, that a point p is an accumulation point of the *set* A, if there are points in $A \setminus \{p\}$ arbitrarily close to p. The following is a related notion for *sequences* of complex numbers, it requires that there are arbitrarily large subscripts for which x_k is close to p, and allows $x_k = p$ for those subscripts:

Definition 9.1.31. Let $x_j \in \mathbb{C}$. A point $p \in \mathbb{C}$ is a *limit point* of (x_j), if

$$\forall \varepsilon > 0, \forall m \in \mathbb{N}, \exists j \in \mathbb{N}, j \geq m \text{ and } |x_j - p| < \varepsilon$$

Example 9.1.32. Some examples of limits points of sequences are:

1. 1 and 2 are limit points of the sequence 0,1,2,1,2,1,2,...
2. 0 and 1 are limit points of the sequence $\frac{1}{2}$, 1, $\frac{1}{4}$, 1, $\frac{1}{8}$, ...; i.e., of the sequence (x_n) determined by $x_n := \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$.
3. 1 is a limit point of the sequence (y_n) determined by

$$y_n := \begin{cases} 1 & \text{if } n \text{ is a power of 2} \\ n & \text{if } n \text{ is not a power of 2} \end{cases}.$$

For the related sets we have:

4. The set $\{0, 1, 2, 1, 2, 1, 2, \ldots\} = \{0, 1, 2\}$ does not have any accumulation points.
5. 0 is the only accumulation point of the set $\{x_n \mid n \in \mathbb{N}\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \ldots\right\}$.
6. The infinite set $\{y_n \mid n \in \mathbb{N}\}$ does not have any accumulation points.

Theorem 9.1.33. *Let (x_n) be a sequence and let p be a point. Then p is a limit point of (x_j) iff there exists a subsequence (x_{i_n}) such that $x_{i_n} \to p$ as $n \to \infty$.*

Proof. Suppose there is a subsequence (x_{i_n}) such that $x_{i_n} \to p$ as $n \to \infty$. Let $\varepsilon > 0$ and $m \in \mathbb{N}$ be given. Pick N such that $|x_{i_n} - p| < \varepsilon$ for all $n \geq N$. Then $n = \max\{N, m\}$ implies $|x_{i_n} - p| < \varepsilon$. Since $i_n \geq n \geq m$ it follows that p is a limit point of (x_n).

Conversely, suppose p is a limit point of (x_n). Setting $\varepsilon = 1$ and $m = 1$ gives $n_1 := j \geq 1$ such that $|p - x_{n_1}| < 1$. Setting $\varepsilon = 1/2$ and $m = 1 + n_1$ gives $n_2 := j \geq 1 + n_1$ such that $|p - x_{n_2}| < 1/2$. Setting $\varepsilon = 1/3$ and $m = 1 + n_2$ gives $n_3 := j \geq 1 + n_2$ such that $\left|p - x_{n_3}\right| < 1/3$. Continuing in this manner we get $n_1 < n_2 < n_3 < \cdots$ such that $\left|p - x_{n_k}\right| < 1/k$. Thus $\left(x_{n_k}\right)$ is a subsequence of (x_k) converging to p. ☺

Suppose (x_k) is a sequence of real numbers. Let

$$y_k := \sup\{x_j \mid j \geq k\}.$$

The sequence (y_k) is decreasing, in fact, $y_{k+1} \leq y_k$, since $\{x_j \mid j \geq k+1\}$ is a subset of $\{x_j \mid j \geq k\}$. By monotone convergence $\lim_{k\to\infty} y_k$ exists, it might equal $-\infty$. We define the *limit superior* of the sequence (x_j) to be

$$\limsup_{j\to\infty} x_j := \lim_{k\to\infty} y_k = \lim_{k\to\infty} \sup_{j\geq k} x_j,$$

where $\sup_{j\geq k} x_j = \sup\left\{x_j \mid j \geq k\right\}$. We will also write

$$\overline{\lim_{j\to\infty}} x_j := \limsup_{j\to\infty} x_j.$$

Example 9.1.34. Let $x_j := 1 + (-1)^j \frac{1}{j}$ for all integers $j \geq 1$. Then

$$y_k = \sup\left\{1 + (-1)^j \frac{1}{j} \mid j \geq k\right\} = \begin{cases} 1 + \frac{1}{k} & \text{if } k \text{ is even} \\ 1 + \frac{1}{k+1} & \text{if } k \text{ is odd} \end{cases},$$

hence $\limsup_{j\to\infty} x_j = \lim_{k\to\infty} y_k = 1$. Mixing in some numbers smaller than one does not change the limit superior. For example, if $b_{2j-1} := 1 + (-1)^j \frac{1}{j}$ and $b_{2j} := (-1)^j \frac{1}{j}$ for all $j \in \mathbb{N}$, then $\limsup_{j\to\infty} b_j = 1$.

Exercise 9.1.35. If $x_k := -k$, then $\overline{\lim}_{j\to\infty} x_j = -\infty$.

Proposition 9.1.36. *If (x_j) is a bounded sequence of real numbers, then $\limsup x_j$ is a limit point of (x_j).*

Proof. Let $y := \limsup_{j\to\infty} x_j$ and $y_k := \sup\left\{x_j \mid j \geq k\right\}$. Let $\varepsilon > 0$ and m be given. Pick N such that $k \geq N$ implies $|y - y_k| < \varepsilon/2$. Let $k := \max\{N, m\}$. Since $y_k - \frac{\varepsilon}{2}$ is not an upper bound for $\{x_j \mid j \geq k\}$ there is a $j \geq k$ such that $y_k - \frac{\varepsilon}{2} < x_j \leq y_k$. Now $j \geq k \geq m$ and

$$|x_j - y| \leq |x_j - y_k| + |y_k - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus y is a limit point of (x_j). ☺

The limit superior of a sequence is its largest limit point:

Proposition 9.1.37. *Let (x_j) be a sequence of real numbers. If p is a limit point of the sequence (x_j), then $p \leq \limsup x_j$.*

Proof. Let p be some limit point of (x_j) and let $y := \limsup_{j\to\infty} x_j$ be the limit superior of (x_j). We need to show that $p \leq y$. Let $y_k := \sup\{x_j \mid j \geq k\}$ and let (x_{n_j}) be some subsequence converging to p. Now $x_{n_k} \in \{x_j \,|\, j \geq k\}$ since $n_k \geq k$. Hence $x_{n_k} \leq y_k$. Since $x_{n_k} \to p$ and $y_k \to y$ as $k \to \infty$ we have $p \leq y$. ☺

Similar to the definition of the limit superior, the sequence

$$z_k = \inf_{j\geq k} x_j := \inf\left\{x_j \mid j \geq k\right\}$$

is an increasing sequence and we define the *limit inferior* by

$$\underline{\lim} x_j = \liminf_{j\to\infty} x_j := \lim_{k\to\infty} \inf_{j\geq k} x_j = \lim_{k\to\infty} z_k.$$

The reader may verify that the limit inferior of a sequence is its smallest limit point. Since $\inf\{x_j \,|\, j \geq k\} \leq \sup\{x_j \,|\, j \geq k\}$ for all k, we have

$$\liminf_{k\to\infty} x_k \leq \limsup_{k\to\infty} x_k.$$

Example 9.1.38. If $x_1, x_2, x_3, \ldots$ is an enumeration of the rationals in $[0,1]$, then density of rationals implies that every point in $[0,1]$ is a limit point of the sequence (x_n).

Below we give alternative characterizations of the limit superior and the limit inferior of sequences. These characterizations are motivated by a restatement of the definition of the limit of a sequence.

Lemma 9.1.39. *If p, x_n are of real numbers, then $x_n \to p$ as $n \to \infty$ iff for any $\varepsilon > 0$ the sets $\{k \mid x_k < p - \varepsilon\}$ and $\{k \mid p + \varepsilon < x_k\}$ both are finite.*

Proof. Follows directly from the definition of the limit of a sequence. ☺

Exercise 9.1.40. Suppose (x_n) is a bounded sequence of real numbers. A real number t is the limit superior of (x_k) iff for any $\varepsilon > 0$ (*i*) the set $\{k \mid t - \varepsilon < x_k\}$ is infinite and (*ii*) the set $\{k \mid t + \varepsilon < x_k\}$ is finite.

Exercise 9.1.41. Suppose (x_n) is a bounded sequence of real numbers. A real number t is the limit inferior of (x_k) iff for any $\varepsilon > 0$ (*i*) the set $\{k \mid t - \varepsilon < x_k\}$ is finite and (*ii*) the set $\{k \mid t + \varepsilon < x_k\}$ is infinite.

The following characterization of the existence of the limit of a sequence in term of lim sup and lim inf is a direct consequence of Lemma 9.1.39, Exercise 9.1.40 and Exercise 9.1.41. We give a different proof below.

Theorem 9.1.42. *Let (x_j) be a sequence of real numbers. The sequence (x_j) is convergent iff $\liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k$.*

Proof. Suppose $\liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k$. Let

$$x := \liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k.$$

Let $y_k := \sup_{j\geq k} x_j$ and $z_k := \inf_{j\geq k} x_j$. Then

$$z_k \leq x_j \leq y_k \text{ for } j \geq k.$$

In particular, $z_k \leq x_k \leq y_k$, since $z_k \to x$ and $y_k \to x$, then $x_k \to x$.

Conversely, suppose $x_n \to x$. Since the limit superior is a limit point there is a subsequence (x_{i_n}) of (x_n) such that $x_{i_n} \to \limsup_{k\to\infty} x_k$. Since a subsequence has the same limit a the "mother" sequence $x = \limsup_{k\to\infty} x_k$. Similarly, the limit inferior equals x. ☺

Example 9.1.43. If $a_n := (-1)^{n+1}$ and $b_n := -a_{n,}$ then $a_n + b_n = 0$ for all n and $\limsup a_n = \limsup b_n = 1$. Consequently, $\limsup(a_n+b_n) = 0 < 2 = (\limsup a_n) + (\limsup b_n)$.

Similarly, $\limsup(a_n b_n) = -1 < 1 = (\limsup a_n)(\limsup b_n)$.

9.2 Sequences of Functions

In this section, we discuss pointwise and uniform convergence of sequences of functions. We also discuss how pointwise and uniform convergence interacts with the basic notions of calculus: continuity, the derivative, and the integral. Uniform convergence is important because it allows us to establish several results about interchanging two limits.

Uniform Convergence

Let D be a subset of the complex plane. Suppose $f_n, f : D \to \mathbb{C}$. If for every x in D, $f_n(x) \to f(x)$ as $n \to \infty$, then f_n *converges pointwise* to f on D. In symbols

$$\forall x \in D, \forall \varepsilon > 0, \exists N = N(x,\varepsilon) > 0, n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Uniform convergence on D requires N to be independent of x. More precisely, the sequence f_n *converges uniformly* to f on D provided

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) > 0, \forall x \in D, n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

See Fig. 9.1. We will write $f_n \to f$, if f_n converges pointwise to f, and $f_n \rightrightarrows f$, if f_n converges uniformly to f. To emphasize this notation:

$$\begin{array}{ll} f_n \to f & \text{pointwise convergence} \\ f_n \rightrightarrows f & \text{uniform convergence} \end{array}$$

Example 9.2.1. Let $f_n(x) := x/n$ and $f(x) := 0$. Then
(i) f_n converges pointwise to f on $\mathbb{R}$,
(ii) f_n does not converge uniformly to f on $\mathbb{R}$, and
(iii) f_n converges uniformly to f on any compact interval $[a,b]$.

Proof. (i) Fix x. Let $\varepsilon > 0$ be given. Pick N such that $\frac{|x|}{N} < \varepsilon$. (For example, let $N := 1 + \frac{|x|}{\varepsilon}$.) Then $n \geq N$ implies $|f_n(x)| = \frac{|x|}{n} \leq \frac{|x|}{N} < \varepsilon$.

(ii) Let $\varepsilon := 1$. Let N be given. Let $n := N$ and $x := N+1$, then $f_N(x) = \frac{N+1}{N} > 1 = \varepsilon$.

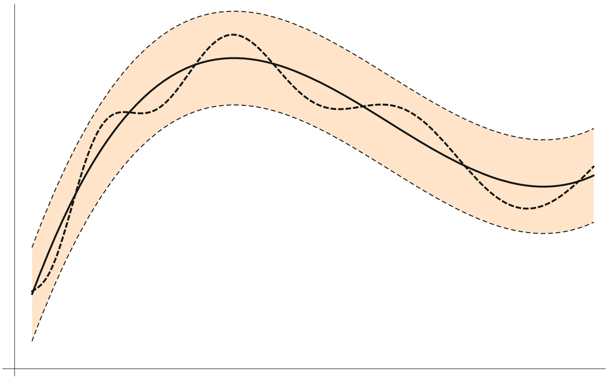

Fig. 9.1 A function f, the band determined by $f(x)-\varepsilon < g(x) < f(x)+\varepsilon$ and a function g inside this bound

(*iii*) Let $\varepsilon > 0$ be given. Pick $k > 0$ such that $[a,b] \subseteq [-k,k]$. Let $N > \frac{k}{\varepsilon}$, e.g., $N := 1+\frac{k}{\varepsilon}$. Let $x \in [a,b]$ and $n \geq N$. Then $|f_n(x)| = \frac{|x|}{n} \leq \frac{k}{N} < \varepsilon$. ☺

Uniform Convergence and Continuity

Uniform convergence is of interest because uniform convergence passes some nice properties of the functions f_n on to the limit function f. We establish some examples below.

Example 9.2.2. Let

$$f_n(x) := \begin{cases} -1 & \text{if } x < -\frac{1}{n} \\ nx & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} < x \end{cases} \text{ and } f(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \end{cases},$$

then each f_n is continuous on $\mathbb{R}$, f is not continuous at 0, and $f_n \to f$. See Fig. 9.2.

The example shows that the *pointwise* limit of a sequence of continuous functions need not be continuous, but the *uniform* limit of a sequence of continuous functions is a continuous function. Since continuity of g at x_0 means $g(x) \to g(x_0)$ as $x \to x_0$ we can think of this result as being about interchanging two limits: the limit in n and the limit in x.

Theorem 9.2.3. *Let $f, f_n : D \to \mathbb{C}$ and let $x_0 \in D$. Suppose $f_n \rightrightarrows f$ on D.*

1. *If each f_n is continuous at x_0, then f is continuous at x_0.*
2. *If each f_n is uniformly continuous on D, then f is uniformly continuous on D.*

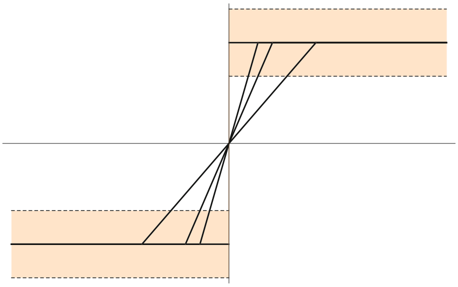

Fig. 9.2 Illustrating that f_n does not converge uniformly to f in Example 9.2.2. The figure shows f, the band $f(x)-\varepsilon < g(x) < f(x)+\varepsilon$ with $\varepsilon = \frac{1}{3}$, and the functions f_n, $n = 1,2,3$

Proof. Let $\varepsilon > 0$ be given. [The proof is in two steps, first we find a good N, then we find δ.] By the triangle inequality

$$|f(x)-f(x_0)| \le |f(x)-f_n(x)| + |f_n(x)-f_n(x_0)| + |f_n(x_0)-f(x_0)|.$$

Since $\varepsilon/3 > 0$, uniform convergence gives an N, such that $|f(t)-f_N(t)| < \varepsilon/3$ for all t in D. Hence

$$|f(x)-f(x_0)| < \frac{\varepsilon}{3} + |f_N(x)-f_N(x_0)| + \frac{\varepsilon}{3}.$$

Since f_N is continuous at x_0 and $\varepsilon/3 > 0$, there is a $\delta > 0$, such that $|x-x_0| < \delta$ implies $|f_N(x)-f_N(x_0)| < \varepsilon/3$. Hence

$$|x-x_0| < \delta \implies |f(x)-f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This proves the first part. ☺

Exercise 9.2.4. Prove the second part of the theorem.

Uniform Convergence and Integrals

We show pointwise limits of integrable functions need not be integrable and that uniform limits of integrable functions are integrable.

Exercise 9.2.5. Let (x_n) be an enumeration of the rationals in the interval $[0,1]$. Let $f_n(x) = 0$ except $f_n(x) = 1$ for $x \in \{x_1,\ldots,x_n\}$. Prove each f_n is integrable and the pointwise limit function is not integrable.

Theorem 9.2.6. *If $f_n : [a,b] \to \mathbb{R}$ are integrable and converge uniformly to f, then f is integrable and $\int_a^b f_n \to \int_a^b f$.*

Proof. We cannot use the Evaluation Theorem for Integrals (Theorem 7.2.5), because we have not assumed that $I = \lim_{n\to\infty} \int_a^b f_n$ exists.

Let $\varepsilon > 0$ be given. Let N be so large that $n \geq N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$, for all $x \in [a,b]$. Let s_n (resp. S_n) be a lower (resp. upper) step function for f_n, such that $\sum S_n - \sum s_n < \frac{\varepsilon}{3}$. Then

$$f(x) < f_n(x) + \frac{\varepsilon}{3(b-a)} \leq S_n(x) + \frac{\varepsilon}{3(b-a)}, \text{for all } x \in [a,b].$$

So $S = S_n + \frac{\varepsilon}{3(b-a)}$ is an upper step function for f. Similarly, $s = s_n - \frac{\varepsilon}{3(b-a)}$ is a lower step function for f. But

$$\begin{aligned}\sum S - \sum s &= \sum \left(S_n + \frac{\varepsilon}{3(b-a)}\right) - \sum \left(s_n - \frac{\varepsilon}{3(b-a)}\right) \\ &= \sum S_n - \sum s_n + \sum \frac{2\varepsilon}{3(b-a)}(x_i - x_{i-1}) \\ &= \sum S_n - \sum s_n + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.\end{aligned}$$

It follows from Theorem 7.2.3 that f is integrable. Furthermore, for $n \geq N$ we have

$$\sum s \leq \sum s_n \leq \int_a^b f_n \leq \sum S_n \leq \sum S$$

and

$$\sum s \leq \int_a^b f \leq \sum S.$$

So both $\int_a^b f$ and $\int_a^b f_n$ are in the interval $[\sum s, \sum S]$. Hence

$$\left| \int_a^b f - \int_a^b f_n \right| \leq \sum S - \sum s < \varepsilon.$$

Consequently, $\int_a^b f_n$ converges to $\int_a^b f$. ☺

Remark 9.2.7. The proof does not use that the f_n's form a sequence. For example, a very similar proof shows that, if f_t is integrable on $[a,b]$ for all t and $f_t \rightrightarrows f$ as $t \to t_0$, then f is integrable on $[a,b]$ and $\int_a^b f_t \to \int_a^b f$.

Here $f_t \rightrightarrows f$ as $t \to t_0$ means

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in [a,b], 0 < |t - t_0| < \delta \implies |f_t(x) - f(x)| < \varepsilon.$$

We just replaced the discrete parameter n with the continuous parameter t in the definition of uniform convergence. This is similar to some of the variations on limits in Sects. 1.5 and 1.6.

Remark 9.2.8. We can rewrite the limit $\int_a^b f_n \to \int_a^b f$ in the theorem as

$$\lim_n \int_a^b f_n = \int_a^b \lim_n f_n.$$

Since the integral is a kind of limit process, this is an example of interchanging two limits.

Uniform Convergence and Derivatives

The following result is about a sequence of differentiable functions passing differentiability on to the limit function.

Corollary 9.2.9. *Let f_n, f, and g be real valued functions defined on some interval I. If each f_n have a continuous derivative on I, the sequence f_n converges pointwise to f on I, and the sequence f_n' converges uniformly to g on I, then f is differentiable on I and $f' = g$.*

One can prove this without using integrals, done this way the proof is a bit tedious. We hide the complications by using the theory of the Riemann integral.

Proof. Let x and x_0 be points in I. By the FTC-Evaluation we can write

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'.$$

Letting $n \to \infty$ we get

$$f(x) = f(x_0) + \int_{x_0}^x g$$

where we used the pointwise convergence of f_n and the uniform convergence of f_n'. Now g is continuous, since it is the uniform limit of continuous functions. Consequently, it follows from FTC-Derivative that $f' = g$. ☺

Partial Derivatives

Given a function $g : [a,b] \times [c,d] \to \mathbb{R}$ of two variables, supposing $y \to g(x,y)$ is integrable for each x we can consider the function

$$f(x) := \int_c^d g(x,y)\,dy$$

defined for $x \in [a,b]$. Imposing suitable conditions on g we establish continuity and differentiability of the function f.

Exercise 9.2.10. If $g : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then $f(x) = \int_c^d g(x,y)\,dy$ is continuous $[a,b] \to \mathbb{R}$.

If g is a function of two variables, for example $g(x,y)$, then $g_x = \partial_x g = \frac{\partial g}{\partial x}$ is the derivative of g with respect to the first variable, that is

$$g_x(s,t) := \lim_{h\to 0} \frac{g(s+h,t)-g(s,t)}{h}.$$

The function g_x is called a *partial derivative* of g. The partial derivative with respect to the second variable g_y is defined in a similar manner. We can now establish a result about interchanging integration and differentiation. This is one of Gottfried Wilhelm von Leibniz (1 July 1646, Leipzig to 14 November 1716, Hanover) many contributions to the early development of Calculus.

Theorem 9.2.11 (Leibniz Integral Rule). *Suppose $g : \mathbb{R}\times[c,d] \to \mathbb{C}$ is continuous and that the partial derivative g_x exists and is continuous on $\mathbb{R}\times[c,d]$. Let $f : \mathbb{R}\to\mathbb{C}$ be determined by*

$$f(t) := \int_c^d g(t,y)\,dy,$$

then f is differentiable on $\mathbb{R}$ and

$$f'(t) := \int_c^d g_x(t,y)\,dy.$$

Proof. Fix a point t_0. Clearly,

$$\frac{f(t)-f(t_0)}{t-t_0} = \int_c^d \frac{g(t,y)-g(t_0,y)}{t-t_0}\,dy.$$

Hence, if we know

$$\frac{g(t,y)-g(t_0,y)}{t-t_0} \rightrightarrows g_x(t_0,y) \text{ as } t\to t_0,$$

the convergence being uniform in y, then we can finish the proof by interchanging the limit and the integral (Theorem 9.2.6).

By the Mean Value Theorem there is an $s_{t,y}$ between t_0 and t such that

$$\frac{g(t,y)-g(t_0,y)}{t-t_0} = g_x(s_{t,y},y).$$

(Note, we do not know anything about $s_{h,y}$ except it is between t and t_0. For example, there is no reason to think $y \to s_{t,y}$ is continuous or even integrable.) To complete the proof, we will show $g_x(s_{t,y},y)$ converges uniformly (in y) to $g_x(t_0,y)$ as $t\to t_0$.

Let $\varepsilon > 0$ be given. We must find a $\delta > 0$, not depending on y, such that

$$|t-t_0| < \delta \text{ implies } \left|g_x(s_{t,y},y) - g_x(t_0,y)\right| < \varepsilon.$$

By uniform continuity of g_x on $[t_0-1,t_0+1]\times[c,d]$ (Exercise 5.4.7), there is a $0<\delta<1$ such that $|(u,v)-(\alpha,\beta)|<\delta$ implies $|g_x(u,v)-g_x(\alpha,\beta)|<\varepsilon$ for all pairs of points (u,v) and (α,β) in $[t_0-1,t_0+1]\times[c,d]$.

If $|t-t_0|<\delta$, then $|s_{t,y}-t_0|<\delta$, since $s_{t,y}$ is between t and t_0. Hence

$$\left|(s_{t,y},y)-(t_0,y)\right|=\left|s_{t,y}-t_0\right|<\delta$$

and consequently $\left|g_x(s_{t,y},y)-g_x(t_0,y)\right|<\varepsilon$ as we needed to show. (We chose $\delta<1$ to force $s_{t,y}$ to be in $[t_0-1,t_0+1]$.) ☺

Fubini's Theorem for Riemann Integrals★

Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous, by Exercise 9.2.10 the function $g(x):=\int_c^d f(x,y)\,dy$ is continuous and therefore integrable. Hence, we can consider the *iterated integral*

$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx.$$

Similarly, we can consider $\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$. The following theorem, named after Guido Fubini (19 January 1879, Venice to 6 June 1943, New York City), shows that these two integrals are equal. The version below was established by Paul David Gustav du Bois-Reymond (2 December 1831, Berlin to 7 April 1889, Freiburg) in 1872, 30 years prior to the publication of Fubini's work.

Theorem 9.2.12 (Fubini). *Suppose f is continuous on $[a,b]\times[c,d]$. Then*

$$\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx.$$

Proof. Let $F(x,y):=\int_a^x f(u,y)\,du$ and $\Phi(x):=\int_c^d F(x,y)\,dy$. By FTC-Derivative $F_x=f$. Hence, by Leibniz' Integral Rule $\Phi'(x)=\int_c^d F_x(x,y)\,dy=\int_c^d f(x,y)\,dy$. The rest of the proof is two applications of FTC-Evaluation:

$$\begin{aligned}\textstyle\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy &= \textstyle\int_c^d\left(\int_a^b F_x(x,y)\,dx\right)dy && \text{since } F_x=f\\ &= \textstyle\int_c^d F(b,y)-F(z,y)\,dy && \text{FTC-Evaluation}\\ &= \Phi(b)-\Phi(a) && \text{definition of } \Phi\\ &= \textstyle\int_a^b \Phi'(x)\,dx && \text{FTC-Evaluation}\\ &= \textstyle\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx && \text{Leibniz.}\end{aligned}$$

This calculation completes the proof. ☺

9.3 Convolution

Let f, g be functions defined on the real line. Suppose f, g are integrable on any compact interval and one is $= 0$ outside some compact interval, then the *convolution* of f and g is

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy. \tag{9.1}$$

The integral is not improper: If $g = 0$ outside $[a,b]$, the integral is over the interval $[a,b]$. If $f = 0$ outside $[a,b]$ the integral is over the interval $[x-b, x-a]$. Equation (9.1) used that the product of two integrable functions is integrable and that if f is integrable, so is $g(x) := f(a-x)$ for any constant a.

The change of variables $u = x - y$ shows that $f * g = g * f$. The reader may state and verify other basic algebraic properties of convolution, for example, $f * (g * h) = (f * g) * h$, $f * (g + h) = (f * g) + (f * h)$, and $f * (ag) = a(f * g)$ for any constant a.

Exercise 9.3.1. Let f and g be continuous functions defined on the real line. If one of f and g is $= 0$ outside some bounded interval, then $f * g$ is a continuous function on the real line.

Approximate Identities

A sequence (g_n) functions on the real line is called an *approximate identity*, if each g_n is integrable on any compact interval and

1. $g_n(x) \geq 0$ for all n [positivity]
2. $\int_{-\infty}^{\infty} g_n(x)\,dx = 1$ for all n [integral one]
3. For any $\delta > 0$, $\int_{|x|\geq\delta} g_n(x)\,dx \to 0$ as $n \to \infty$ [concentrated near the origin]

Where $\int_{|x|\geq\delta} f := \int_{-\infty}^{-\delta} f + \int_{\delta}^{\infty} f$.

Exercise 9.3.2. If f is a continuous on $\mathbb{R}$ and $f = 0$ outside some compact interval, then f is uniformly continuous on $\mathbb{R}$.

Exercise 9.3.3. If f is a continuous on $\mathbb{R}$ and $f = 0$ outside some compact interval, then f is bounded on $\mathbb{R}$.

Theorem 9.3.4 (Approximate Identity Lemma). *Let (g_n) be an approximate identity and let f be continuous on $\mathbb{R}$. If $f = 0$ outside some compact interval, then $f * g_n$ converges uniformly (on $\mathbb{R}$) to f as $n \to \infty$.*

Proof. Since f is bounded on $\mathbb{R}$, there is an M be such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$ be given. Use the uniform continuity of f to pick $\delta > 0$ so that $|x - y| \leq \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. Use property (3) of (g_n) to pick N so that $n \geq N$ implies $2M \int_{|x|\geq\delta} g_n(x)\,dx < \varepsilon/2$.

Property (2) of (g_n) implies $f(x) = f(x)\int_{-\infty}^{\infty} g_n(y)\,dy = \int_{-\infty}^{\infty} f(x)g_n(y)\,dy$, hence

$$f * g_n(x) - f(x) = \int_{-\infty}^{\infty} (f(x-y) - f(x))g_n(y)\,dy.$$

Writing $\int_{-\infty}^{\infty} = \int_{|y|\geq\delta} + \int_{|y|\leq\delta}$ gives the equality below.

$$\begin{aligned}
|f * g_n(x) - f(x)| &= \left| \int_{|y|\geq\delta} (f(x-y) - f(x))g_n(y)\,dy \right. \\
&\qquad \left. + \int_{|y|\leq\delta} (f(x-y) - f(x))g_n(y)\,dy \right| \\
&\leq \int_{|y|\geq\delta} 2Mg_n(y)\,dy + \int_{|y|\leq\delta} \frac{\varepsilon}{2} g_n(y)\,dy \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The first inequalities uses $|f(x-y) - f(x)| \leq 2M$ to bound the first term and that $|y| \leq \delta$ implies $|f(x) - f(y)| < \varepsilon/2$ to bound the second term.

The second inequality uses $2M\int_{|x|\geq 1/n} g_n(x)\,dx < \varepsilon/2$ to bound the first term and that properties (1) and (2) of (g_n) implies

$$\int_{|y|\leq 1/n} \frac{\varepsilon}{2} g_n(y)\,dy \leq \frac{\varepsilon}{2}\int_{-\infty}^{\infty} g_n = \frac{\varepsilon}{2}$$

to bound the second term. This completes the proof. ☺

One reason that convolution is important is that the convolution of two functions tends to inherit the "good" properties of both functions. For example:

Proposition 9.3.5. *Suppose $f \in \mathscr{C}^1(\mathbb{R})$ (i.e., f' exists and is continuous on $\mathbb{R}$). If g is continuous on $\mathbb{R}$ and $= 0$ outside some compact interval, then $f * g \in \mathscr{C}^1(\mathbb{R})$ and $(f * g)' = f' * g$.*

Proof. This is essentially a special case of Theorem 9.2.11. Suppose $g = 0$ outside $[c,d]$.

Let $\phi(x,y) := f(x-y)g(y)$, then ϕ and $\phi_x(t,y) = f'(t-y)g(y)$ are continuous on $\mathbb{R}\times\mathbb{R}$. Since $(f * g)(t) = \int_c^d \phi_x(t,y)\,dy$ Theorem 9.2.11, tells us that

$$(f * g)'(t) = \int_c^d \phi_x(t,y)\,dy = f' * g(t).$$

Finally, $f' * g$ is continuous by Exercise 9.3.1. ☺

Exercise 9.3.6. Suppose $f \in \mathscr{C}^1(\mathbb{R})$ and $= 0$ outside some compact interval and g is continuous on $\mathbb{R}$, then $f * g$ is differentiable on $\mathbb{R}$ and $(f * g)' = f' * g$ on $\mathbb{R}$.

Combining this with the Approximate Identity Lemma we see that if the approximate identity (g_n) consists of $\mathscr{C}^1-$ functions, then $f * g_n$ is a uniform approximation of the continuous function f by $\mathscr{C}^1-$ functions.

9.4 The Fundamental Theorem of Algebra*

This section is an application of iterated integrals and the uniform convergence theorem for integrals. We assume familiarity with polar coordinates $z = r e^{i\theta}$ for $z \in \mathbb{C}$ and with the exponential function $e^{i\theta}$, see Chap. 11.

Theorem 9.4.1 (Fundamental Theorem of Algebra). *Let p be a polynomial of degree at least one. Then $p(z) = 0$ for some $z \in \mathbb{C}$.*

Proof. Let $p(z) = \sum_{k=0}^{n} a_k z^k$. Suppose $a_n \neq 0$ and $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $g : [0,\infty) \times [0,2\pi] \to \mathbb{C}$ determined by

$$g(r,\vartheta) = \frac{1}{p\left(re^{i\vartheta}\right)}$$

has continuous partial derivatives

$$g_r = \frac{-p'\left(re^{i\vartheta}\right)e^{i\vartheta}}{p\left(re^{i\vartheta}\right)^2} \text{ and } g_\vartheta = \frac{-p'\left(re^{i\vartheta}\right)ire^{i\vartheta}}{p\left(re^{i\vartheta}\right)^2}$$

in particular

$$g_\vartheta = ir\, g_r. \tag{9.2}$$

Let $f(r) = \int_0^{2\pi} g(r,\vartheta)\, d\vartheta$. Then $f'(r) = \int_0^{2\pi} g_r(r,\vartheta)\, d\vartheta$, by Theorem 9.2.11. Multiplying by ir and using (9.2) gives the first equality below.

$$ir\, f'(r) = \int_0^{2\pi} g_\vartheta(r,\vartheta)\, d\vartheta = g(r,2\pi) - g(r,0) = 0,$$

The middle equality uses FTC-Evaluation and the last equality is a consequence of $e^{i2\pi} = e^{i0}$.

Hence $f'(r) = 0$ for all r, thus f is constant. Specifically,

$$f(r) = f(0) = \frac{2\pi}{p(0)} \neq 0 \text{ for all } r.$$

On the other hand

$$\left|p\left(re^{i\vartheta}\right)\right| = \left|\sum_{k=0}^{n} a_k r^k e^{ik\vartheta}\right| \geq |a_n|\, r^n - \left|\sum_{k=0}^{n-1} a_k r^k e^{ik\vartheta}\right|$$

so

$$\left|p(re^{i\vartheta})\right| \geq |a_n|r^r - \sum_{k=0}^{n-1} |a_k|r^k = r^n \left(|a_n| - \sum_{k=0}^{n-1} |a_k|r^{k-n}\right). \tag{9.3}$$

Since $|a_n| - \sum_{k=0}^{n-1} |a_k|r^{k-n} \to |a_n|$ as $r \to \infty$, if follows that $\left|p(re^{i\vartheta})\right| \rightrightarrows \infty$ as $r \to \infty$, the convergence is uniform in θ, since the right hand side of (9.3) does not depend on θ. Consequently,

$$g(r, \vartheta) = \frac{1}{p(re^{i\theta})} \rightrightarrows 0 \text{ as } r \to \infty$$

uniformly in ϑ. Hence

$$f(r) = \int_0^{2\pi} g(r, \vartheta)\, d\vartheta \to 0 \text{ as } r \to \infty$$

contradicting that $f(r) = 2\pi/p(0) \neq 0$ for all r. ☺

By repeated applications of the Fundamental Theorem of Algebra and Lemma 1.4.13 it follows that any polynomial is a product of linear factors:

Corollary 9.4.2. *If* $p(z) = \sum_{k=0}^{n} a_k z^k$, $n > 0$ *and* $a_n \neq 0$, *then*

$$p(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)$$

for some $z_1, z_2, \ldots, z_n$ *in* $\mathbb{C}$.

Proof. Let z_1 be a root of p. By Lemma 1.4.13 there is a polynomial p_1 of degree $n-1$ such that $p(z) = (z - z_1)p_1(z)$. Similarly, if z_2 is a root of p_1, then $p_1(z) = (z - z_2)p_2(z)$ where p_2 has degree $n-2$. Hence $p(z) = (z - z_1)(z - z_2)p_2(z)$. Continuing in this manner we see that

$$p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)p_n(z)$$

where p_n has degree 0, hence p_n is a constant. Expanding the product, and comparing coefficients to z^n, it follows that $p_n(z) = a_n$. ☺

Problems

Problems for Sect. 9.1

1. Prove the following claims.
   (i) If $a_n = p$ for all n, then (a_n) converges to p. [*Hint*: Set $N := 1$.]
   (ii) If $p \neq q$, $a_{2n-1} = p$, and $a_{2n} = q$, then (a_n) is divergent. [*Hint*: Suppose (a_n) is convergent to a. Consider the cases $a = p$ and $a \neq p$.]
   (iii) If $\lim a_n = p$ and $\lim a_n = q$, then $p = q$. [*Hint*: If $p \neq q$, then setting $\varepsilon := |p - q|/2$ leads to a contradiction.]

2. Give a direct proof of limit point compactness by mimicking the proof of sequential compactness. [*Hint*: One step is to find an integer m such that $A_0 := \{x \in D \mid m \le x < m+1\}$ is infinite.]
3. If (x_n) is not bounded, then there is a subsequence (x_{i_n}) such that $|x_{i_n}| \to \infty$.
4. For a sequence (x_n) of real or complex numbers, let $L[(x_n)]$ be the set of limits of all convergent subsequences of (x_n), hence

$$L[(x_n)] := \{b \in \mathbb{R} \mid x_{i_n} \to b, \text{ for some subsequence of } (x_n)\}.$$

Let (a_n) be a sequence of real numbers. Let (b_n) be a sequence of points in $L[(a_n)]$. Suppose $b_n \to b$. Show $b \in L[(a_n)]$ by completing the following steps:
(*a*) If $t \in L[(a_n)]$ and $\varepsilon > 0$, then the set

$$\{n \in \mathbb{N} \mid |a_n - t| < \varepsilon\}$$

is infinite.
(*b*) Why is there an integer $n_1 \ge 1$ such that $|a_{n_1} - b| < \frac{1}{2}$?
(c) Why is there an integer $n_2 > n_1$ such that $|a_{n_2} - b| < \frac{1}{4}$?
(*d*) Suppose we have constructed $n_1 < n_2 < \ldots < n_k$ such that $\left|a_{n_j} - b\right| < \frac{1}{2^j}$ for $j = 1, 2, \ldots, k$. Why is there an integer $n_{k+1} > n_k$ such that $\left|a_{n_{k+1}} - b\right| < \frac{1}{2^{k+1}}$?
(*e*) Why does $a_{n_k} \to b$?

Cauchy Sequences

5. Let (x_n) be a sequence of complex numbers. If there is a constant $c < 1$, such that

$$|x_{n+1} - x_n| \le c|x_n - x_{n-1}|$$

for all $n \ge 2$, then (x_n) is convergent.
6. If $x_n := \log(n)$, then $x_{n+1} - x_n \to 0$ and (x_n) is not Cauchy.
7. Give an example of a sequence (x_n) such that $|x_{n+1} - x_n| \le 1/n$ and (x_n) is not Cauchy.

Sequences and Continuity

8. Give an example of a continuous function f and a divergent sequence (a_n) such that $(f(a_n))$ is convergent.

To simplify the notation we will say that a sequence of points in A is *convergent in* A, if the sequence is convergent and the limit is a point in A. That is, if A is a subset of the complex plane, $a_n \in A$ for all n, and $a_n \to a$ for some $a \in A$, then

we say (a_n) is convergent *in* A. For example, $(1/n)$ is convergent in $[0,2]$ and $(1/n)$ is not convergent in $[0,2]$.
The goal of the following problems is to extend the sequential continuity theorem slightly. Namely to establish

Corollary 9.4.3. *Let M and N be subsets of the complex plane. Let $f : M \to N$ be a function and let $a \in M$. Suppose that $(f(a_n))$ is convergent in N, whenever $a_n \to a$ in M. Then f is continuous at a.*

9. Let $f : M \to N$ be a function and let $a \in M$. Suppose $x_n \to a \implies (f(x_n))$ is convergent. Let $a_n \to a$ and $b_n \to a$. If $f(a_n) \to c$ and $f(b_n) \to d$, prove that $c = d$. [*Hint*: Let $x_1 := a_1$,$x_2 := b_1$, $x_3 := a_2$, $x_4 := b_2$, etc.]
10. Let $f : M \to N$ be a function and let $a \in M$. Suppose $x_n \to a \implies (f(x_n))$ is convergent. Let $a_n \to a$. Prove $f(a_n) \to f(a)$. [*Hint*: Use the previous problem with $b_n := a$ for all n.]
11. Prove the corollary above.
    The next group of problems explore connections between uniform continuity and sequences.
12. Let $f : D \to \mathbb{C}$. Then f is uniformly continuous on D iff for any sequences (x_n) and (y_n) is D, $x_n - y_n \to 0$ implies $f(x_n) - f(y_n) \to 0$.
13. Let$f : D \to \mathbb{C}$ be uniformly continuous on D. If (x_n) is a Cauchy sequence in D, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{C}$.
14. Let $f(x) := x^2$. Show f maps Cauchy sequences onto Cauchy sequences and f is not uniformly continuous. [You may take the domain of f to be $\mathbb{R}$ or $\mathbb{C}$.] Hence the converse in the previous problem fails.

Monotone Sequences

15. Let $x_n := \frac{1+n}{1+2n}$.
    (a) Use Monotone Convergence to prove that (x_n) is convergent.
    (b) Use the definition of the limit of a sequence to show that $x_n \to \frac{1}{2}$ as $n \to \infty$.
16. Let
    $$x_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$
    Use Monotone Convergence to show that (x_n) is convergent. The limit is $\log(2)$, but you need not prove this fact.
17. Let $(T_n \exp)(x)$ be the nth Taylor polynomial of exp at $x_0 = 0$. Prove the sequence $((T_n \exp)(1))$ is an increasing sequence converging to $\exp(1) = e$.

Limit Superior and Limit Inferior

18. Give an example of a sequence whose set of limits points equals the set
    $$\{0\} \cup \left\{\tfrac{1}{k} \mid k = 1,2,\dots\right\}.$$

19. (i) If $a_n := 1 + (-1)^n$ and $b_n := 1 - (-1)^n$ calculate

$$\begin{aligned}&\limsup\,(a_n b_n)\\&(\limsup a_n)\,(\limsup b_n)\\&\limsup\,(a_n + b_n)\ \text{ and}\\&(\limsup a_n) + (\limsup b_n)\end{aligned}$$

(ii) Repeat part (i) for the limit inferior.
20. $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$
21. $\liminf a_n + \liminf b_n \le \liminf(a_n + b_n)$.
22. If t is a limit point of (x_k) then

$$\liminf x_k \le t \le \limsup x_k.$$

23. Suppose $a_n > 0$. Prove $\frac{a_{n+1}}{a_n} \to L$ implies $a_n^{1/n} \to L$.
24. If $\limsup a_n = \infty$, then there is a subsequence (a_{i_n}) such that $a_{i_n} \to \infty$.
25. Formulate and prove the analogue of Exercise 9.1.40 for the limit inferior.
26. Verify the claim in Example 9.1.38.
27. Give a detailed proof of Lemma 9.1.39.
28. Use Lemma 9.1.39, Exercises 9.1.40 and 9.1.41 to give an alternative proof of Theorem 9.1.42.
29. For a sequence (a_n) let $L(a_n)$ denote the set of limits of all convergent subsequences of (a_n). Does there exists a sequence (a_n) in the closed interval $[0,1]$ such that
a. $L(a_n)$ has three elements?
b. $L(a_n)$ is countably infinite?
c. $L(a_n)$ is the set of rational numbers in $[0,1]$?
If you answers "yes" to one of the parts above, provide an example of such a sequence. If you answer "no" prove that no such sequence can exist.

Problems for Sect. 9.2

1. Let $f_n(x) := x\left(\frac{1}{n} - x\right)$ and $f(x) := -x^2$.
(i) $f_n \to f$ on $\mathbb{R}$.
(ii) $f_n \not\rightrightarrows f$ on $\mathbb{R}$.
(iii) $f_n \rightrightarrows f$ on the closed interval $[0,1]$.
2. Let $f_n(x) := x^n$ and $f(x) := 0$.
(i) Let $0 < a < 1$. Prove $f_n \rightrightarrows f$ on $[0,a]$.
(ii) Find a function g such that $f_n \to g$ on $[0,1]$.
(iii) Prove $f_n \not\rightrightarrows g$ on $[0,1]$.

3. [Dini's Theorem] If $f_n \searrow 0$ pointwise on $[a,b]$, then $f_n \rightrightarrows 0$ on $[a,b]$. Consequently:

Theorem 9.4.4. *Monotone convergence to a* continuous *function implies uniform convergence.*

4. Give an example of functions $f, f_n : [0,1] \to \mathbb{R}$ and $x, x_n \in [0,1]$, such that each f_n is continuous, $f_n \to f$, $x_n \to x$ and $f_n(x_n) \not\to f(x)$.
5. Let $f_n, f : [0,1] \to \mathbb{R}$ and $x, x_n \in [0,1]$, such that each f_n is continuous, $f_n \rightrightarrows f$, and $x_n \to x$. Prove $f_n(x_n) \to f(x)$.
6. Prove that a continuous function on a closed bounded interval is the uniform limit of step functions, thereby obtaining a second proof that a continuous function is integrable.
7. If $f : [0,1] \to \mathbb{R}$ is continuous, then $\int_0^1 f = \lim_{n\to\infty} \sum_{k=1}^n f\left(\frac{k}{n}\right)\frac{1}{n}$.
8. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Suppose $f(x) = 0$ for all $|x| > 1$. Then
(a) f is uniformly continuous.
(b) Suppose $y_n \to y_\infty$. If $g_n(x) := f(x - y_n)$, show $g_n \rightrightarrows g_\infty$ on $\mathbb{R}$.
9. Let $f_n : [0,1] \to \mathbb{R}$ be determined by

$$f_n(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } q < n \\ 0 & \text{otherwise} \end{cases},$$

then f_∞ is the Riemann function. Prove f_n converges uniformly to f_∞. Why is $\int_0^1 f_n = 0$ for all $n \in \mathbb{N}$? Consequently, Theorem 9.2.6 shows f_∞ is integrable with integral $= 0$.
10. Suppose $g : \mathbb{R} \times [a,b] \to \mathbb{R}$ is continuous. Show $f : \mathbb{R} \to \mathbb{R}$ determined by

$$f(x) := \int_a^b g(x,y)\,dy$$

is continuous.
11. If f_n converges uniformly to f on some open interval I and each f_n is differentiable at $x_0 \in I$, must f be differentiable at x_0?
12. Show

$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dy \right) dx \neq \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dx \right) dy.$$

Why does this not contradict Fubini's Theorem?
13. For each $n = 1, 2, 3, \ldots$ let

$$f_n(x) = \begin{cases} 1 & \text{when } n-1 \le x < n \\ 0 & \text{otherwise} \end{cases}.$$

Prove the following claims:
(a) $f_n \to 0$ pointwise on $\mathbb{R}$.
(b) $f_n \not\rightrightarrows 0$ on $\mathbb{R}$. [Does there exist a function g such that $f_n \not\rightrightarrows g$ on $\mathbb{R}$?]
(c) $\int_0^\infty f_n = 1$ for all n.
(d) $\lim_{n\to\infty}\left(\int_0^\infty f_n\right) \neq \int_0^\infty \left(\lim_{n\to\infty} f_n\right)$. [Where $\left(\lim_{n\to\infty} f\right)(x) := \lim_{n\to\infty} \left(f_n(x)\right)$.]

Problems for Sect. 9.3

1. If f is continuous on $\mathbb{R}$ and $= 0$ outside some compact interval and g is integrable on any compact interval, then $f * g$ is continuous on $\mathbb{R}$.
2. If f is continuous on $\mathbb{R}$, g is integrable on $[a,b]$, and $g = 0$ outside $[a,b]$, then $f * g$ is continuous on $[a,b]$.
3. Let $g_n(x) := \begin{cases} n & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Show that if f is continuous on $\mathbb{R}$ and $= 0$ on $\mathbb{R} \setminus [a,b]$ for some real numbers $a < b$, then $f \star g_n \rightrightarrows f$ on $\mathbb{R}$.
4. If φ is the bump function from Sect. 8.4, then $g_n(x) := n\alpha\varphi(nx)$ is an approximate identity, when $\alpha \int \varphi = 1$.
5. If f is $\mathscr{C}^2$ and g is $\mathscr{C}^3$, then $f * g$ is $\mathscr{C}^5$.
   A function is $\mathscr{C}^\infty$ if it can be differentiated arbitrarily many times. Continuous functions can be uniformly approximated by $\mathscr{C}^\infty$-functions:
6. Let $g_n(x) := n\varphi(nx)$, where φ is the bump function from Sect. 8.4. Let f be uniformly continuous on $\mathbb{R}$, prove $f * g_n \rightrightarrows (\int \varphi) f$ on $\mathbb{R}$ and that $f * g_n$ is $\mathscr{C}^\infty$.
7. Let $g : \mathbb{R} \to \mathbb{R}$ be $\mathscr{C}^1$. For any real number a, show
$$f(x) := \int_a^x g(x-t)\,dt$$
   is differentiable and find a formula for f'. [*Hint*: You can check your formula by evaluating the integral for specific functions g.]
8. Let $a, b \in \mathbb{R}$. Suppose $y = f(x)$ satisfies the differential equation
$$y'' + ay' + by = 0.$$
   In particular, we assume $f'(x)$ and $f''(x)$ exist for all $x \in \mathbb{R}$. Show $f(x)$ is $\mathscr{C}^\infty$.
9. Let $a, b \in \mathbb{R}$ and let $r : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $g(x)$ satisfies the initial value problem
$$y'' + ay' + by = 0, \quad y(0) = 0, y'(0) = 1.$$
   Show $y = f(x) := \int_0^x g(x-t)r(t)\,dt$ satisfies the initial value problem
$$y'' + ay' + by = r(x), \quad y(0) = y'(0) = 0.$$
10. Let (g_n) be an approximate identity. Let f be bounded and uniformly continuous on $\mathbb{R}$. If there is a compact interval $[a,b]$ such that all the $g_n = 0$ outside $[a,b]$, then $f * g_n$ converges uniformly (on $\mathbb{R}$) to f as $n \to \infty$.

Problems for Sect. 9.4

Solutions and Hints for the Exercises

Exercise 9.1.1. $2 \neq 1$, the fourth terms disagree. Both sets equal $\{1,2,3\}$.

Exercise 9.1.4. p is an accumulation point of A iff $A \cap B'_\varepsilon(p) \neq \emptyset$ for all ε. So, if p is an accumulation point of A, then for each n we can pick a point a_n in $A \cap B'_{1/n}(p)$.

Exercise 9.1.6. $\varphi(1) \geq 1$ since $\varphi(1) \in \mathbb{N}$. Suppose $n \in \mathbb{N}$ and $\varphi(n) \geq n$. Then $\varphi(n+1) > \varphi(n)$ since φ is strictly increasing and $n+1 > n$. Hence, $\varphi(n+1) > n$, and consequently, $\varphi(n+1) \geq n+1$.

Exercise 9.1.7. Let $\varepsilon > 0$ be given. Pick N such that $n \geq N$ implies $|a_n - L| < \varepsilon$. Then $n \geq N$ implies $i_n \geq n \geq N$, hence $|a_{i_n} - L| < \varepsilon$.

Exercise 9.1.8. (1) $\varphi(n) = n$. (2) By composition. (3) $\varphi(n) = n+m$.

Exercise 9.1.10. Since (x_n) is bounded and $x_n \geq 0$, there is an integer L, such that $0 \leq x_n < L$ for all n. Hence, $\mathbb{N} = \bigcup_{j=0}^{L-1} \{n \in \mathbb{N} \mid j \leq x_n < j+1\}$. Since a finite union of finite sets is a finite set, at least one of the sets $\{n \in \mathbb{N} \mid j \leq x_n < j+1\}$, $j = 0,1,\ldots,L-1$ must be infinite.

Exercise 9.1.11. Similar to Exercise 9.1.10 using

$$A_0 = \bigcup_{m=0}^{9} \{n \in \mathbb{N} \mid d_0.m \leq x_n < d_0.m + 1/10\}.$$

Exercise 9.1.12. We must show $i_1 < i_2 < i_3 < \cdots$. Now i_2 is the smallest element of $A_2 \setminus \{i_1\} \subseteq A_1 \setminus \{i_1\}$. Hence i_2 is in A_1. But $i_2 \neq i_1$ and i_1 is the smallest element of A_1, consequently $i_1 < i_2$.

Similarly, i_3 is the smallest element of $A_3 \setminus \{i_1,i_2\} \subseteq A_2 \setminus \{i_1,i_2\}$. Hence i_3 is in $A_2 \setminus \{i_1\}$. But $i_3 \neq i_2$ and i_2 is the smallest element of $A_2 \setminus \{i_1\}$, consequently $i_2 < i_3$. The desired sequence of inequalities follows by induction.

Exercise 9.1.13. By construction y_n and b are in the interval $[d_0.d_1 \cdots d_n, d_0.d_1 \cdots d_n + 1/10^n]$ for all n. Since the interval has length $1/10^n$, we conclude $|y_n - b| \leq 1/10^n$ for all n.

Exercise 9.1.14. If $(x_n + iy_n)$ is bounded, then (x_n) and (y_n) are bounded. Since (x_n) is a bounded sequence of real numbers we just showed it has a convergent subsequence (x_{i_n}). Since (y_{i_n}) is a bounded sequence of real numbers it has a convergent subsequence $(y_{i_{j_n}})$. It follows that $(x_{i_{j_n}} + iy_{i_{j_n}})$ is convergent.

Exercise 9.1.16. Let $b := \lim x_{i_n}$. Let $\varepsilon > 0$ be given. Pick N such that $n \geq N$ implies $|x_{i_n} - b| < \varepsilon$. Then x_{i_N} and $x_{i_{N+1}}$ both are in $B_b(\varepsilon)$ and at least one of them is $\neq b$.

Exercise 9.1.17. If $x_n \to x$, then $|x_n - x_m| \leq |x_n - x| + |x - x_m|$.

Exercise 9.1.19. If $x_{i_n} \to x$, then $|x - x_m| \leq |x - x_{i_n}| + |x_{i_n} - x_m|$.

Exercise 9.1.20. There is an N such that $m, n \geq N$ implies $|x_m - x_n| < 1$. Hence $|x_m| < 1 + |x_N|$ for all $m \geq N$.

Exercise 9.1.22. Cauchy $\implies$ bounded $\implies$ convergent subsequence. And convergent subsequence implies convergent.

Exercise 9.1.40. Let $t := \limsup_{k\to\infty} x_k$. Recall $y_k := \sup\{x_j \mid k \leq j\}$. Since $t - \varepsilon < t$ and $y_k \to t$ as $k \to \infty$, there is an N such that $y_k > t - \varepsilon$ when $k \geq N$. Since $y_k = \sup\{x_j \mid k \leq j\}$ there is a $j \geq k$ such that $x_j > t - \varepsilon$. This gives (i).

Pick N such that $b_N < t + \varepsilon$. Since $x_j \leq b_N < t + \varepsilon$ for all $j \geq N$, we get (ii).

Conversely suppose (i) and (ii). By (i) $y_k = \sup\{x_j \mid k \leq j\} > t - \varepsilon$. Hence $\lim y_k \geq t - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we concluded $\lim y_k \geq t$. Similarly, (ii) implies $\lim y_k \leq t$.

Exercise 9.2.4. Similar to the first part. Since f_N is uniformly continuous the same δ works for all x_0.

Exercise 9.2.5 To see that f_n is integrable use step functions with partition points at the x_i's. The limit function is zero on the rationals and one on the irrationals. By density of rationals any lower step functions has sum ≤ 0. Similarly, any upper step function has sum ≥ 1.

Exercise 9.2.10. Let $\varepsilon > 0$ be given. By Exercise 5.4.7 g is uniformly continuous. Hence there is a $\delta > 0$ such that $|(x,y) - (u,v)| < \delta$ implies $|g(x,y) - g(u,v)| < \varepsilon/(d-c)$. If $|x - x_0| < \delta$, then $|(x,y) - (x_0,y)| = |x - x_0| < \delta$.

Exercise 9.3.1 Similar to part of the proof of Theorem 9.2.11.

Exercise 9.3.2 Suppose $f = 0$ outside $[a,b]$. Then f is uniformly continuous on $[a-1, b+1]$. Hence there is a $0 < \delta < 1$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all x and y in $[a-1, b+1]$. Since $\delta < 1$, we get $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all x and y in $\mathbb{R}$.

Exercise 9.3.3 If $|f(x)| \leq M$ for $x \in [a,b]$, then $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

Exercise 9.3.6 One way is to mimic the proof of Theorem 9.2.11. Fix x_0 we will show $f * g$ is differentiable at x_0 with derivative $f' * g(x_0)$. Let $\phi(x,y) := f(x-y)g(y)$, then ϕ and $\phi_x(t,y) = f'(t-y)g(y)$ are continuous on $\mathbb{R} \times \mathbb{R}$. Note that

$$f * g(t) = \int_{x_0-1-b}^{x_0+1-a} f(t-y)g(y)\,dy = \int_{x_0-1-b}^{x_0+1-a} \phi(t,y)\,dy$$

for $t \in [x_0 - 1, x_0 + 1]$. By the Mean Value Theorem

$$\frac{f(t-y) - f(x_0-y)}{t - x_0} = f'(x_t - y)$$

for some x_t between t and x_0. And so on.

Chapter 10
Series

We study infinite sums (series) of numbers and of functions. Among the topics are products of series, the Riemann rearrangement theorem, and the theory of power series. As applications we construct a space filling curve, construct a continuous nowhere differentiable functions, and prove the Weierstrass Approximation Theorem.

10.1 Series of Numbers

We begin by recalling some of Sect. 1.7 where we briefly considered infinite series. The main aim in Sect. 1.7 was to establish convergence of the geometric series $\sum_{k=0}^{\infty} z^k$ for $|z| < 1$.

Let $x = (x_j)$ be a sequence of complex numbers. We will assign a meaning to the *infinite* sum

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + x_3 + \cdots$$

To do so we consider the (finite) partial sums s_n of the first n terms, that is

$$s_n = s_n[x] = s_n[(x_j)] := \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_{n-1} + x_n, n \in \mathbb{N}.$$

Each of these is a finite sum so we can, at least in principle, calculate each of them.

We say that $\sum_{k=1}^{\infty} x_k$ is *convergent*, if the sequence (s_n) is convergent to some complex number. Otherwise we say that $\sum_{k=1}^{\infty} x_k$ is *divergent*. When $\sum_{k=1}^{\infty} x_k$ is convergent, we will write

$$\sum_{k=1}^{\infty} x_k := \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \lim_{n\to\infty} s_n.$$

Hence, we can make use of our discussion of sequences to establish results about series.

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If the x_k are real numbers and $s_n \to \infty$, we will write $\sum_{k=1}^{\infty} x_k = \infty$ and we will say that $\sum_{k=1}^{\infty} x_k$ diverges to ∞. Similarly for divergence to $-\infty$.

Exercise 10.1.1 (Linearity). If $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ are convergent and a and b are complex numbers, then $\sum_{k=1}^{\infty} (ax_k + by_k)$ is convergent and

$$\sum_{k=1}^{\infty} (ax_k + by_k) = a \sum_{k=1}^{\infty} x_k + b \sum_{k=1}^{\infty} y_k.$$

Proposition 10.1.2. *The convergence of an infinite series does not depend on any finite number of terms.*

Proof. If $y_k = x_k$ when $k > n$ and $m > n$, then

$$s_m[(x_k)] = \sum_{k=1}^{m} x_k = c + \sum_{k=1}^{m} y_k = c + s_m[(y_k)]$$

where

$$c := \sum_{k=1}^{n} (x_k - y_k).$$

Hence, $s_m[(x_k)] = c + s_m[(y_k)]$ for all $m > n$. Thus, $s_m[(x_k)]$ is convergent as $m \to \infty$ iff $s_m[(y_k)]$ is convergent as $m \to \infty$. ☺

We considered geometric series in Sect. 1.7. Our results are summarized in:

Example 10.1.3 (Geometric Series). Consider the sum $\sum_{k=0}^{\infty} z^k$. In this case, $s_n = 1 + z + z^2 + \cdots + z^{n-1}$ and $zs_n = z + z^2 + \cdots + z^{n-1} + z^n$, hence, $s_n - zs_n = 1 - z^n$. Solving for s_n gives

$$s_n = \sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z}.$$

It follows that $\sum_{k=0}^{\infty} z^k$ is convergent and

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}$$

when $|z| < 1$ and divergent when $1 \leq |z|$. Furthermore, $\sum_{k=0}^{\infty} z^k = \infty$ when z is real and $1 \leq z$.

Remark 10.1.4. Setting $z = 2$, the argument above shows the geometric series $\sum_{k=0}^{\infty} 2^k$ is divergent. In fact, the partial sums grow without bound. Hence the series diverges to infinity. Is there a way to get a finite sum for this series? Yes! In fact, using the 2-adic metric, a distance function different from the absolute value, the sum $\sum_{k=0}^{N} 2^k$ converges to $\frac{1}{1-2} = -1$ as $N \to \infty$. This is possible, in part, because the 2-adic metric is not related to the order on $\mathbb{R}$.

In addition to considering notions of distance different from absolute values, there are many other ways of assigning a finite sum to an ordinarily divergent series, see Hardy (1949). Among such methods are summability methods. We will use the Cesàro summability method in Chap. 12, as part of our discussion of convergence of Fourier series.

Exercise 10.1.5. Some simple facts.

1. (Test for Divergence) If $\sum_{k=1}^{\infty} x_k$ is convergent, then $x_n \to 0$.
2. If $\sum_{k=1}^{\infty} x_k$ is convergent, then $\sum_{k=N}^{\infty} x_k \to 0$ as $N \to \infty$.
3. If each $x_k \geq 0$, then $\sum_{k=1}^{\infty} x_k$ is convergent if and only if the sequence of partial sums (s_n) is bounded.

One of the "big" theorem in Sect. 9.1 is that a sequence of numbers is convergent iff it is a Cauchy sequence. In the context of series, this result takes the form:

Proposition 10.1.6 (Cauchy Criterion). *$\sum_{k=1}^{\infty} x_k$ is convergent if and only if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n \in \mathbb{N} : m \geq n \geq N \implies \left| \sum_{k=n}^{m} x_k \right| < \varepsilon. \tag{10.1}$$

Proof. Since $s_m - s_n = \sum_{k=n+1}^{m} x_k$, when $n < m$, the sequence (s_n) is Cauchy iff (10.1) holds. ☺

We will say that $\sum_{k=1}^{\infty} x_k$ is *absolutely convergent*, if $\sum_{k=1}^{\infty} |x_k|$ is convergent.

Exercise 10.1.7. If $\sum_{k=1}^{\infty} x_k$ is absolutely convergent, then $\sum_{k=1}^{\infty} x_k$ is convergent.

Convergence Tests

The following result is sometimes called the *Comparison Test*, using it is the most important way to show that a series is convergent.

Theorem 10.1.8 (Dominated Convergence Theorem). *If $|x_k| \leq y_k$ for all k and $\sum_{k=1}^{\infty} y_k$ is convergent, then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent.*

Proof. Let $\varepsilon > 0$ be given. Since $\sum_{k=1}^{\infty} y_k$ is convergent, the Cauchy criterion gives us an N, such that $m \geq n \geq N$ implies $\sum_{k=n}^{m} y_k < \varepsilon$. Since $|x_k| \leq y_k$ implies $\sum_{k=n}^{m} |x_k| \leq \sum_{k=n}^{m} y_k$, we conclude $m \geq n \geq N$ implies $\sum_{k=n}^{m} |x_k| < \varepsilon$. So, by the Cauchy criterion, $\sum_{k=1}^{\infty} |x_k|$ is convergent. ☺

The ratio and root tests follow from the Dominated Convergence Theorem by dominating by suitable geometric series.

Exercise 10.1.9 (Ratio Test). If there exists an $r < 1$ such that $\left|\frac{x_{n+1}}{x_n}\right| \leq r$ for all sufficiently large n, then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent.

Exercise 10.1.10 (Root Test). If there exists an $r < 1$ such that $\sqrt[n]{|x_n|} \leq r$ for all sufficiently large n, then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent.

Example 10.1.11 (The Riemann Zeta Function). The sum $\zeta(a) := \sum_{n=1}^{\infty} \frac{1}{n^a}$ is convergent, when $a > 1$.

Proof. The basic estimate needed is

$$\sum_{n=1}^{\infty} \frac{1}{n^a} = \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n^a} \leq \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{2^{ak}}$$
$$= \sum_{k=0}^{\infty} \frac{2^k}{2^{ak}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{a-1}}\right)^k.$$

The last sum is a convergent geometric series with sum $2^{a-1}/\left(2^{a-1}-1\right)$. It follows that the partial sums $s_m := \sum_{n=1}^{m} \frac{1}{n^a}$ are bounded by $2^{a-1}/\left(2^{a-1}-1\right)$. ☺

Exercise 10.1.12. Give a similar proof that $\sum_{n=1}^{\infty} \frac{1}{n^a}$ is divergent when $a \leq 1$.

Example 10.1.13 (Euler Constant). Let

$$\gamma_n := \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) = \int_1^{n+1} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx = \sum_{k=1}^{n} \int_k^{k+1} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx.$$

Note, $0 \leq \frac{1}{\lfloor x \rfloor} - \frac{1}{x} \leq \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} < \frac{1}{k^2}$, when $k \leq x \leq k+1$. Hence, (γ_n) is an increasing sequence bounded by $\sum_{k=1}^{\infty} \frac{1}{k^2}$. In particular, (γ_n) is convergent. The Euler constant γ is the limit of the sequence (γ_n).

Exercise 10.1.14 (Integral Test). Suppose $f : [1,\infty) \to [0,\infty[$ is decreasing. Let $x_k = f(k)$. Then $\int_1^{\infty} f$ is convergent if and only if $\sum_{k=1}^{\infty} x_k$ is convergent.

If (a_k) is a sequence, then we can think of the partial sum $A_n := \sum_{k=1}^{n} a_k$ as the integral of (a_k) and of the difference $\alpha_n := a_n - a_{n+1}$ as (the negative of) the derivative of (a_k) at n. If we do so, then the following is an analogues of the Fundamental Theorem of Calculus.

Lemma 10.1.15 (Fundamental Theorem of Discrete Calculus). *Let (a_k) be as sequence, if $\alpha_k := a_k - a_{k+1}$, then*

$$\sum_{k=1}^{m} \alpha_k = a_1 - a_{m+1}$$

for all $m \geq 1$.

Proof. The left hand side is

$$(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_m - a_{m+1}).$$

So the result follows by cancellation. ☺

We will also use the analogue of integration by parts:

Lemma 10.1.16 (Summation by Parts). *Let (a_k) and (b_k) be sequences. Let $\alpha_n := a_n - a_{n+1}$ and $B_n := \sum_{k=1}^{n} b_k$, then*

$$\sum_{k=1}^{n} \alpha_k B_k = \sum_{k=1}^{n} a_k b_k - a_{n+1} B_n.$$

Proof. $\alpha_1 B_1 = (a_1 - a_2) b_1 = a_1 b_1 - a_2 B_1$ so the claim is true if f $n = 1$. Suppose $n \geq 1$ and

$$\sum_{k=1}^{n} \alpha_k B_k = \sum_{k=1}^{n} a_k b_k - a_{n+1} B_n.$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n+1} \alpha_k B_k &= \sum_{k=1}^{n} \alpha_k B_k + \alpha_{n+1} B_{n+1} \\ &= \sum_{k=1}^{n} a_k b_k - a_{n+1} B_n + \alpha_{n+1} B_{n+1} \\ &= \cdots\end{aligned}$$

Exercise 10.1.17. Fill in the missing details in the proof of the Summation by Parts rule.

☺

The following is a general form of the alternating series test.

Theorem 10.1.18 (Dirichlet's Test). *Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and $a_n \to 0$. If (b_k) be a sequence of complex numbers such that the sequence of partial sums $B_n := \sum_{k=1}^{n} b_k$ is bounded, then $\sum_{k=1}^{\infty} a_k b_k$ is convergent.*

Proof. Since (B_n) is bounded, there is an M such that $|B_n| \leq M$ for all n. Let $\alpha_k := a_k - a_{k+1}$, then $\alpha_k \geq 0$ and by the Fundamental Theorem of Discrete Calculus

$$\sum_{k=1}^{m} \alpha_k = a_1 - a_{m+1}.$$

Hence, $\sum_{k=1}^{\infty} \alpha_k$ is convergent with sum a_1. Now $|\alpha_k B_k| \leq M \alpha_n$, so the Dominated Convergence Theorem, implies $\sum_{k=1}^{\infty} \alpha_k B_k$ is absolutely convergent

By Summation by Parts,

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} \alpha_k B_k + a_{n+1} B_n.$$

We will show that the right hand side is convergent at $n \to \infty$.

Since, $a_n \to 0$, we have $|a_{n+1} B_n| \leq M a_{n+1} \to 0$. Consequently, $\sum_{k=1}^{n} a_k b_k \to \sum_{k=1}^{\infty} \alpha_k B_k$. ☺

Leibniz is sometimes credited with the discovery of:

Corollary 10.1.19 (Alternating Series Test). *Suppose* $a_1 \geq a_2 \geq a_3 \geq \cdots$ *and* $a_n \to 0$*. Then* $\sum_{k=1}^{\infty}(-1)^k a_k$ *is convergent.*

Proof. Set $b_k = (-1)^{k+1}$ in Dirichlet's Test. ☺

Example 10.1.20. We saw above that $\sum_{k=1}^{\infty}\frac{1}{k}$ is not convergent. Setting $a_k := \frac{1}{k}$ in the alternating series test shows that $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ is convergent. Consequently, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ is convergent and not absolutely convergent.

A series that is convergent and not absolutely convergent is called *conditionally convergent*.

Products of Series

Formally, that is without paying attention to convergence issues, we have

$$\left(\sum_{j=0}^{\infty} a_j x^j\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{n=0}^{\infty}\left(\sum_{j+k=n} a_j b_k\right) x^n$$

where $\sum_{j+k=n} a_j b_k = \sum_{j=0}^{n} a_j b_{n-j}$ is the sum of the terms whose subscripts add up to n, that is the coefficient to x^n. Setting $x = 1$, this suggests the formula in the next theorem.

Theorem 10.1.21 (Cauchy Product). *Suppose* $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{k=0}^{\infty} b_k$ *are absolutely convergent. If*

$$c_n := \sum_{j=0}^{n} a_j b_{n-j} = \sum_{j+k=n} a_j b_k, \tag{10.2}$$

then $\sum_{n=0}^{\infty} c_n$ *is absolutely convergent and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right).$$

Proof. We begin by showing that the partial sums $\sum_{n=0}^{N}|c_n|$ are bounded. Once we have established that we know $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

$$\begin{aligned}
\sum_{n=0}^{N}|c_n| &= \sum_{n=0}^{N}\left|\sum_{j+k=n} a_j b_k\right| \leq \sum_{n=0}^{N}\sum_{j+k=n}|a_j b_k| \\
&= \sum_{j+k\leq N}|a_j b_k| \leq \left(\sum_{j=0}^{N}|a_j|\right)\left(\sum_{k=0}^{N}|b_k|\right) \qquad (10.3)\\
&\leq \left(\sum_{j=0}^{\infty}|a_j|\right)\left(\sum_{k=0}^{\infty}|b_k|\right).
\end{aligned}$$

The middle inequality used that the triangle $\{(j,k) \mid j+k \leq N\}$ is contained in the

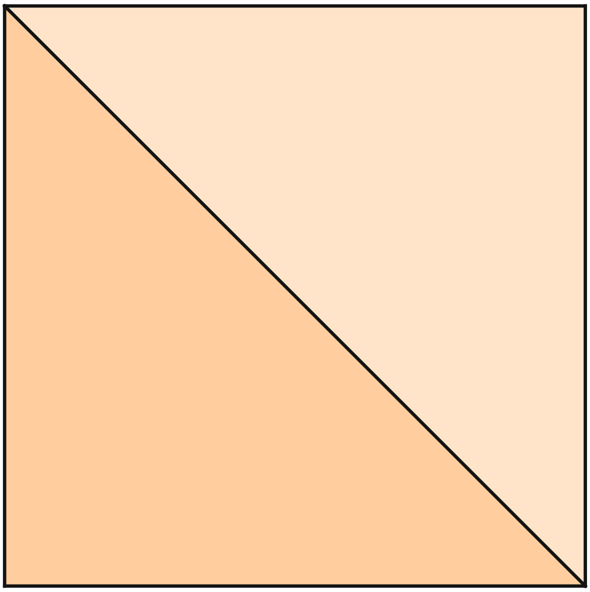

Fig. 10.1 The *square* is the region $\{(j,k) \mid 0 \leq j,k \leq N\}$ corresponding to the sum $\left(\sum_{j=0}^{N} a_j\right)\left(\sum_{k=0}^{N} b_k\right) = \sum_{j=0}^{N}\sum_{k=0}^{N} a_j b_k$. The *boundary* between the two *triangles* is the line $j+k=N$, corresponding to the sum $c_N = \sum_{j+k=N} a_j b_k$. The *lower triangle* is the region $\{(j,k) \mid 0 \leq j+k \leq N\}$ corresponding to the sum $\sum_{n=0}^{N} c_n = \sum_{j+k\leq N} a_j b_k$. The *upper triangle* corresponds to the sum $\sum_{\substack{j,k=0\\ j+k>N}}^{N} a_j b_k$

square $[0,N]\times[0,N]$, see Fig. 10.1. Considering this square and triangle also shows

$$\sum_{n=0}^{N} c_n = \left(\sum_{j=0}^{N} a_j\right)\left(\sum_{k=0}^{N} b_k\right) - \left(\sum_{\substack{j,k=0\\ j+k>N}}^{N} a_j b_k\right).$$

Where the notation $\sum_{\substack{j,k=0\\ j+k>N}}^{N}$ means we are summing over the triangle

$$\left\{(j,k) \in \mathbb{N}_0^2 \mid 0 \leq j \leq N, 0 \leq k \leq N, N < j+k\right\}.$$

It remains to show that the last term $\sum_{\substack{j,k=0\\ j+k>N}}^{N} a_j b_k \to 0$ as $N \to \infty$. But

$$\left|\sum_{\substack{j,k=0\\ j+k>N}}^{N} a_j b_k\right| \le \sum_{\substack{j,k=0\\ j+k>N}}^{N} \left|a_j b_k\right| \le \sum_{j+k>N} \left|a_j b_k\right| \to 0$$

as $N \to \infty$. To verify the $\to 0$, note the sequence $s_N := \sum_{j+k\le N} \left|a_j b_k\right|$ is bounded above by $\left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right)$ by (10.3). Hence, (s_N) is convergent, because it is bounded and increasing. Let $s := \lim_{N\to\infty} s_N$, then $\sum_{j+k>N} \left|a_j b_k\right| = s - s_N \to 0$. ☺

Rearrangements★

The purpose of this section is to expose one of the benefits of working with absolutely convergent series. For simplicity we only consider series of real numbers.

If $\sum_{k=1}^{\infty} x_k$ is an infinite series, then $\sum_{k=1}^{\infty} y_k$ is a *rearrangement* of $\sum_{k=1}^{\infty} x_k$, if there is a one-to-one and onto function $\phi : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{\phi(n)}$.

Theorem 10.1.22. *If $\sum_{k=1}^{\infty} x_k$ is absolutely convergent, then every rearrangement $\sum_{k=1}^{\infty} y_k$ is also absolutely convergent and $\sum_{k=1}^{\infty} y_k = \sum_{k=1}^{\infty} x_k$.*

Proof. Let $s := \sum_{k=1}^{\infty} x_k$. Let $\varepsilon > 0$ be given. Pick N, such that $\sum_{k=N}^{\infty} |x_k| < \varepsilon/2$. Pick M such that each $x_1, x_2, \ldots, x_N$ is one of $y_1, y_2, \ldots, y_M$. If $n > M$, then the terms $x_1, x_2, \ldots, x_N$ in $\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} y_k$ cancel out, the surviving terms are $\pm x_k$ with $k > N$. Hence,

$$\left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} y_k\right| \le \sum_{k=N}^{\infty} |x_k| < \frac{\varepsilon}{2}.$$

Consequently, the calculation

$$\begin{aligned}\left|\sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n} y_k\right| &\le \left|\sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n} x_k\right| + \left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} y_k\right| \\ &< \sum_{k=n+1}^{\infty} |x_k| + \frac{\varepsilon}{2} \le \sum_{k=N}^{\infty} |x_k| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

completes the proof. ☺

Remark 10.1.23. It can be shown that if every rearrangement of $\sum_{k=1}^{\infty} x_k$ is convergent, then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent. However, we will not prove this result.

Note this does not assume the rearrangement converge to the same limit, that is part of the conclusion.

Theorem 10.1.24 (Riemann Rearrangement Theorem). *Let $x_n \in \mathbb{R}$ for all n. Suppose $\sum x_n$ converges, but does not converge absolutely. Then $\sum x_n$ can be rearranged to have any real number as its sum, to be divergent to ∞, and to be divergent to $-\infty$.*

Proof. Let

$$x_n^+ := \begin{cases} x_n & \text{when } 0 \le x_n \\ 0 & \text{when } x_n < 0 \end{cases}$$

be the positive part of x_n and let

$$x_n^- := \begin{cases} 0 & \text{when } 0 \le x_n \\ x_n & \text{when } x_n < 0 \end{cases}$$

be the negative part of x_n. Then

$$x_n = x_n^+ + x_n^- \text{ and } |x_n| = x_n^+ - x_n^-.$$

Suppose $\sum x_n^-$ is convergent. Then $\sum x_n^+ = \sum (x_n - x_n^-)$ shows that $\sum x_n^+$ is convergent, and consequently, $\sum |x_n| = \sum (x_n^+ - x_n^-)$ is convergent, a contradiction. A similar argument shows that $\sum x_n^+$ is not convergent. Hence,

$$\sum x_n^- = -\infty \text{ and } \sum x_n^+ = \infty.$$

Let $a_1, a_2, \ldots$ be the positive terms and let $b_1, b_2, \ldots$ be the negative terms. We saw $\sum a_n = \infty$ and $\sum b_n = -\infty$. Let t be some real number. The desired rearrangement $\sum y_n$ is achieved as follows. Pick positive terms such that

$$\begin{gathered} a_1 + a_2 + \cdots + a_{N_1 - 1} < t \text{ and} \\ a_1 + a_2 + \cdots + a_{N_1} > t \end{gathered}$$

then pick negative terms such that

$$\begin{gathered} a_1 + a_2 + \cdots + a_{N_1 - 1} + b_1 + \cdots + b_{M_1 - 1} > t \text{ and} \\ a_1 + a_2 + \cdots + a_{N_1 - 1} + b_1 + \cdots + b_{M_1} < t \end{gathered}$$

then pick positive terms such that

$$\begin{gathered} a_1 + \cdots + a_{N_1 - 1} + b_1 + \cdots + b_{M_1 - 1} + a_{N_1 + 1} + \cdots + a_{N_2 - 1} < t \text{ and} \\ a_1 + \cdots + a_{N_1 - 1} + b_1 + \cdots + b_{M_1} + a_{N_1 + 1} + \cdots + a_{N_2} > t \end{gathered}$$

continuing in this way gives the desired rearrangement.

By construction $0 < \sum_{k=1}^{N_1} y_k - t < a_{N_1}$. For $N_1 < n < N_1 + M_1$,

$$0 < \sum_{k=1}^{n} y_k - t \leq \sum_{k=1}^{N_1} y_k - t < a_{N_1}$$

while

$$0 < t - \sum_{k=1}^{N_1+M_1} y_k < -b_{M_1}.$$

For $N_1 + M_1 < n < N_1 + M_1 + N_2$,

$$0 < t - \sum_{k=1}^{n} y_k \leq t - \sum_{k=1}^{N_1+M_1} y_k < -b_{M_1}$$

while

$$0 < \sum_{k=1}^{N_1+M_1+N_2} y_k - t < a_{N_2}.$$

It remains to show that $a_{N_j} \to 0$ and $b_{M_j} \to 0$. Since $\sum x_n$ is convergent, $x_n \to 0$, hence $a_n \to 0$ and $b_n \to 0$. Consequently, $a_{N_j} \to 0$ and $b_{M_j} \to 0$.

Remark 10.1.25. To summarize: addition is commutative when working with absolutely convergent series, but not when working with conditionally convergent series.

Example 10.1.26. Let $\sum_{k=1}^{\infty} y_k$ be a rearrangement of the alternating Harmonic series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$. Assume the positive terms in $\sum_{k=1}^{\infty} y_k$ are in the same order as in the original series and that the negative terms are in the same order as in the original series. For example, $\sum_{k=1}^{\infty} y_k$ could be

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots \tag{10.4}$$

but, for example, could not be

$$1 + \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + \frac{1}{9} - \frac{1}{4} + \frac{1}{7} + \frac{1}{11} - \frac{1}{6} + \cdots.$$

Let P_n be the number of positive terms in $\sum_{k=1}^{n} y_k$, then $N_n := n - P_n$ is the number of negative terms in $\sum_{k=1}^{n} y_k$. Recall, $\gamma_n := \sum_{k=1}^{n} \frac{1}{k} - \log(n+1)$ converges by Example 10.1.13. Calculating we get

$$\begin{aligned}
\sum_{k=1}^{n} y_k &= \sum_{k=1}^{P_m} \frac{1}{2k-1} - \sum_{k=1}^{N_n} \frac{1}{2k} \\
&= \left(\sum_{k=1}^{2P_n} \frac{1}{k} - \sum_{k=1}^{P_n} \frac{1}{2k} \right) - \sum_{k=1}^{N_n} \frac{1}{2k} \\
&= \gamma_{2P_n} + \log(2P_n + 1) - \tfrac{1}{2}(\gamma_{P_n} + \log(P_n + 1)) - \tfrac{1}{2}(\gamma_{N_n} + \log(N_n + 1)) \\
&= \gamma_{2P_n} - \tfrac{1}{2}(\gamma_{P_n} + \gamma_{N_n}) + \tfrac{1}{2} \log\left(\frac{(2P_n + 1)(2P_n + 1)}{(P_n + 1)(N_n + 1)} \right).
\end{aligned}$$

Hence, if $P_n \to \infty$, $N_n \to \infty$, and $\frac{P_n}{N_n} \to a$, then $\sum_{k=1}^{\infty} y_k = \log(2) + \frac{1}{2}\log(a)$. In particular, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \log(2)$ and the sum of the rearrangement in (10.4) equals $\frac{3}{2}\log(2)$.

10.2 Series of Functions

Let S be some set. Let $f_n : S \to \mathbb{C}$ for all $n \in \mathbb{N}$. Let $g_n(x) = \sum_{k=1}^{n} f_k(x)$. We say that $\sum_{k=1}^{\infty} f_k(x)$ is *convergent*, if $(g_n(x))$ is convergent for all $x \in S$, and we set

$$\sum_{k=1}^{\infty} f_k(x) := \lim_{n\to\infty}\sum_{k=1}^{n} f_k(x).$$

We say that $\sum_{k=1}^{\infty} f_k(x)$ is *uniformly convergent* on S, if $(g_n(x))$ is uniformly convergent on S.

The following is a version of the Dominated Convergence Theorem.

Theorem 10.2.1 (Weierstrass M-Test). *Let S be some set and let $f_n : S \to \mathbb{C}$ be a sequence of functions. Suppose there exists a sequence (M_n) of positive real numbers such that*

$$|f_n(x)| \leq M_n, \text{ for all } n \in \mathbb{N} \text{ and all } x \in S.$$

If $\sum_{n=1}^{\infty} M_n$ is convergent, then $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on S.

Proof. Fix $x \in S$. Since $|f_k(x)| \leq M_k$ and $\sum M_k$ is convergent the series $\sum_{k=1}^{\infty} f_k(x)$ is convergent by Dominated Convergence. Hence, setting $f(x) := \sum_{k=1}^{\infty} f_k(x)$ determines a function $f : S \to \mathbb{C}$. To complete the proof we need to show $g_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly to f on S.

Let $\varepsilon > 0$ be given. We must find an N such that $n \geq N \implies |f(x) - g_n(x)| < \varepsilon$ for all $x \in S$. Pick N such that $\sum_{k=N}^{\infty} M_n < \varepsilon$. For $n \geq N$ we have

$$|f(x) - g_n(x)| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \leq \sum_{k=n+1}^{\infty} M_k < \varepsilon$$

for any x in S. ☺

Corollary 10.2.2. *If each f_n in the Weierstrass M-test is continuous, then the sum $f(x) = \sum_{k=1}^{\infty} f_k(x)$ is continuous.*

Proof. The functions $g_n = \sum_{k=1}^{n} f_k$ are continuous, since they are *finite* sums of continuous functions. Using $g_n \rightrightarrows f$ and that uniform limits of continuous functions are continuous completes the proof. ☺

Example 10.2.3. We showed in Example 8.2.6 that $\sum_{k=0}^{\infty} x^k/k!$ converges uniformly to e^x on any interval $[-M, M]$.

10.3 Space Filling Curves★

A function $f : [0,1] \to [0,1]^2$ is called *space filling* if it is onto. A continuous function $f : [0,1] \to \mathbb{R}^2$ is called a *curve*.

Theorem 10.3.1. *Space filling curves exists.*

To prove this theorem we will construct such a function. Curiously enough, this will use a function $C \to [0,1]^2$ we considered in Exercise 3.6.2, here C denotes the Cantor set. This curve was discovered by Isaac Jacob Schoenberg (21 April 1903, Galaţi to 21 February 1990).

Example 10.3.2 (Schoenberg). Let

$$f(t) := \begin{cases} 0 & \text{if } 0 \le t \le 1/3 \\ 3t-1 & \text{if } 1/3 \le t \le 2/3 \\ 1 & \text{if } 2/3 \le t \le 1. \end{cases} \tag{10.5}$$

This determines a continuous function f defined on $[0,1]$. Extend f to $[-1,1]$ by setting $f(-t) := f(t)$ for $0 \le t \le 1$ and then to all of $\mathbb{R}$, by periodicity, setting $f(t+2k) = f(t)$, for $-1 < t \le 1$ and $k \in \mathbb{Z}$. Since $f(-1) = f(1)$ we arrive at a continuous function $f : \mathbb{R} \to [0,1]$, see Fig. 10.2.

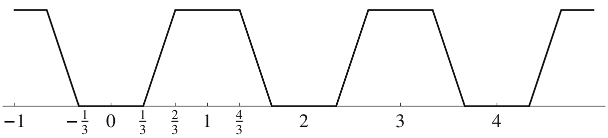

Fig. 10.2 The periodic function $f(t)$

For $t \in \mathbb{R}$ let

$$x(t) := \sum_{j=1}^{\infty} 2^{-j} f\left(3^{2j-2}t\right) \text{ and} \tag{10.6}$$

$$y(t) := \sum_{k=1}^{\infty} 2^{-k} f\left(3^{2k-1}t\right)$$

By the Weierstrass $M-$test with $M_j := 2^{-j}$ the functions x and y are continuous. Let $g(t) = (x(t), y(t))$, then g is continuous. We will show that g maps the Cantor set C onto the square $[0,1]^2$. Hence, writing a point $t \in C$ as

$$t = \sum_{n=1}^{\infty} \frac{d_n}{3^n}, \text{ where } d_n \in \{0,2\},$$

we will show

$$g\left(\sum_{n=1}^{\infty}\frac{d_n}{3^n}\right)=\left(\sum_{j=1}^{\infty}\frac{d_{2j-1}}{2^{j+1}},\sum_{k=1}^{\infty}\frac{d_{2k}}{2^{k+1}}\right). \tag{10.7}$$

Consequently, on the Cantor set g equals the function from Exercise 3.6.2. In particular, g maps the Cantor set C onto the closed unit square $[0,1]^2$. In particular, g is a continuous function mapping the interval $[0,1]$ onto the square $[0,1]^2$. Hence, establishing (10.7) completes the proof.

Proof. (Proof of (10.7)) Clearly, (10.7) is equivalent to

$$x\left(\sum_{n=1}^{\infty}\frac{d_n}{3^n}\right)=\sum_{j=1}^{\infty}\frac{d_{2j-1}}{2^{j+1}} \tag{10.8}$$

and

$$y\left(\sum_{n=1}^{\infty}\frac{d_n}{3^n}\right)=\sum_{k=1}^{\infty}\frac{d_{2k}}{2^{k+1}}. \tag{10.9}$$

To establish (10.8) we must show

$$f\left(3^{2j-2}\sum_{n=1}^{\infty}\frac{d_n}{3^n}\right)=\frac{d_{2j-1}}{2}, \tag{10.10}$$

since $x\left(\sum_{n=1}^{\infty}d_n3^{-n}\right)=\sum_{j=1}^{\infty}2^{-j}f\left(3^{2j-2}\sum_{n=1}^{\infty}d_n3^{-n}\right)$. So (10.10) must be a consequence of the construction of f. Since f has period two, writing $3^{2j-2}\sum_{n=1}^{\infty}d_n3^{-n}$ as an even integer plus a remainder is useful:

$$\begin{aligned}3^{2j-2}\sum_{n=1}^{\infty}\frac{d_n}{3^n}&=\sum_{n=1}^{\infty}\frac{d_n3^{2j-2}}{3^n}=\sum_{n=1}^{2j-2}\frac{d_n3^{2j-2}}{3^n}+\frac{d_{2j-1}}{3}+\sum_{n=2j}^{\infty}\frac{d_n3^{2j-2}}{3^n}\\&=2k+\frac{d_{2j-1}}{3}+\delta,\end{aligned}$$

where $k:=\sum_{n=1}^{2j-2}\frac{d_n}{2}3^{2j-2-n}$ and $\delta:=\sum_{n=2j}^{\infty}d_n3^{2j-2-n}$. Using $d_n\in\{0,2\}$ for all n, it follows that k is an integer and that

$$0\le\sum_{m=2}^{\infty}\frac{d_{2j-2+m}}{3^m}\le\sum_{m=2}^{\infty}\frac{2}{3^m}=\frac{2/9}{1-1/3}=\frac{1}{3}.$$

In particular, $0\le\delta\le\frac{1}{3}$.

Using f has period 2 and k is an integer we arrive at

$$f\left(3^{2j-2}\sum_{n=1}^{\infty}\frac{d_n}{3^n}\right)=f\left(\frac{d_{2j-1}}{3}+\delta\right). \tag{10.11}$$

Using (10.5) and $0 \le \delta \le \frac{1}{3}$, we have $f(\delta) = 0$ and $f\left(\frac{2}{3} + \delta\right) = 1$. Hence, it follows from (10.11) that

$$f\left(3^{2j-2}\sum_{n=1}^{\infty}\frac{d_n}{3^n}\right) = \begin{cases} 0 & \text{if } d_{2j-1} = 0 \\ 1 & \text{if } d_{2j-1} = 1 \end{cases},$$

but this is (10.10).

A similar argument establishes (10.9). We have established (10.7), hence, we shown g is a space filling curve. ☺

Exercise 10.3.3. Show that (10.9) holds.

We saw in Sect. 4.2 that there are bijections $f : [0,1] \to [0,1]^2$. The following shows that you cannot have everything. In particular, a space-filling curve cannot be one-to-one. This is due to Eugen Otto Erwin Netto (30 June 1848, Halle to 13 May 1919, Giessen).

Theorem 10.3.4 (Netto's Theorem). *No function $f : [0,1] \to [0,1]^2$ is 1-1, onto, and continuous.*

This is Theorem 13.4.3.

10.4 Power Series

Given a sequence of constants a_n and a point z_0 we can form a *power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots,$$

z_0 is the point about which the power series is *expanded*. By definition $(z - z_0)^0 = 1$ so the first term is the constant a_0. We allow a_n, z, and z_0 to be complex numbers.

For Taylor series we start with a function f and a point x_0, set $a_k := \frac{f^{(k)}(x_0)}{k!}$ and then ask for which x the Taylor series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

equals $f(x)$. Example 8.2.6 illustrates this. When studying power series we start with a sequence a_k and a point z_0 and then define a function f by setting $f(z) := \sum_{k=0}^{\infty} a_k (z - z_0)^k$. We study properties of functions defined in terms of power series, these properties depend on the given data: the sequence a_n and the point z_0. We begin by studying the domain of such functions. We also study continuity, derivatives, and integrals of these functions.

Convergence

The most basic question is: given a_k and z_0 for which z does the series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converge? Clearly $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ is a translate of $\sum_{k=0}^{\infty} a_k z^k$, so we will consider the case $z_0 = 0$.

It turns out that there is a R such that $\sum_{k=0}^{\infty} a_k z^k$ is convergent for all $z \in \mathbb{C}$ with $|z| < R$ and divergent for all $z \in \mathbb{C}$ with $|z| > R$. The following formula for this R was discovered by Cauchy and rediscovered by Jacques Salomon Hadamard (8 December 1865 Versailles, France to 17 October 1963 Paris, France).

Theorem 10.4.1 (Cauchy–Hadamard Formula). *Given a power series* $\sum_{k=0}^{\infty} a_k z^k$ *let*

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}. \tag{10.12}$$

- *If* $|z| < R$, *then* $\sum_{k=0}^{\infty} a_k z^k$ *is absolutely convergent.*
- *If* $R < |z|$, *then* $\sum_{k=0}^{\infty} a_k z^k$ *is divergent.*

We use the interpretations $\frac{1}{0} = \infty$ *and* $\frac{1}{\infty} = 0$.

Proof. The proof relies on two facts about the limit superior. Suppose $\limsup b_k = L$, then (i) for any $M > L$ the set $\{k \mid b_k > M\}$ is finite and (ii) there is a subsequence $b_{n_k} \to L$.

Let $L := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ then $R = 1/L$. Let z be some complex number. We must show that, if $|z| < R$ then $\sum_{n=0}^{\infty} a_n z^n$ is convergent, and if $R < |z|$ then $\sum_{n=0}^{\infty} a_n z^n$ is not convergent.

Suppose $|z| < R$, in particular $0 < R$, hence $L < \infty$. Pick S, such that $|z| < S < R$. Then $\sum_{k=0}^{\infty} (|z|/S)^k$ is a convergent geometric series and

$$\frac{1}{S} > \frac{1}{R} = L.$$

So, by one of the characterizations of the limit superior, for all but a finite number of k we have

$$|a_k|^{1/k} \le \frac{1}{S}.$$

Multiplying by $|z|$ and raising both sides to the k^{th} power yields

$$\left|a_k z^k\right| \le \left(\frac{|z|}{S}\right)^k.$$

Since $\sum_{k=0}^{\infty} (|z|/S)^k$ is convergent, Dominated Convergence shows $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.

Conversely, suppose $R < |z|$, in particular $R \neq \infty$, hence $0 < L$. Pick S, such that $R < S < |z|$. Hence $\frac{1}{S} < L$. Let a_{n_k} be a subsequence such that

$$\left|a_{n_k}\right|^{1/n_k} \to L \text{ as } k \to \infty.$$

Since $\frac{1}{s} < L$, for all but a finite number of k, we have $\left|a_{n_k}\right|^{1/n_k} > \frac{1}{S}$. It follows that

$$\left|a_{n_k} z^{n_k}\right| = \left(\left|a_{n_k}\right|^{1/n_k} |z|\right)^{n_k} \geq \left(\frac{|z|}{s}\right)^{n_k} \to \infty$$

as $k \to \infty$. Consequently, $a_k z^k \not\to 0$ as $k \to \infty$. Thus, $\sum_{n=0}^{\infty} a_n z^n$ is not convergent. ☺

Definition 10.4.2. The number R in Eq. (10.12) is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n z^n$.

Remark 10.4.3. If $0 < R < \infty$, the power series $\sum_{k=0}^{\infty} a_k z^k$ may converge at none, some, or all points on the circle $|z| = R$.

Due to the usefulness of uniform convergence, we make the following observation.

Lemma 10.4.4. *Let R be the radius of convergence of the power series $\sum a_k x^k$. If $0 < S < R$, then $\sum a_k x^k$ converges uniformly in the closed disk $|x| \leq S$.*

Proof. By the Cauchy–Hadamard Theorem $\sum a_k S^k$ is absolutely convergent. For x in the disk $|x| \leq S$, $\left|a_k x^k\right| \leq \left|a_k S^k\right|$, hence the result follows from the Weierstrass $M-$test with $M_k := \left|a_k S^k\right|$.

☺

Calculus

The following lemma is useful in establishing some of the results below.

Lemma 10.4.5. *If $M > 0$, then $M^{1/n} \to 1$ as $n \to \infty$.*

Proof. Suppose $M > 1$, then $M^{1/n} > 1$ for all n. Let $\varepsilon > 0$. By Bernoulli's inequality $(1+\varepsilon)^n \geq 1 + n\varepsilon$, for any $n \in \mathbb{N}$. Hence,

$$1+\varepsilon \geq (1+n\varepsilon)^{1/n}, \text{ for any } n \in \mathbb{N}.$$

Pick $N \in \mathbb{N}$ with $1 + N\varepsilon \geq M$. Let $n \geq N$. Then $1 + n\varepsilon \geq M$ and therefore

$$0 < M^{1/n} - 1 \leq (1+n\varepsilon)^{1/n} - 1 \leq (1+\varepsilon) - 1 = \varepsilon.$$

Consequently, $M^{1/n} \to 1$ as $n \to \infty$.

If $M = 1$, then $M^{1/n} = 1$ for all n. If $M < 1$, then $1/M^{1/n} = (1/M)^{1/n} \to 1$. ☺

Remark 10.4.6. Alternative proofs of the lemma. (a) If $|z| < 1$, the series $\sum_{k=0}^{\infty} z^k$ is a convergent geometric series. When $z = 1$, the series $\sum_{k=0}^{\infty} z^k$ is divergent. Hence the power series $\sum_{k=0}^{\infty} z^k$ has radius of convergence $R = 1$. Consequently, so does $\sum_{k=0}^{\infty} M z^k = M \sum_{k=0}^{\infty} z^k$. By Cauchy–Hadamard $\limsup M^{1/k} = 1$. Since the sequence $\left(M^{1/k}\right)$ is monotone, $M^{1/k} \to 1$. (b) Using exponential functions and logarithms: $M^{1/n} = \exp\left(\frac{1}{n}\log(M)\right) \to e^0 = 1$ as $n \to \infty$.

Exercise 10.4.7. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R and $[a,b] \subset]-R,R[$, then $\int_a^b \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n \, dx$.

Exercise 10.4.8. $n^{1/n} \to 1$ as $n \to \infty$.

Exercise 10.4.9. $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence.

Exercise 10.4.10. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R and $|y| < R$, then $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is differentiable at y and $f'(y) = \sum_{n=1}^{\infty} n a_n y^{n-1}$, that is

$$\left(\sum_{n=0}^{\infty} a_n y^n \right)' = \sum_{n=0}^{\infty} (a_n y^n)' .$$

10.5 The Weierstrass Approximation Theorem★

In this section, we will reproduce Weierstrass' original proof of the Weierstrass Approximation Theorem. Weierstrass proved this theorem when he was 70 years old. There are several other proofs of this important theorem.

The Weierstrass Approximation Theorem states that given any continuous function f defined on a compact interval $[a,b]$, there is a sequence of polynomials p_n such that $p_n \rightrightarrows f$ on $[a,b]$.

Theorem 10.5.1 (Weierstrass Approximation Theorem). *Let $f : [a,b] \to \mathbb{C}$ be continuous and let $\varepsilon > 0$. There exists a polynomial p such that*

$$|f(t) - p(t)| < \varepsilon, \textit{ for all } t \in [a,b] .$$

Picking a p_n for each $\varepsilon = 1/n$ yields:

Corollary 10.5.2. *If f is continuous on $[a,b]$, then there is a sequence of polynomials p_n, such that p_n converges uniformly to f on $[a,b]$.*

The rest of this section contains the proof of the Weierstrass Approximation Theorem. Apart from some preparatory work, that was well known at the time Weierstrass proved this result, the proof is about one page long.

Convolution by a Polynomial

Below, we will need that convolution of a function and a polynomial is a polynomial.

Lemma 10.5.3. *Let f be a continuous function on $\mathbb{R}$ that equals zero outside some bounded interval and let p be a polynomial, then $f * p$ is a polynomial.*

Proof. Suppose $p(x) = \sum_{k=0}^{n} a_k x^k$ has degree n. Since $a_k (x-y)^k = \sum_{j=0}^{k} \beta_{k,j}(y) x^j$, where each $\beta_{k,j}(y)$ is a polynomial of degree $k - j \leq k$ (explicit expressions for the $\beta_j(y)$ can be read off from the binomial theorem, but are not needed here), then

$$\begin{aligned} p(x-y) &= \sum_{k=0}^{n} a_k (x-y)^k = \sum_{k=0}^{n} \sum_{j=0}^{k} \beta_{k,j}(y) x^j \\ &= \sum_{j=0}^{n} \sum_{k=j}^{n} \beta_{k,j}(y) x^j = \sum_{j=0}^{n} b_j(y) x^j \end{aligned}$$

where $b_j(y) := \sum_{k=j}^{n} \beta_{k,j}(y)$ is a polynomial of degree $\leq n$. Choose R so that $f(x) = 0$ if $|x| \geq R$. The calculation

$$\begin{aligned} f * p(x) &= \int_{-\infty}^{\infty} f(y) p(x-y)\, dy \\ &= \int_{-R}^{R} f(y) \left(\sum_{k=0}^{n} b_k(y) x^k \right) dy \\ &= \sum_{k=0}^{n} \left(\int_{-R}^{R} f(y) b_k(y)\, dy \right) x^k \end{aligned}$$

shows that $f * p$ is a polynomial in x of degree $\leq n$. ☺

Remark 10.5.4. Using the binomial theorem, one can verify that

$$b_j(y) = \sum_{k=j}^{n} a_k (-1)^{k-j} \binom{k}{j} y^{k-j}.$$

But we do not need this explicit formula in the proof above.

Similar calculations can be found in our proof that $\sqrt{2}$ is irrational (Theorem 3.5.4) and in the proof of Lemma 1.4.12.

An Approximate Identity

We use the function $e^{-x^2/2}$ to construct an approximate identity.

Lemma 10.5.5. *The improper integral* $\int_{-\infty}^{\infty} \exp(-x^2/2)\, dx$ *is convergent.*

Proof. Since $(-x)^2 = x^2$ conclude $\int_{-\infty}^{0} \exp(-x^2/2)\, dx = \int_{0}^{\infty} \exp(-x^2/2)\, dx$. Therefore $\int_{-\infty}^{\infty} \exp(-x^2/2)\, dx = 2\int_{0}^{1} \exp(-x^2/2)\, dx + 2\int_{1}^{\infty} \exp(-x^2/2)\, dx$. We must show that the improper integral $\int_{1}^{\infty} \exp(-x^2/2)\, dx$ is convergent.

Since $x^2 \geq x$ when $x \geq 1$ we have $\int_{1}^{N} \exp(-x^2/2)\, dx \leq \int_{1}^{N} \exp(-x/2)\, dx$. By the FTC-Evaluation we have $\int_{1}^{N} \exp(-x/2)\, dx = 2e^{-1/2} - 2e^{-N/2} \to 2e^{-1/2}$ as $N \to \infty$.

So, the sequence $\int_1^N \exp(-x^2/2)\,dx$ is increasing and bounded by $2/\sqrt{e}$, hence

$$\int_1^\infty \exp(-x^2/2)\,dx = \lim_{N\to\infty} \int_1^N \exp(-x^2/2)\,dx$$

is convergent. ☺

Let $A := \int_{-\infty}^\infty \exp(-x^2/2)\,dx$. Then A is some strictly positive real number. It can be shown that $A = \sqrt{2\pi}$, but the value of A is not of importance to us.

Let $E(x) = A^{-1}\exp(-x^2/2)$, for $x \in \mathbb{R}$. Then $E(x) > 0$ for all x since $A > 0$ and $\exp(t) > 0$ for all t. Also, we choose A such that $\int_{-\infty}^\infty E(x)dx = 1$.

Let

$$g_n(x) := nE(nx) = \frac{n}{A}\exp\left(-\frac{n^2x^2}{2}\right)$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Lemma 10.5.6. *(g_n) is an approximate identity.*

Proof. (1) Positivity: $g_n(x) = nE(nx) > 0$ for all n and x. (2) Integral:

$$\int_{-\infty}^\infty g_n(x)\,dx = \int_{-\infty}^\infty nE(nx)\,dx = \int_{-\infty}^\infty E(u)\,du = 1.$$

using the change of variables $u = nx$. (3) It remains to check that g_n concentrated near the origin. Let $\delta > 0$ be given. We must show that $\int_{|x|\geq\delta} g_n \to 0$ as $n \to \infty$. For any integer $n \geq 1$ we have

$$\begin{aligned}\int_{|x|\geq\delta} g_n(x)\,dx &= 2\int_\delta^\infty nE(nx)\,dx \\ &= 2\int_{n\delta}^\infty E(u)\,du\end{aligned}$$

by the change of variables $u = nx$. If $n\delta \geq 1$, then

$$\begin{aligned}\int_{n\delta}^\infty E(u)\,du &= \frac{1}{A}\int_{n\delta}^\infty \exp(-u^2/2)\,du \\ &\leq \frac{1}{A}\int_{n\delta}^\infty \exp(-u/2)\,du \\ &= \frac{2}{A}\exp(-n\delta/2).\end{aligned}$$

Since $\exp(-n\delta/2) \to 0$ as $n \to \infty$ we are done. ☺

Proof of the Weierstrass Approximation Theorem

Having taken care of the preliminaries, we are now ready to give the proof of the Weierstrass Approximation Theorem.

Let $\varepsilon > 0$ be given.

Let $F(x) := f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$ for $x \in [a,b]$. Then $F(a) = F(b) = 0$. So setting $F(x) = 0$ for $x \in \mathbb{R} \setminus [a,b]$ makes F a continuous function on $\mathbb{R}$. F is uniformly continuous (Exercise 9.3.2) and bounded (Exercise 9.3.3) on $\mathbb{R}$.

We will find a polynomial Q such that $|F(x) - Q(x)| < \varepsilon$ on $[a,b]$. Setting $p(x) := Q(x) + f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ then completes the proof.

Remark 10.5.7. The idea is: (i) Fix a large n so $F \star g_n$ is uniformly close to F, this is possible due to the Approximate Identity Lemma. (ii) Find a Taylor polynomial q that is uniformly close to g_n, this is possible by Example 8.2.6. (iii) Verify $F \star q$ is uniformly close to $F \star g_n$. (iv) Since q is a polynomial, $Q := F \star q$ is also a polynomial. Apart from "book keeping" all of the work is in (iii) and the lemmas above.

Let $\varepsilon > 0$ be given. Since (g_n) is an approximate identity (Lemma 10.5.6), it follows from the Approximate Identity Lemma that we can pick n_0 so large that

$$\left|F * g_{n_0}(x) - F(x)\right| < \frac{\varepsilon}{2} \text{ for all } x \in \mathbb{R}. \tag{10.13}$$

Let $R > 0$ be an integer, such that $-R \le a < b \le R$. Let $M > 0$ be a real number such that $|F(t)| \le M$ for all $t \in \mathbb{R}$.

Recall, from Sect. 10.2 that for any $S > 0$ the polynomials $p_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!} \xrightarrow[n\to\infty]{} \exp(x)$ uniformly on $[-S,0]$. In fact, if $K_S := S^{S+1}/S!$, we showed in Example 8.2.6 that

$$|e^x - p_n(x)| \le e^c \frac{K_S}{n+1} < \frac{K_S}{n+1}$$

for all $x \in [-S,0]$. Where we used that if $x \in [-S,0]$ and c is between x and 0, then $c < 0$, so $e^c < 1$.

Let $S := 2R^2 n_0^2$. Pick N so large that $\frac{K_S}{N+1} \le \frac{A}{n_0} \frac{\varepsilon}{4RM}$, then

$$|p_N(x) - \exp(x)| < \frac{A}{n_0} \frac{\varepsilon}{4RM}$$

for all $x \in \left[-2R^2 n_0^2, 0\right]$. Let $q(x) := \frac{n_0}{A} p_N\left(-n_o^2 x^2/2\right)$. Then q is a polynomial of degree $2N$. Since $-2R \le x \le 2R$ implies $-2R^2 n_0^2 \le -\frac{n_0^2 x^2}{2} \le 0$ we have

$$\begin{aligned} \left|q(x) - g_{n_0}(x)\right| &= \left|\frac{n_0}{A} p_N\left(-\frac{n_o^2 x^2}{2}\right) - \frac{n_0}{A} \exp\left(-\frac{n_o^2 x^2}{2}\right)\right| \\ &< \frac{\varepsilon}{4RM} \end{aligned} \tag{10.14}$$

for all x in $[-2R, 2R]$.

For $t \in [-R, R]$. We have the equalities (the third uses $F(x) = 0$ for $|x| \geq R$).

$$\begin{aligned}
|F * g_{n_0}(t) - F * q(t)| &= |F * (g_{n_0} - q)(t)| \\
&= \left| \int_{-\infty}^{\infty} F(x) \left(g_{n_0}(t-x) - q(t-x) \right) dx \right| \\
&= \left| \int_{-R}^{R} F(x) \left(g_{n_0}(t-x) - q(t-x) \right) dx \right| \\
&\leq \int_{-R}^{R} |F(x)| \left| g_{n_0}(t-x) - q(t-x) \right| dx \\
&\leq \int_{-R}^{R} M \frac{\varepsilon}{4RM} dx \\
&= M \frac{\varepsilon}{4RM} 2R = \frac{\varepsilon}{2}.
\end{aligned}$$

The first inequality is a special case of $\left| \int_a^b h \right| \leq \int_a^b |h|$. The second inequality used $|F(t)| \leq M$ and that $x, t \in [-R, R]$ implies $t - x \in [-2R, 2R]$ so can use (10.14) to conclude $\left| g_{n_0}(t-x) - q(t-x) \right| \leq \varepsilon/2$ for all $x, t \in [-R, R]$.

Using this inequality and (10.13) we have

$$\begin{aligned}
&|F * q(t) - f(t)| = |F * q(t) - F(t)| \\
&\leq |F * q(t) - F * g_{n_0}(t)| + |F * g_{n_0}(t) - F(t)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

for $t \in [a, b] \subseteq [-R, R]$

It remains to note that $Q := F * q$ is a polynomial by Lemma 10.5.3.

Problems

Problems for Sect. 10.1

1. Suppose $\sum a_n$ is convergent. Let $b_n := a_{2n-1} + a_{2n}$ for all n. Show $\sum b_n$ is convergent.
2. Suppose $a_n \geq 0$, $b_n \geq 0$, $\sum a_n^2$ is convergent, and $\sum b_n^2$ is convergent. Show $\sum a_n b_n$ is convergent. [*Hint*: $(a-b)^2 \geq 0$.]
3. Look up Raabe's test and give a proof of this test.

4. If $a_n = b_n := (-1)^{n+1} 1/\sqrt{n+1}$, then $\sum a_n$ and $\sum b_n$ are convergent by the Alternating Series Test. Let c_n be determined by (10.2). Show that $\sum c_n$ is divergent.

5. If $\sum a_n$ is absolutely convergent and $\sum b_n$ is convergent. Let c_n be determined by (10.2). Must $\sum c_n$ be convergent?

6. Let $a_k := \frac{1}{k} - \log\left(1 + \frac{1}{k}\right)$.
   a. Show $a_k > 0$.
   b. Prove $\sum_1^\infty a_k$ is convergent. [The sum $\gamma := \sum_{k=1}^\infty a_k$ is Euler's constant.]

7. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and let (b_k) be a sequence of complex numbers such that the sequence of partial sums $B_n := \sum_{k=1}^n b_k$ is convergent. Show $\sum_{k=1}^\infty a_k b_k$ is convergent. [*Hint*: Modify the proof of Dirichlet's Test.]

8. Suppose $a_n \geq 0$ and $\sum a_n$ is convergent.
   a. Assuming there is a real number B such that $|b_n| \leq B$ for all n, show $\sum a_n b_n$ is convergent.
   b. Assuming $\sum b_n$ is convergent, show $\sum a_n b_n$ is convergent.

Problems for Sect. 10.2.

1. Let f_n be a sequence of integrable functions on the interval $[a,b]$. If $\sum_{n=1}^\infty f_n$ converges uniformly to f on $[a,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b \sum_n f_n = \sum_n \int_a^b f_n.$$

2. Show $\sum_{k=0}^\infty x^k \sin(kx)$ is convergent for $x \in]-1,1[$.

Problems for Sect. 10.3

1. Show that y is not differentiable at any point.

Problems for Sect. 10.4

1. Show the Cauchy–Hadamard formula implies the root test.
2. Use Lemma 10.4.4 and that a uniform limit of continuous functions is a continuous function to show: If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on the open disk $\{x \mid |x| < R\}$.
3. Suppose $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ on some interval $]-\delta, \delta[$. Show that $a_n = b_n$ for all n.
4. Find the radius of convergence of $\sum_{k=0}^{\infty} k^k z^k$.
5. Find the radius of convergence of $\sum_{k=0}^{\infty} \frac{1}{k!} z^k$.
6. Find the radius of convergence of $\sum_{k=0}^{\infty} k! z^k$.

Problems for Sect. 10.5

1. We did not prove a change of variables formula for improper integrals. Carefully verify the claim: $\int_{-\infty}^{\infty} nE(nx)\,dx = \int_{-\infty}^{\infty} E(u)\,du$ in the proof of Lemma 10.5.6.

Solutions and Hints for the Exercises

Exercise 10.1.1. $\sum_{k=1}^{n}(ax_k + by_k) = a\sum_{k=1}^{n} x_k + b\sum_{k=1}^{n} y_k$ the right hand side converges to $a\sum_{k=1}^{\infty} x_k + b\sum_{k=1}^{\infty} y_k$.

Exercise 10.1.5. (1) $x_{n+1} = s_{n+1} - s_n \to s - s = 0$. (2) $\sum_{k=N+1}^{\infty} x_k = \sum_{k=1}^{\infty} x_k - s_N \to \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} x_k = 0$. (3) $s_{n+1} = s_n + x_{n+1}$ so (s_n) is increasing, hence convergent iff bounded.

Exercise 10.1.7. Since $|\sum_{k=n}^{m} x_k| \leq \sum_{k=n}^{m} |x_k|$, this follows from the Cauchy criterion.

Exercise 10.1.9. We may assume the inequality holds for all n. It follows that $|x_{n+1}| \leq |x_1| r^n$, hence an application of the Dominated Convergence Theorem completes the proof.

Exercise 10.1.10. Similar to the previous exercise.

Exercise 10.1.12. This really is similar to Example 10.1.11. The change is we want the inequality to go the other way, so the sequence of partial sums will be unbounded. This can be accomplished by using $\frac{1}{n^a} \geq \frac{1}{2^{a(k+1)}}$ for n between 2^k and $2^{k+1} - 1$.

Exercise 10.1.14. Suppose $\int_1^{\infty} f$ is convergent. Since f is decreasing $s = \sum_{k=1}^{n} f(k)\mathbb{1}_{]k,k+1[}$ is a lower step function for f on the interval $[1, n+1]$. Hence $\sum_{k=1}^{n} x_k = \sum s \leq \int_1^{n+1} f$. Consequently, the partial sums $(\sum_{k=1}^{n} x_k)$ is bounded by $\int_1^{\infty} f$. The converse is similar, using $S = \sum_{k=2}^{n} f(k)\mathbb{1}_{]k-1,k[}$ is an upper step function for f on the interval $[1, n]$.

Exercise 10.1.17. Show $-a_{n+1}B_n + \alpha_{n+1}B_{n+1} = a_{n+1}b_{n+1} - a_{n+2}B_{n+1}$.

Exercise 10.3.3. Similar to what we did for x.

Exercise 10.3.3 For each n either $a_n < t$ or $a_n = t$. For n with $a_n \neq t$ we have

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{b_n - t}{b_n - a_n} \frac{f(b_n) - f(t)}{b_n - t} + \frac{t - a_n}{b_n - a_n} \frac{f(t) - f(a_n)}{t - a_n}.$$

Exercise 10.4.7. This is a consequence of Theorem 9.2.6.

Exercise 10.4.8. $(1+\varepsilon)^n \geq n\varepsilon$ [why?]. So $n^{1/n} \leq (1/\varepsilon)^{1/n}(1+\varepsilon)$. Now use that $(1/\varepsilon)^{1/n} \to 1$ as $n \to \infty$. Other proofs are also possible.

Exercise 10.4.9. Follows from the previous exercise and the Cauchy–Hadamard formula.

Exercise 10.4.10. This is a simple consequence of Corollary 9.2.9.

Chapter 11
Trigonometric Functions and Applications

We investigate the sine and cosine functions, show the Weierstrass function is continuous and nowhere differentiable, construct the number π, establish that π is an irrational number, and give a brief treatment of polar coordinates. We also discuss arc length, in particular, we show and circumference of the unit circle is 2π. Finally, we show the area of the unit circle equals π.

The basic transcendental functions are the logarithm $\log(x)$, the exponential function e^x, and the sine and cosine functions $\sin(x)$ and $\cos(x)$. We investigated $\log(x)$ and e^x in Chap. 8.

In the first two sections our treatment is precise, but somewhat informal, with many results and proofs stated in passing.

11.1 Exponential Function

We begin by using power series to extend the exponential function to a function of a complex variable. For a complex number $z \in \mathbb{C}$, let

$$\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}, \tag{11.1}$$

where the conventions $z^0 = 1$ and $0! = 1$ tell us that the first term is equal to 1 for any z. When z is a real number the right-hand side is the power series for the usual exponential function, hence (11.1) extends the usual exponential function from $\mathbb{R}$ to $\mathbb{C}$. We will sometimes write $e^z := \exp(z)$ for $z \in \mathbb{C}$.

Exercise 11.1.1. The radius of convergence is $R = \infty$.

Since the radius of convergence is $R = \infty$, the series is uniformly convergent on any bounded subset of $\mathbb{C}$. Hence, the *exponential function* exp determined by (11.1) is a continuous function mapping $\mathbb{C} \to \mathbb{C}$.

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By manipulating the power series we see that

$$\exp(z)\exp(w) = \exp(z+w).$$

In fact, this is an application of the Cauchy product:

$$\begin{aligned}\sum_j \frac{z^j}{j!} \sum_k \frac{w^k}{k!} &= \sum_n \left(\sum_{j+k=n} \frac{z^j w^k}{j!\,k!} \right) \\ &= \sum_n \frac{(z+w)^n}{n!}\end{aligned}$$

the last equality is the Binomial Theorem: $(z+w)^n = \sum_{j+k=n} \frac{n!}{j!k!} z^j w^k$.

11.2 Trigonometric Functions

By the power series expansion (11.1)

$$\exp(-iy) = \sum \frac{(-iy)^n}{n!} = \sum \frac{\left(\overline{iy}\right)^n}{n!} = \overline{\sum \frac{(iy)^n}{n!}} = \overline{\exp(iy)},$$

for $y \in \mathbb{R}$. Hence,

$$|\exp(iy)|^2 = \exp(iy)\overline{\exp(iy)} = \exp(0) = 1.$$

So $\exp(iy)$ is a point on the unit circle $a^2 + b^2 = 1$ in the plane $\mathbb{R}^2 = \mathbb{C}$. For $y \in \mathbb{R}$, we define the *sine* and *cosine* functions as the real and imaginary parts of $\exp(iy)$:

$$\begin{aligned}\cos(y) &:= \operatorname{Re}\exp(iy) \text{ and} \\ \sin(y) &:= \operatorname{Im}\exp(iy),\end{aligned}$$

for $y \in \mathbb{R}$. Thus, $\exp(iy) = \cos(y) + i\sin(y)$. Since $\exp(i0) = 1 = 1 + i0$ we have $\cos(0) = 1$ and $\sin(0) = 0$.

Since $|e^{iy}| = 1$ we have

$$\cos^2(y) + \sin^2(y) = 1,$$

in particular, $-1 \le \cos(y) \le 1$ and $-1 \le \sin(y) \le 1$. By taking the real and imaginary parts of

$$e^{i(x+y)} = e^{ix} e^{iy}$$

we see that

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \cos(x)\sin(y) + \sin(x)\cos(y).\end{aligned}$$

Setting $x = y$ gives $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\sin(x)\cos(x)$.

Exercise 11.2.1. Prove

$$\cos(-y) = \cos(y)$$
$$\sin(-y) = -\sin(y)$$

for all $y \in \mathbb{R}$.

Exercise 11.2.2. Use the power series expansion of $\exp(iy)$ to verify that

$$\cos(y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!}$$
$$\sin(y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}.$$

Hence, $\cos' = -\sin$ and $\sin' = \cos$. This can also be verified by differentiating $\exp(iy)$ with respect to y:

$$\exp'(iy) = i\exp(iy)$$

by the chain rule, so

$$\cos'(y) + i\sin'(y) = i(\cos(y) + i\sin(y)).$$

We could use the power series to define cos and sin of a complex variable, but we will not need these extensions of sine and cosine from $\mathbb{R}$ to $\mathbb{C}$.

Construction of π

Consider only $x \geq 0$. By the Mean Value Theorem $\sin(x) - \sin(0) = \cos(a)(x - 0)$ for some c between 0 and x, hence

$$-x \leq \sin(x) \leq x \quad \forall x \geq 0, \tag{11.2}$$

since $-1 \leq \cos(c) \leq 1$.

Lemma 11.2.3. *For all $x \geq 0$,*

$$1 - \tfrac{1}{2}x^2 \leq \cos(x) \leq 1. \tag{11.3}$$

Proof. Let $f(x) = \cos(x)$. Then $f(0) = 1$, $f(x) \leq 1, \forall x$, and $f'(x) = -\sin(x)$. By (11.2) we have $f'(x) \geq -x$. Integrating this inequality over the interval $[0,x]$ gives $f(x) - f(0) \geq -\frac{1}{2}x^2$. ☺

Exercise 11.2.4. Prove

$$x - \tfrac{1}{6}x^3 \leq \sin(x) \leq x \quad \forall x \geq 0 \tag{11.4}$$

and

$$1-\tfrac{1}{2}x^2 \le \cos(x) \le 1-\tfrac{1}{2}x^2+\tfrac{1}{24}x^4 \quad \forall x \ge 0. \tag{11.5}$$

Lemma 11.2.5. $\cos(x) > 0$ *for* $0 \le x < \sqrt{2}$ *and* $\cos(x) = 0$ *has a solution between* $\sqrt{2}$ *and* $\sqrt{6-2\sqrt{3}}$.

Proof. Since $1-\frac{1}{2}x^2 > 0$ for $0 \le x < \sqrt{2}$, it follows from the first inequality in (11.5) that $\cos(x) > 0$, when $0 \le x < \sqrt{2}$. In particular, $\cos\left(\sqrt{2}\right) \ge 0$.

Let $a := \sqrt{6-2\sqrt{3}}$ and $b := \sqrt{6+2\sqrt{3}}$. Then $0 < a < b$ and

$$x^4-12x^2+24 = \left(x^2-a^2\right)\left(x^2-b^2\right) = (x-a)(x+a)(x-b)(x+b).$$

Hence, $x^4-12x^2+24 < 0$ when $a < x < b$. It follows from the second inequality in (11.5) that $\cos(a) \le 0$.

If $\cos\left(\sqrt{2}\right) = 0$ or $\cos(a) = 0$ we are done. If $\cos\left(\sqrt{2}\right) > 0$ and $\cos(a) < 0$, then the Intermediate Value Theorem gives a c between $\sqrt{2}$ and a, such that $\cos(c) = 0$. ☺

Let ω be the smallest positive root of $\cos(x) = 0$. By Lemma 11.2.5

$$\sqrt{2} \le \omega \le \sqrt{6-2\sqrt{3}}.$$

Exercise 11.2.6. $\sqrt{6-2\sqrt{3}} < \sqrt{3}$.

Let $\pi := 2\omega$. Then $\cos(\pi/2) = \cos(\omega) = 0$. Since $\cos(0) = 1 > 0$ and $\pi/2$ is the smallest positive root of $\cos(x) = 1$ we have

$$\cos(x) > 0 \quad \text{for } 0 < x < \pi/2. \tag{11.6}$$

Lemma 11.2.7. $\sin(\pi) = 0$ *and*

$$\sin(x) > 0 \quad \textit{for } 0 < x < \pi. \tag{11.7}$$

Proof. Since $\sin(\pi) = 2\sin(\pi/2)\cos(\pi/2) = 0$ we have shown that π is a root of $\sin(x) = 0$. It remains to establish (11.7). By (11.4) $\sin(x) \ge x(1-x^2/6)$ hence

$$\sin(x) > 0 \quad \text{for } 0 < x < \sqrt{6}. \tag{11.8}$$

Suppose $\sin(t) = 0$ for some t in the open interval $\left]\sqrt{6},\pi\right[$. Then

$$0 = \sin(t) = 2\sin(t/2)\cos(t/2)$$

means that $\sin(t/2) = 0$ or $\cos(t/2) = 0$. Since $t/2 < \pi/2$, $\cos(t/2) = 0$ contradicts (11.6). Since $t/2 < \pi/2 < \sqrt{3} < \sqrt{6}$, $\sin(t/2) = 0$ contradicts (11.8). ☺

Exercise 11.2.8. $\sin(\pi/2) = 1$.

We can now verify all sort of properties of sin and cos, for example,

$$\cos(\pi) = \cos^2(\pi/2) - \sin^2(\pi/2) = 0 - 1 = -1.$$

Similarly,

Exercise 11.2.9. $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$.

Periodicity: $\cos(x+2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) = \cos(x)$.

Exercise 11.2.10. $\sin(x+2\pi) = \sin(x)$.

Exercise 11.2.11. Prove

$$\begin{aligned} \cos(x) &= \sin(x+\pi/2) \text{ and} \\ \sin(x) &= -\cos(x+\pi/2). \end{aligned}$$

Exercise 11.2.12. Prove

$$\begin{aligned} \cos(x+\pi) &= -\cos(x) \text{ and} \\ \sin(x+\pi) &= -\sin(x). \end{aligned}$$

Example 11.2.13. Since the unit circle is determined by the equation $x^2 + y^2 = 1$, the area of the unit circle is

$$\begin{aligned} 4\int_0^1 \sqrt{1-x^2}dx &= 4\int_0^{\pi/2} \sqrt{1-\sin^2(t)}\cos(t)\,dt \\ &= 4\int_0^{\pi/2} \cos^2(t)\,dt \\ &= 4\int_0^{\pi/2} \tfrac{1}{2}(1+\cos(2t))\,dt \\ &= \left(2\left(\tfrac{\pi}{2}\right) - \sin(\pi)\right) - (2(0) - \sin(0)) = \pi, \end{aligned}$$

where the first equality used the change of variables formula (7.4) with $x = g(t) = \sin(t)$ and the fourth equality used the derivative of $(2t + \sin 2t)$ equals $2(1 + \cos(2t))$ and the Fundamental Theorem of Calculus.

In Sect. 11.3, it is shown that the unit circle has length 2π.

Polar Coordinates

Given a point (x,y) with $x, y \in \mathbb{R}$. We will show there is exactly one θ such that $-\pi < \theta \leq \pi$ and

$$(x,y) = r(\cos(\theta), \sin(\theta)),$$

where $r := \sqrt{x^2+y^2}$.

Fix a point $(x,y) \in \mathbb{R}^2$. Let $r := \sqrt{x^2+y^2}$, $\alpha := \frac{x}{r}$, and $\beta := \frac{y}{r}$. Then $(x,y) = r(\alpha,\beta)$. Hence, the following lemma completes the proof.

Lemma 11.2.14. *There is exactly one number θ, with $-\pi < \theta \leq \pi$ such that*

$$\alpha = \cos(\theta) \text{ and } \beta = \sin(\theta).$$

Proof. By construction $\alpha^2+\beta^2 = 1$. If $\alpha = 1$ then $\beta = 0$, so $\theta = 0$ and if $\alpha = -1$ then $\beta = 0$, so $\theta = \pi$. Hence, we will consider $-1 < \alpha < 1$.

Since $\cos'(\theta) = -\sin(\theta) < 0$ for $0 < \theta < \pi$, the function $\cos : [0,\pi] \to [-1,1]$ is a strictly decreasing continuous function mapping the interval $[0,\pi]$ onto the interval $[-1,1]$. Since $-1 < \alpha < 1$, it follows from the Intermediate Value Theorem that there is a θ, $0 < \theta < \pi$, such that $\cos(\theta) = \alpha$. Using $\cos^2(\theta)+\sin^2(\theta) = 1$ and $\alpha^2+\beta^2 = 1$ it follows that

$$\sin(\theta) = \pm\beta.$$

Hence, either $\sin(\theta) = \beta$ or $\sin(-\theta) = -\sin(\theta) = \beta$. Using $\cos(-\theta) = \cos(\theta) = \alpha$, it follows that either $(\cos(\theta),\sin(\theta)) = (\alpha,\beta)$ or $(\cos(-\theta),\sin(-\theta)) = (\alpha,\beta)$. This established the existence of θ. The uniqueness follows from $\cos : [0,\pi] \to [-1,1]$ being strictly decreasing, hence one-to-one. ☺

Remark 11.2.15. Using $z = x+iy$, $|z| = \sqrt{x^2+y^2} = r$, and $e^{i\theta} = (\cos(\theta),\sin(\theta))$ we can write

$$(x,y) = r(\cos(\theta),\sin(\theta))$$

as

$$z = re^{i\theta} = |z|\,e^{i\theta}.$$

11.3 Arc Length★

A curve is a continuous function $\phi : [a,b] \to \mathbb{C}$. For example, if $f : [a,b] \to \mathbb{R}$ is a real valued continuous function, then $\phi(t) := (t, f(t))$ determines a curve. The length, called *arc length*, of the curve ϕ is

$$\operatorname{length}(\phi) := \sup\left\{ \sum_{k=1}^{n} |\phi(t_k) - \phi(t_{k-1})| \;\middle|\; n \in \mathbb{N} \text{ and } a = t_0 < t_1 < \cdots < t_n = b \right\},$$

where the supremum is over all partitions of the closed interval $[a,b]$. A curve whose length is finite is called *rectifiable*. The sum, corresponds to approximating the curve by line segments connecting the partition points (see Fig. 11.1). The following lemma shows that the sum gets larger when we refine the partition, hence taking the supremum in the definition of arc length makes sense.

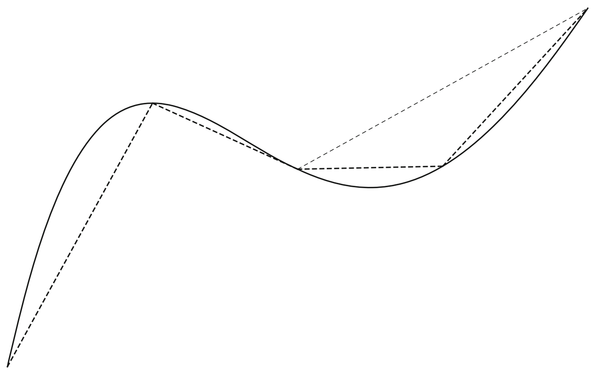

Fig. 11.1 Illustrating the definition of arc length and Lemma 11.3.1

Lemma 11.3.1. *If $a = t_0 < t_1 < \cdots < t_n = b$ is a refinement of $a = s_0 < s_1 < \cdots < s_m = b$, then*

$$\sum_{j=1}^{m} \left|\phi\left(s_j\right) - \phi\left(s_{j-1}\right)\right| \leq \sum_{k=1}^{n} \left|\phi\left(t_k\right) - \phi\left(t_{k-1}\right)\right|.$$

Proof. This is a direct consequence of the triangle inequality (see Fig. 11.1). ☺

Theorem 11.3.2 (Arc Length Formula). *If $x, y : [a,b] \to \mathbb{R}$ have continuous derivatives on $[a,b]$, then*

$$\text{length}\left(\phi\right) = \int_a^b \sqrt{x'\left(t\right)^2 + y'\left(t\right)^2}\, dt,$$

where $\phi\left(t\right) = \left(x\left(t\right), y\left(t\right)\right)$. In particular, the curve ϕ is rectifiable.

Proof. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. By the Mean Value Theorem

$$\begin{aligned}
\sum_{k=1}^{n} \left|\phi\left(t_k\right) - \phi\left(t_{k-1}\right)\right| &= \sum_{k=1}^{n} \sqrt{\left(x\left(t_k\right) - x\left(t_{k-1}\right)\right)^2 + \left(y\left(t_k\right) - y\left(t_{k-1}\right)\right)^2} \\
&= \sum_{k=1}^{n} \sqrt{\left(\frac{x\left(t_k\right) - x\left(t_{k-1}\right)}{t_k - t_{k-1}}\right)^2 + \left(\frac{y\left(t_k\right) - y\left(t_{k-1}\right)}{t_k - t_{k-1}}\right)^2}\left(t_k - t_{k-1}\right) \qquad (11.9) \\
&= \sum_{k=1}^{n} \sqrt{\left(x'\left(c_k\right)\right)^2 + \left(y'\left(d_k\right)\right)^2}\left(t_k - t_{k-1}\right)
\end{aligned}$$

for some c_k, d_k in $[x_{k-1}, x_k]$.

By the Extreme Value Theorem both x' and y' have a largest value on $[a,b]$. Let A be the largest value of x' and let B be the largest value of y'. Using (11.9), we get

$$\sum_{k=1}^{n} |\phi(t_k) - \phi(t_{k-1})| \leq \sum_{k=1}^{n} \sqrt{A^2+B^2}\,(t_k - t_{k-1}) = \sqrt{A^2+B^2}\,(b-a).$$

Hence, ϕ is rectifiable and $\operatorname{length}(\phi) \leq \sqrt{A^2+B^2}\,(b-a)$.

Let $L := \operatorname{length}(\phi)$ and $I := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$. We must show $L = I$. Let $\varepsilon > 0$ be given. By uniform continuity of x' and of y', there is a $\delta > 0$, such that $|x'(s) - x'(t)| < \varepsilon/(b-a)$ and $|y'(s) - y'(t)| < \varepsilon/(b-a)$ whenever $|s-t| < \delta$. Since L is finite, there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ such that

$$L - \varepsilon < \sum_{k=1}^{n} |\phi(t_k) - \phi(t_{k-1})| \leq L. \tag{11.10}$$

By the lemma inserting additional partition points increases the sum. Hence, the inequalities hold for any refinement of the given partition. In particular, given any $\delta > 0$, we may assume $t_k - t_{k-1} < \delta$ for all k.

Since x' and y' are continuous, it follows from the Extreme Value Theorem that there are $r'_k, r''_k\ s'_k, s''_k$ in $[t_{k-1}, t_k]$, such that

$$\begin{aligned} \left|x'\left(r'_k\right)\right| &\leq \left|x'(t)\right| \leq \left|x'\left(r''_k\right)\right| \text{for all } t \in [t_{k-1}, t_k] \\ \left|y'\left(s'_k\right)\right| &\leq \left|y'(t)\right| \leq \left|y'\left(s''_k\right)\right| \text{for all } t \in [t_{k-1}, t_k]. \end{aligned}$$

Then,

$$s := \sum_{k=1}^{n} \sqrt{\left(x'\left(r'_k\right)\right)^2 + \left(y'\left(s'_k\right)\right)^2}\,(t_k - t_{k-1})$$

is a lower sum for I,

$$S := \sum_{k=1}^{n} \sqrt{\left(x'\left(r''_k\right)\right)^2 + \left(y'\left(s''_k\right)\right)^2}\,(t_k - t_{k-1})$$

is an upper sum for I, and applying (11.9)

$$s \leq \sum_{k=1}^{n} |\phi(t_k) - \phi(t_{k-1})| \leq S. \tag{11.11}$$

Using the reverse triangle inequality, Exercise E.1.8, i.e.,

$$\left|\sqrt{a_1^2 + b_1^2} - \sqrt{a_2^2 + b_2^2}\right| \leq \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2},$$

we get

$$S - s = \sum_{k=1}^{n} \left(\sqrt{\left(x'\left(r''_k\right)\right)^2 + \left(y'\left(s''_k\right)\right)^2} - \sqrt{\left(x'\left(r'_k\right)\right)^2 + \left(y'\left(s'_k\right)\right)^2}\right)(t_k - t_{k-1})$$

$$\le \sum_{k=1}^{n} \sqrt{\left(x'\left(r_k''\right) - x'\left(r_k'\right)\right)^2 + \left(y'\left(s_k''\right) - y'\left(s_k'\right)\right)^2} \left(t_k - t_{k-1}\right)$$
$$\le \sum_{k=1}^{n} \sqrt{\left(\frac{\varepsilon}{(b-a)}\right)^2 + \left(\frac{\varepsilon}{(b-a)}\right)^2} \left(t_k - t_{k-1}\right)$$
$$= \sum_{k=1}^{n} \frac{\varepsilon\sqrt{2}}{(b-a)} \left(t_k - t_{k-1}\right) = \varepsilon\sqrt{2}.$$

Hence, as $\varepsilon \to 0$ both $s \to I$ and $S \to I$. So by (11.11) $\sum_{k=1}^{n} |\phi(t_k) - \phi(t_{k-1})| \to I$. But $\sum_{k=1}^{n} |\phi(t_k) - \phi(t_{k-1})| \to L$ by (11.10). Thus $I = L$ as we needed to show. ☺

The following example shows that arc $e^{i\theta}$, $a \le \theta \le b$ of the unit circle has length $b-a$.

Example 11.3.3. Let $0 \le a < b \le 2\pi$. Consider $\phi(t) := (\cos(t), \sin(t))$ as a curve $\phi : [a,b] \to \mathbb{C}$. Then

$$\begin{aligned}\operatorname{length}(\phi) &= \int_a^b \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt \\ &= \int_a^b 1\, dt = b - a.\end{aligned}$$

In particular, the unit circle has length 2π.

11.4 Weierstrass' Nowhere Differentiable Function★

A continuous nowhere differentiable function was first constructed by Bolzano. Bolzano used a geometric construction utilizing a limit of piecewise linear functions. Bolzano's construction is similar to a construction that is used to generate fractal sets, sometimes called an Iterated Function System. The following is essentially a reproduction of an argument given by Weierstrass in 1872 in a lecture to the Royal Academy of Science in Berlin. Before Weierstrass publishes his example, most mathematicians including Johann Carl Friedrich Gauss (30 April 1777 Braunschweig to 23 February 1855 Göttingen) thought that a continuous function would have a derivative at most points. We presented a different example in Sect. 10.3.

Theorem 11.4.1. *Let* $0 < b < 1$ *a real number and let a be a positive odd integer. If* $ab > 1$ *and* $\frac{2}{3} > \frac{\pi}{ab-1}$, *then*

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x\pi)$$

is continuous on $\mathbb{R}$ *and not differentiable at any point in* $\mathbb{R}$.

In what follows a and b will be fixed. For example, $a = 13$ and $b = 1/2$ satisfied the stated assumptions. The example in Sect. 10.3 was of the form $f(x) = \sum_{n=0}^{\infty} 2^{-n} g(9^n x)$ where g is a continuous function of period two which is not differentiable at all points. By contrast $x \to \cos(x\pi)$ is a continuous function of period two which is differentiable at all points.

Since $|b^n \cos(a^n x\pi)| \leq b^n$ and $\sum_n b^n$ is a convergent geometric series, it follows from the Weierstrass M-test that f is uniformly continuous on $\mathbb{R}$. So only the non-differentiability claim is interesting.

Remark 11.4.2. If $f_N(x) := \sum_{n=0}^{N} b^n \cos(a^n x\pi)$, then f_N converges uniformly to f as $N \to \infty$. Each f_N is $\mathscr{C}^\infty$, yet the limit function f does not have a derivative even at one point.

Fix x_0. For each $m \in \mathbb{N}_0$, let β_m be the integer satisfying

$$\frac{1}{2} \leq \beta_m - a^m x_0 < \frac{3}{2}. \tag{11.12}$$

Since $a > 1$ we have, $\beta_m / a^m \to x_0$. Hence, if $f'(x_0)$ exists, then

$$\frac{f\left(\frac{\beta_m}{a^m}\right) - f(x_0)}{\frac{\beta_m}{a^m} - x_0} \to f'(x_0).$$

We will show that

$$(-1)^{\beta_m} \frac{f\left(\frac{\beta_m}{a^m}\right) - f(x_0)}{\frac{\beta_m}{a_m} - x_0} \to \infty.$$

Consequently, $f'(x_0)$ does not exist.

Write

$$\begin{aligned}
(-1)^{\beta_m} \frac{f\left(\frac{\beta_m}{a^m}\right) - f(x_0)}{\frac{\beta_m}{a_m} - x_0} &= \sum_{n=0}^{\infty} (-1)^{\beta_m} b^n \frac{\cos\left(a^n \frac{\beta_m}{a^m} \pi\right) - \cos(a^n x_0 \pi)}{\frac{\beta_m}{a^m} - x_0} \\
&= \sum_{n=0}^{m-1} \cdots + \sum_{n=m}^{\infty} \cdots \\
&= A_m + B_m.
\end{aligned}$$

We will show that

$$|A_m| \leq (ab)^m \frac{\pi}{ab - 1} \tag{11.13}$$

and

$$B_m \geq (ab)^m \frac{2}{3}. \tag{11.14}$$

Using (11.13) and (11.14), it follows that

$$(-1)^{\beta_m} \frac{f\left(\frac{\beta_m}{a^m}\right) - f(x_0)}{\frac{\beta_m}{a_m} - x_0} \geq B_m - |A_m| \geq (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab-1}\right) \to \infty.$$

Hence, we just need to verify (11.13) and (11.14).

(11.13): By the Mean Value Theorem

$$\begin{aligned}
A_m &= \sum_{n=0}^{m-1} (-1)^{\beta_m} b^n \frac{\cos\left(a^n \frac{\beta_m}{a^m}\pi\right) - \cos(a^n x_0 \pi)}{\frac{\beta_m}{a^m} - x_0} \\
&= \sum_{n=0}^{m-1} (-1)^{\beta_m} a^n b^n \pi \frac{\cos\left(a^n \frac{\beta_m}{a^m}\pi\right) - \cos(a^n x_0 \pi)}{a^n \left(\frac{\beta_m}{a^m} - x_0\right)\pi} \\
&= \sum_{n=0}^{m-1} (-1)^{\beta_m} a^n b^n \pi \sin(c_{n,m}),
\end{aligned}$$

for some $c_{n,m}$. Taking the absolute value and using the triangle inequality we get

$$\begin{aligned}
|A_m| &= \left|\sum_{n=0}^{m-1} (-1)^{\beta_m} a^n b^n \pi \sin(c_{n,m})\right| \\
&\leq \sum_{n=0}^{m-1} a^n b^n \pi = \pi \frac{(ab)^m - 1}{ab-1} \\
&< \pi \frac{(ab)^m}{ab-1},
\end{aligned}$$

by evaluating the geometric series. Consequently, (11.13) holds.

(11.14): For $n \geq m$, we have

$$(-1)^{\beta_m} \cos\left(a^n \frac{\beta_m}{a^m}\pi\right) = (-1)^{\beta_m} \cos\left(a^{n-m}\beta_m \pi\right) = (-1)^{\beta_m}(-1)^{\beta_m} = 1$$

since a is odd. For the same reason

$$\begin{aligned}
(-1)^{\beta_m} \cos(a^n x_0 \pi) &= (-1)^{\beta_m} \cos\left(a^{n-m} a^n x_0 \pi\right) \\
&= (-1)^{\beta_m} \cos\left(a^{n-m}\beta_m \pi + a^{n-m}(a^m x_0 - \beta_m)\pi\right) \\
&= (-1)^{\beta_m}(-1)^{\beta_m} \cos\left(a^{n-m}(a^m x_0 - \beta_m)\pi\right) \\
&= \cos\left(a^{n-m}(a^m x_0 - \beta_m)\pi\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
B_m &= \sum_{n=m}^{\infty} (-1)^{\beta_m} b^n \frac{\cos\left(a^n \frac{\beta_m}{a^m}\pi\right) - \cos(a^n x_0 \pi)}{\frac{\beta_m}{a^m} - x_0} \\
&= \sum_{n=m}^{\infty} b^n \frac{1 - \cos\left(a^{n-m}\left(a^m x_0 - \beta_m\right)\pi\right)}{\frac{\beta_m}{a^m} - x_0} \\
&= b^m \frac{1 - \cos\left(\left(a^m x_0 - \beta_m\right)\pi\right)}{\frac{\beta_m}{a^m} - x_0} + \sum_{n=m+1}^{\infty} b^n \frac{1 - \cos\left(a^{n-m}\left(a^m x_0 - \beta_m\right)\pi\right)}{\frac{\beta_m}{a^m} - x_0}.
\end{aligned}
$$

Since β_m satisfies (11.12), we have $\cos\left(\left(a^m x_0 - \beta_m\right)\pi\right) \le 0$ and $\frac{\beta_m}{a^m} - x_0 \ge 0$. (Note, this is the only time we used (11.12), the rest of the proof just need the β_m's to be integers.) Consequently,

$$
\begin{aligned}
B_m &\ge b^m \frac{1-0}{\frac{\beta_m}{a^m} - x_0} + \sum_{n=m+1}^{\infty} b^n \frac{0}{\frac{\beta_m}{a^m} - x_0} \\
&= a^m b^m \frac{1}{\beta_m - a^m x_0} > (ab)^m \frac{1}{3/2}.
\end{aligned}
$$

This verifies (11.14).

Remark 11.4.3. One might think that continuous functions without derivatives are exceptional. In fact, the opposite is true. The set of continuous functions on an interval with a derivative at some point in that interval is a "vanishingly small" subset of the set of all continuous functions on that interval. The author likes the Baire category interpretation of "vanishingly small." Baire category is named after René-Louis Baire (21 January 1874, Paris to 5 July 1932, Chambéry). Other mathematicians prefer other interpretations.

11.5 The Number Pi is Irrational★

Irrationality of π was first proven by Johann Heinrich Lambert (26 August 1728 Mülhausen, Elsaß to 25 September 1777 Berlin) in 1761. Lambert actually proved: if x is rational, then $\tan(x)$ is irrational. So $\tan(\pi/4) = 1$ implies π is irrational. Carl Louis Ferdinand von Lindemann (12 April 1852 Hanover to 6 March 1939 Munich) proved in 1882 that e^a is transcendental for every nonzero algebraic number a. Setting in $a = i\pi$ in this result shows that π is transcendental.

Remark 11.5.1. It is not known whether or not any the numbers $\pi \pm e, \pi e$, or $\frac{\pi}{e}$ are rational. On the other hand, e is a root of

$$x^2 - (\pi + e)x + \pi e = 0.$$

Hence, if $\pi + e$ and πe both are rational, then e would be algebraic of order 2, contradicting Theorem 8.3.3. Thus, at least one of $\pi + e$ and πe is irrational.

Theorem 11.5.2 (Lambert). *The number π is irrational.*

Proof. Suppose $\pi = a/b$, where a and b are positive integers. Let

$$f_n(x) := \frac{1}{n!} x^n (a - bx)^n.$$

Then, $f_n(0) = f_n(\pi) = 0$ and $f_n(x) > 0$ for $0 < x < \pi$. Hence,

$$I_n := \int_0^\pi f_n(x) \sin(x)\, dx > 0.$$

Now,

$$0 < I_n \le \int_0^\pi \frac{\pi^n a^n}{n!}\, dx = \frac{\pi\,(\pi a)^n}{n!} \to 0 \text{ as } n \to \infty.$$

Hence, the proof is completed by showing that each I_n is an integer for all $n \ge 0$. Clearly,

$$I_0 = \int_0^\pi \sin(x)\, dx = -\cos(x)|_0^\pi = 2.$$

Integrations by parts leads to

$$\begin{aligned}
I_1 &= \int_0^\pi x(a - bx) \sin(x)\, dx = \int_0^\pi (ax - bx^2) \sin(x)\, dx \\
&= -(ax - bx^2) \cos(x)\big|_0^\pi + \int_0^\pi (a - 2bx) \cos(x)\, dx \\
&= 0 - 2b \left(x \sin(x)|_0^\pi - \int_0^\pi \sin \right) = 4b.
\end{aligned}$$

Integrating by parts twice using $f_{n+2}(0) = f_{n+2}(\pi) = 0$, and $f'_{n+2}(0) = f'_{n+2}(\pi) = 0$ we see

$$I_{n+2} = \int_0^\pi f_{n+2}(x) \sin(x)\, dx = -\int_0^\pi f''_{n+2}(x) \sin(x)\, dx,$$

hence we need a formula for f''_{n+2}. For $k \ge 0$,

$$\begin{aligned}
f'_{k+1}(x) &= \left(\frac{1}{(k+1)!} x^{k+1} (a - bx)^{k+1} \right)' \\
&= \frac{1}{k!} \left(x^k (a - bx)^{k+1} - bx^{k+1} (a - bx)^k \right) \\
&= f_k(x) \left((a - bx) - bx \right) \\
&= f_k(x) \left(a - 2bx \right).
\end{aligned}$$

Using

$$(a - 2bx)^2 = a^2 - 4abx + 4bx^2 = a^2 - 4bx(a - bx)$$

implies

$$f_n(x)\,(a-2bx)^2 = a^2 f_n(x) - 4b(n+1)f_{n+1}(x)$$

we get

$$\begin{aligned} f''_{n+2}(x) &= (f_{n+1}(x)\,(a-2bx))' \\ &= f_n(x)\,(a-2bx)^2 - 2bf_{n+1}(x) \\ &= a^2 f_n(x) - 2b(2n+3)f_{n+1}(x). \end{aligned}$$

It follows that

$$\begin{aligned} I_{n+2} &= -\int_0^\pi \left(a^2 f_n(x) - 2b(2n+3)f_{n+1}(x)\right)\sin(x)\,dx \\ &= 2b(n+3)I_{n+1} - a^2 I_n. \end{aligned}$$

Thus, I_n is an integer for all n. ☺

Problems

Problems for Sect. 11.1

1. Let $f(x) := \log(1+x)$ for $x > -1$. Let

$$T(x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

be the Taylor series of f. Use the series to define a function

$$g(z) := \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

of a complex variable z.

(a) Find the radius of convergence R of the power series.
(b) Show $f(x) = T(x)$ for $-R < x < R$.
(c) Conclude $f(x) = g(x)$ for $-R < x < R$.

Problems for Sect. 11.2

1. Prove that $\int_0^\infty \frac{sin(xt)}{1+t^2}\,dt$ converges uniformly. That is prove the improper integral exists and

$$\forall\varepsilon, \exists M, \forall R, R > M \implies \left[\forall x, \int_0^\infty \frac{\sin(xt)}{1+t^2}\,dt - \int_0^R \frac{\sin(xt)}{1+t^2}\,dt < \varepsilon\right].$$

2. Prove $e^{\pi} - \pi \neq 20$.
3. Suppose f is continuous. Let $g(x) := \int_0^x \sin(x-t) f(t) dt$. Prove $g'' + g = f$. [*Hint*: Begin by showing that g' exists and finding a formula for g'.]
4. Let $f(x) := \frac{\cos(x)}{1+x^2}$. Prove $\int_0^1 f < 1$.

Problems for Sect. 11.3

1. Fix $0 \leq a < b \leq 2\pi$. Find the length of the curve $\phi(t) := e^{it}$, $t \in [a,b]$.

Problems for Sect. 11.4

1. Does $\left(\sum_{n=0}^{N} b^n \cos(a^n x \pi)\right)'$ converge as $N \to \infty$ for all x?
2. Use Corollary 9.2.9 to find a sufficient condition for the Weierstrass function to be differentiable.
3. If $x_0 = 0$, the proof in the text simplifies. Write out this simplification. [For example, $\beta_m = 1$ for all m and $\cos(a^n x_0 \pi) = 1$.]
4. Continuation of Problem 3. Find integers β_m such that
$$(-1)^{\beta_m} \frac{f\left(\frac{\beta_m}{a^m}\right) - f(0)}{\frac{\beta_m}{a_m} - 0} \to -\infty.$$
[*Hint*: Equation (11.12) must be replaced.]
5. Show it is possible to choose β_m such that $\beta_m / a^m \to x_0$ and
$$(-1)^{\beta_m} \frac{f\left(\frac{\beta_m}{a^m}\right) - f(x_0)}{\frac{\beta_m}{a_m} - x_0} \to -\infty.$$
Thus, $f'(x_0)$ cannot be a finite or even an infinite value.

Problems for Sect. 11.5

An alternative proof that π is irrational is outlined as sequence of problems below. This proof is due to Ivan Morton Niven (25 October 1915, Vancouver to 9 May 1999, Eugene).

Assume $\pi = a/b$ for some positive integers a and b. For any natural number $n \geq 1$, let

$$f_n(x) := \frac{x^n(a-bx)^n}{n!} \text{ and } F_n(x) := \sum_{j=0}^{n} (-1)^j f_n^{(2j)}(x),$$

where $f^{(k)}$ is the kth derivative of f. Note that $a - b\pi = 0$. Expanding the product in the definition of f_n we see that $f_n(x) = \frac{1}{n!}\sum_{k=n}^{2n} c_k x^k$ for some integers c_k. Differentiating this polynomial we get

$$(**) \quad f_n^{(j)}(x) = \frac{1}{n!} \sum_{k=\max\{j,n\}}^{2n} c_k \frac{k!}{(k-j)!} x^{k-j},$$

for $0 \leq j \leq 2n$.

The proof is now completed in 11 easy steps:

1. $0 \leq f_n(x) \leq \pi^n a^n / n!$ for $0 \leq x \leq \pi$ and all n [Directly from the definition of f_n.]
2. $0 < \int_0^\pi f_n \sin$ for all n. [Since both f_n and sin are > 0 on $]0, \pi[$.]
3. There is an N such that $\int_0^\pi f_n \sin < 1$. [Is a consequence of 1]
4. $\frac{k!}{n!(k-j)!}$ is an integer for all $n \leq j \leq k$. [$\frac{k!}{n!(k-n)!}$ is a binomial coefficient, hence an integer. $\frac{k!}{n!(k-(n+1))!} = (k-n)\frac{k!}{n!(k-n)!}$, etc.]
5. $f_n(x) = f_n(\pi - x)$ for all x and all n. [Directly from the definition of f_n, since $a - b(\pi - x) = x$.]
6. $f_n^j(0) = f_n^{(j)}(\pi) = 0$, for all n, j such that $0 \leq j < n$. [By 5 and $(**)$.]
7. $f_n^{(j)}(0)$ and $f_n^{(j)}(\pi)$ are integers, for all n, j such that $n \leq j \leq 2n$. [By 4, 5, and $(**)$.]
8. $F_n(0)$ and $F_n(\pi)$ are integers for all n. [By the definition of F_n, 6, and 7]
9. $F_n + F_n'' = f_n$ for all n. [The point of the $(-1)^j$ in the definition of F_n.]
10. $(F_n' \sin - F_n \cos)' = f_n \sin$ for all n. [Derived using 9]
11. $\int_0^\pi f_n \sin = F_n(0) - F_n(\pi)$ is an integer for all n. [By 8 and 10]

By 2, 3, and 11 $\int_0^\pi f_N \sin$ is an integer in $]0, 1[$. This contradiction completes the proof that π is irrational.

Solutions and Hints for the Exercises

Exercise 11.1.1. The ratio test show that the series is absolutely convergent for all z.

Exercise 11.2.1. $e^{-iy} = \overline{e^{iy}}$.

Exercise 11.2.2. $\sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \cdots + i \cdots$.

Exercise 11.2.4. Similar to our proof of (11.3).

Exercise 11.2.6. $\sqrt{6-2\sqrt{3}} < \sqrt{3} \Leftarrow 6-2\sqrt{3} < 3 \Leftarrow 3 < 2\sqrt{3} \Leftarrow 9 < 12$.

Exercise 11.2.8. $1 = \cos^2(\pi/2) + \sin^2(\pi/2) = 0 + \sin^2(\pi/2)$ and $\sin(\pi/2) > 0$.

Exercise 11.2.9. $\sin(2\pi) = 2\sin(\pi)\cos(\pi) = 2 \cdot 0 \cdot (-1) = 0$ and $\cos^2(2\pi) = \cos^2(\pi) - \sin^2(\pi) = (-1)^2 - 0^2 = 1$.

Exercise 11.2.10. Similar to $\cos(x+2\pi) = \cos(x)$.

Exercise 11.2.11. Similar to $\cos(x+2\pi) = \cos(x)$ and $\sin(x+2\pi) = \sin(x)$.

Exercise 11.2.12. Use Exercise 11.2.11.

Chapter 12
Fourier Series

Our approach to Fourier series is based on some rudimentary facts about linear spaces equipped with an inner product. Our approach to pointwise convergence is based on Dini's criterion. We discuss uniform convergence and Cesàro summability of Fourier series. We also show the Fourier series of a Riemann integrable function convergences in the mean. We establish Weyl's criterion for uniform distribution of sequences. As an application, we establish the uniform distribution in the unit interval of the fractional parts of the integer multiples of an irrational number.

12.1 Introduction

Attempting to diagonalize the derivative $f \to -if'$ acting on function is the interval $[0,1]$ leads to the Fourier series for f.

Diagonalization★

Recall from linear algebra that given a linear transformation L on $\mathbb{R}^d$ or $\mathbb{C}^d$ it is often useful to find an orthonormal basis e_k, $k = 1,2,\ldots,d$ such that

$$Le_k = \lambda_k e_k$$

for some scalars λ_k. The e_k are called *eigenvectors* and the λ_k are the corresponding *eigenvalues*. The matrix of the transformation L with respect to the basis e_k, $k = 1,2,\ldots,d$ is a diagonal matrix with the λ_k's as the entries on the diagonal. We will focus on the $\mathbb{C}^d$ case. For x,y in $\mathbb{C}^d$ the dot product or inner product is

$$x \cdot y = \sum_{k=1}^{d} x_k \overline{y_k},$$

S. Pedersen, *From Calculus to Analysis*, DOI 10.1007/978-3-319-13641-7_12

where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. The existence of an orthonormal basis diagonalizing the transformation L is implied by

$$Lx \cdot y = x \cdot Lx \tag{12.1}$$

for all x, y in $\mathbb{C}^d$. That the e_k's form an orthonormal basis means that

$$e_j \cdot e_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and for any $x \in \mathbb{C}^d$ there are scalars a_k such that

$$x = \sum_{k=1}^{d} a_k e_k.$$

For an orthonormal basis there is a formula for the scalars:

$$a_k = x \cdot e_k.$$

That is

$$x = \sum_{k=1}^{d} (x \cdot e_k) e_k \tag{12.2}$$

for all x in $\mathbb{C}^d$.

Let $\mathscr{R}$ denote the set of Riemann integrable functions $f : [0,1] \to \mathbb{C}$. Since the product of two integrable functions is an integrable function, we can introduce an *inner product* on $\mathscr{R}$ by setting

$$\langle f \mid g \rangle = \langle f \mid g \rangle_{\mathscr{R}} = \langle f \mid g \rangle_2 := \int_0^1 f\overline{g} = \int_0^1 f(x)\overline{g(x)}\, dx, \tag{12.3}$$

for f and g in $\mathscr{R}$.

We study the diagonalization problem for

$$L := -i\frac{d}{dx} \text{ that is for } L : f \to -if',$$

where $f : [0,1] \to \mathbb{C}$. The i is there to make the analogue of (12.1) possible. Specifically, suppose L is defined on the set $\mathscr{D}(L)$ of differentiable functions f on $[0,1]$ for which f' is integrable and $f(0) = f(1)$. If f and g are in $\mathscr{D}(L)$, then integration by parts shows

$$\begin{aligned} \langle Lf \mid g \rangle = \left\langle -if' \mid g \right\rangle &= \int_0^1 -if'\,\overline{g} \\ &= \int_0^1 f\,\overline{(-ig')} = \left\langle f \mid -ig' \right\rangle = \langle f \mid Lg \rangle, \end{aligned}$$

since $-if(1)\overline{g(1)}+if(0)\overline{g(0)}=0$. Hence, the analogue of (12.1) holds. The eigenvector equation $Lf=\lambda f$ is

$$-if'=\lambda f$$

so $f'=i\lambda f$, thus $f(x)=ce^{i\lambda x}$. The boundary condition $f(0)=f(1)$ means $c=ce^{i\lambda}$, hence $\lambda=2\pi k$ for some integer k. Setting $c=1$, we arrive at the eigenfunctions

$$e_k(x)=e^{i2\pi kx}, k\in\mathbb{Z}.$$

Note the eigenfunctions are orthonormal because

$$\langle e_j \mid e_k\rangle=\int_0^1 e^{i2\pi(j-k)x}\,dx=\begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}. \tag{12.4}$$

It remains to investigate the analogue of (12.2), that is, can we write any f as

$$f=\sum_{k\in\mathbb{Z}}a_ke_k:=\lim_{N\to\infty}\sum_{k=-N}^{N}a_ke_k$$

for some scalars a_k and if the scalars a_k are determined by the formula

$$a_k=\langle f \mid e_k\rangle .$$

Note, the sum is infinite, hence there are convergence questions that needs to be investigated.

Fourier Series

The major contribution of Jean Baptiste Joseph Fourier (21 March 1768, Auxerre to 16 May 1830, Paris) to mathematics is now known as Fourier series. He also made other major contributions to science, for example, he discovered the green house effect.

Let $e_k(x):=e^{i2\pi x}$. If f is integrable on $[0,1]$, then

$$\widehat{f}(k):=\langle f \mid e_k\rangle=\int_0^1 f\overline{e_k}=\int_0^1 f(x)e^{-i2\pi kx}\,dx \tag{12.5}$$

are the *Fourier coefficients* of f. Since the integral is linear, $f\to\widehat{f}$ is linear, in the sense that if f and g are integrable and a and b are constants, then $(af+bg)\widehat{\ }(k)=a\widehat{f}(k)+b\widehat{g}(k)$ for all k.

The N^{th} partial sum of the Fourier series for f at x is

$$S_Nf(x)=(S_Nf)(x):=\sum_{k=-N}^{N}\widehat{f}(k)e_k(x)=\sum_{k=-N}^{N}\widehat{f}(k)e^{i2\pi kx}. \tag{12.6}$$

Finally,

$$\sum_{k\in\mathbb{Z}} \langle f \mid e_k\rangle\, e_k = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\, e_k := \lim_{N\to\infty} S_N f \tag{12.7}$$

is the *Fourier series* associated with f. The limit in (12.7) can have several different interpretations. Each interpretations leads to a different convergence question. We investigate the following convergence questions:

- Pointwise convergence, i.e., for which f does $S_N f(x) \underset{N\to\infty}{\longrightarrow} f(x)$ for all x? See Theorem 12.4.1 and Theorem 12.4.4.
- Uniform convergence, i.e., for which f does $S_N f \underset{N\to\infty}{\rightrightarrows} f$? See Theorem 12.4.5 and Theorem 12.5.5.
- Convergence in the mean, i.e., convergence with respect to the norm

$$\|f\| = \|f\|_{\mathscr{R}} = \|f\|_2 := \langle f \mid f\rangle^{1/2} = \left(\int_0^1 |f|^2\right)^{1/2}, \tag{12.8}$$

  i.e., for which f does $\|f - S_N f\| \underset{N\to\infty}{\longrightarrow} 0$? See Theorem 12.7.3.

Convergence of (f_n) to g in the sense that $\|g - f_n\| \to 0$ is called L^2-*convergence* or *convergence in the mean*.

12.2 Linear Algebra

The diagonalization problem belongs to linear algebra. This section develops the linear algebra we will need.

Let A be a set and let $\mathscr{V}$ be a set of complex valued functions defined on A. If $af + cg \in \mathscr{V}$ for all $a, b \in \mathbb{C}$ and all $f, g \in \mathscr{V}$, then $\mathscr{V}$ is a *vector space*.

A complex valued map $f, g \in \mathscr{V} \to \langle f \mid g\rangle \in \mathbb{C}$ is an *inner product* on $\mathscr{V}$, if for all f, g, and h in $\mathscr{V}$ and all complex numbers a and b

$$\langle af + bg \mid h\rangle = a\langle f \mid h\rangle + b\langle g \mid h\rangle \tag{12.9}$$

$$\langle f \mid g\rangle = \overline{\langle g \mid f\rangle} \tag{12.10}$$

$$\langle f \mid f\rangle \ge 0 \tag{12.11}$$

$$\langle f \mid f\rangle = 0 \implies f = 0 \tag{12.12}$$

Equation (12.9) states the map $f \in \mathscr{V} \to \langle f \mid g\rangle \in \mathbb{C}$ is linear for each fixed $g \in \mathscr{V}$.

Lemma 12.2.1. *If f, g, and h are in $\mathscr{V}$ and a is a complex number, then*

$$\langle f \mid g + h\rangle = \langle f \mid g\rangle + \langle f \mid h\rangle$$
$$\langle f \mid ag\rangle = \overline{a}\langle f \mid g\rangle.$$

Thus, for fixed f, the map $g \in \mathscr{V} \to \langle f \mid g \rangle \in \mathbb{C}$ is conjugate linear.

Proof. The calculations

$$\begin{aligned}\langle f \mid g+h \rangle &= \overline{\langle g+h \mid f \rangle} = \overline{\langle g \mid f \rangle + \langle h \mid f \rangle} = \langle f \mid g \rangle + \langle f \mid h \rangle \text{ and}\\ \langle f \mid ag \rangle &= \overline{\langle ag \mid f \rangle} = \overline{a \langle g \mid f \rangle} = \overline{a} \langle f \mid g \rangle\end{aligned}$$

based on (12.9) and (12.10), constitutes the proof. ☺

Riemann Integrable Functions

Comparing to (12.5) it is natural to consider

$$\langle f \mid g \rangle = \langle f \mid g \rangle_{\mathscr{R}} := \int_0^1 f\overline{g} \tag{12.13}$$

on the vector space $\mathscr{R}$ of Riemann integrable functions. Since the product of two integrable functions is an integrable function, the integral makes sense. The properties (12.9–12.11) are simple consequences of properties of the integral. For continuous f we have (12.12), hence (12.13) determines an inner product on the set of continuous functions. There are positive nonzero integrable functions with integral zero, hence (12.12) does not hold. Thus $\langle \cdot \mid \cdot \rangle$ is an inner product on $\mathscr{R}$, except (12.12) does not hold. The following is an adequate substitute for our purposes.

Exercise 12.2.2. Suppose f and g are integrable. If $\|f\| = 0$, then $\langle f \mid g \rangle = 0$. Where $\|f\| := \sqrt{\langle f \mid f \rangle}$.

For simplicity, we refer to $\langle \cdot \mid \cdot \rangle$ determined by (12.13) as an inner product also on $\mathscr{R}$.

Remark 12.2.3. If $\|f - g\| = 0$, then

$$\widehat{f}(k) - \widehat{g}(k) = \int_0^1 (f(x) - g(x))\, e^{-i2\pi kx} dx = \langle f - g, e_k \rangle = 0,$$

the last equality is Exercise 12.2.2. Hence $S_N f = S_N g$ for all N. Consequently, Fourier series cannot distinguish between f and g.

Remark 12.2.4. One can eliminate nonzero functions with norm zero by considering the equivalence classes

$$[f] := \{g \mid \|f - g\| = 0\}$$

of functions instead of working with the functions themselves.

The Vector Space of Square Summable Sequences

For any integrable f, the Fourier coefficients $\widehat{f}(k)$, $k \in \mathbb{Z}$ form a sequence. To investigate these sequences we need an inner product on an appropriate vector space of sequences.

Remark 12.2.5. Equation (12.10) holds for the inner product on $\mathbb{C}^d$. In that case of $\mathbb{C}^d$ it is customary to set $e_1 = (1,0,0,\ldots,0)$, $e_2 = (0,1,0,0,\ldots,0)$, $\cdots$, and $e_d = (0,0,\ldots,0,1)$.

We extend Remark 12.2.5 to $d = \infty$. In this extension we replace $\mathbb{C}^d$ by the set ℓ^2 of all sequences $x_k \in \mathbb{C}$, $k \in \mathbb{Z}$, such that

$$\sum_{k=-\infty}^{\infty} |x_k|^2 := \lim_{N\to\infty} \sum_{k=-N}^{N} |x_k|^2 < \infty.$$

In particular, $\sum_{k=-\infty}^{\infty} x_k^2$ is absolutely convergent and consequently convergent. Sequences in ℓ^2 are called *square summable*.

Example 12.2.6. The sequence (x_k) determined by $x_0 = 0$, and $x_k = 1/|k|$, if $k \neq 0$, is in ℓ^2, but $\sum_{k=-\infty}^{\infty} x_k$ is divergent. The sequence (y_k) determined by $y_0 = 0$, and $y_k = 1/\sqrt{|k|}$, if $k \neq 0$, is not in ℓ^2.

Lemma 12.2.7. *If (x_k) and (y_k) are in ℓ^2, then $\sum_{k=-\infty}^{\infty} x_k \overline{y_k}$ is convergent.*

Proof. Let $c_k := \max\{|x_k|, |y_k|\}$. Then

$$|x_k \overline{y_k}| \leq c_k^2 \leq |x_k|^2 + |y_k|^2,$$

for all k. Consequently, $\sum_{k=-\infty}^{\infty} x_k \overline{y_k}$ is absolutely convergent by dominated convergence. ☺

Analogous to (12.3) we define

$$\langle (x_k) \mid (y_k) \rangle\rangle = \langle (x_k) \mid (y_k) \rangle\rangle_{\ell^2} := \sum_{k=-\infty}^{\infty} x_k \overline{y_k}. \tag{12.14}$$

We will show this defines an inner product on ℓ^2. To verify the first property in (12.9) we need to know that ℓ^2 is a vector space:

Lemma 12.2.8. *If (x_k) and (y_k) are in ℓ^2, then $(x_k) + (y_k) := (x_k + y_k)$ is in ℓ^2. Hence, ℓ^2 is a vector space.*

Proof. As in the proof of the previous lemma let $c_k := \max\{|x_k|, |y_k|\}$ be in ℓ^2. Using $a + \overline{a} \le 2|a|$ and the definition of $c_k$we have

$$\begin{aligned}
|x_k + y_k|^2 &= (x_k + y_k)\,\overline{(x_k + y_k)} \\
&= x_k\overline{x_k} + x_k\overline{y_k} + \overline{x_k}y_k + y_k\overline{y_k} \\
&\le |x_k|^2 + 2|x_k||y_k| + |y_k|^2 \\
&\le |x_k|^2 + 2c_k^2 + |y_k|^2 .
\end{aligned}$$

In the proof of the previous lemma we showed that (c_k) is square summable. Hence, summing the right-hand side gives a convergent series. Consequently, by dominated convergence, summing the left-hand side also gives a convergent series. This yields the desired conclusion. ☺

It is now easy to see that (12.9–12.12) hold. Hence (12.14) determines an inner product. Similar to (12.8) we set

$$\|(x_k)\| = \|(x_k)\|_{\ell^2} := \langle (x_k) \mid (x_k)\rangle_{\ell^2}^{1/2} = \left(\sum_{k=-\infty}^{\infty} |x_k|^2\right)^{1/2}$$

for (x_k) in ℓ^2.

Analogously with the case of $\mathbb{C}^d$, it is customary to let $e_k \in \ell^2$ be the sequence that consist entirely of zeros, except there is a one in the k^{th} place.

Some Properties of Inner Products

Properties of interest to us are the Projection Theorem, the Pythagorean Theorem, Bessel's Inequality, and the Riemann–Lebesgue Lemma.

For the case of $\mathscr{R} = \mathscr{V}$ and $e_k(x) = e^{i2\pi x}$, $k \in \mathbb{Z}$, we have

$$\langle e_j \mid e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases}. \tag{12.15}$$

by (12.4). In the case of $\mathscr{V} = \ell^2$ equation (12.15) is obvious for the standard e_k sequence $k \in \mathbb{Z}$, where the sequence e_k consist entirely of zeros, except there is a one in the k^{th} place. When (12.15) holds for j,k in some set S, we say the set $\{e_k \mid k \in S\}$ is orthonormal.

The rest of this section is based on (12.15), on the properties (12.9–12.11) of the inner product, and on the definition $\|f\| = \sqrt{\langle f \mid f\rangle}$. Hence, the following does not depend on the way we defined the inner product or on the expressions for the e_k. In particular, the results below are true both for the set of integrable functions $\mathscr{R}$ and for the set ℓ^2 of infinite sequences.

In the following we assume $\mathscr{V}$ is a vector space equipped with an inner product $f, g \to \langle f \mid g \rangle$ satisfying (12.9–12.11), that S is a subset of $\mathbb{Z}$ and there are vectors e_k, $k \in S$ such that (12.15) holds for all $j, k \in S$. When (12.15) holds for all $j, k \in S$ we say the set $\{e_k \mid k \in S\}$ is *orthonormal*. Let $\|f\| := \sqrt{\langle f \mid f \rangle}$ for all $f \in \mathscr{V}$.

The basic result is an equality:

Lemma 12.2.9. *If the set $\{e_k \mid k \in S\}$ is orthonormal and the integers $m, m+1, \ldots, n$ are in S, then*

$$\left\| f - \sum_{k=m}^{n} a_k e_k \right\|^2 = \langle f \mid f \rangle - \sum_{k=m}^{n} |\langle f \mid e_k \rangle|^2 + \sum_{k=m}^{n} |\langle f \mid e_k \rangle - a_k|^2 \tag{12.16}$$

for all f in $\mathscr{V}$ and all complex numbers a_k.

Proof. For scalars a_k and integers $m \leq n$ we have

$$\begin{aligned}
\left\| f - \sum_{k=m}^{n} a_k e_k \right\|^2 &= \left\langle f - \sum_{k=m}^{n} a_k e_k \,\middle|\, f - \sum_{j=m}^{n} a_j e_j \right\rangle \\
&= \langle f \mid f \rangle - \sum_{k=m}^{n} a_k \langle e_k \mid f \rangle - \sum_{j=m}^{n} \overline{a_j} \langle f \mid e_j \rangle + \sum_{k=m}^{n} \sum_{j=m}^{n} a_k \overline{a_j} \langle e_k \mid e_j \rangle \\
&= \langle f \mid f \rangle - \sum_{k=m}^{n} (a_k \langle e_k \mid f \rangle + \overline{a_k} \langle f \mid e_k \rangle) + \sum_{k=m}^{n} a_k \overline{a_k}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\langle f \mid f \rangle - \sum_{k=m}^{n} |\langle f \mid e_k \rangle|^2 + \sum_{k=m}^{n} |\langle f \mid e_k \rangle - a_k|^2 \\
&= \langle f \mid f \rangle - \sum_{k=m}^{n} \langle f \mid e_k \rangle \langle e_k \mid f \rangle + \sum_{k=m}^{n} (\langle f \mid e_k \rangle - a_k)(\langle e_k \mid f \rangle - \overline{a_k}) \\
&= \langle f \mid f \rangle - \sum_{k=m}^{n} \langle f \mid e_k \rangle \langle e_k \mid f \rangle \\
&\quad + \sum_{k=m}^{n} (\langle f \mid e_k \rangle \langle e_k \mid f \rangle - \langle f \mid e_k \rangle \overline{a_k} - a_k \langle e_k \mid f \rangle + a_k \overline{a_k}) \\
&= \langle f \mid f \rangle + \sum_{k=m}^{n} (-\langle f \mid e_k \rangle \overline{a_k} - a_k \langle e_k \mid f \rangle + a_k \overline{a_k}).
\end{aligned}$$

Since the two sides of (12.16) are equal to the same thing they are equal. ☺

There are several interesting consequences of this calculation.

Theorem 12.2.10 (Projection Theorem). *Let f be in $\mathscr{V}$. If the set $\{e_k \mid k \in S\}$ is orthonormal and the integer $m, m+1, \ldots, n$ are in S, then the norm $\|f - \sum_{k=m}^{n} a_k e_k\|$*

is minimized by choosing $a_k = \langle f \mid e_k \rangle$. *More precisely,*

$$\left\| f - \sum_{k=m}^{n} a_k e_k \right\|^2 \geq \|f\|^2 - \sum_{k=m}^{n} |\langle f \mid e_k \rangle|^2$$

for all complex numbers a_k, *with equality when* $a_k = \langle f \mid e_k \rangle$.

Proof. Since $\sum_{k=m}^{n} |\langle f \mid e_k \rangle - a_k|^2 \geq 0$ for all a_k, and we have equality when all $a_k = \langle f \mid e_k \rangle$, this follows from (12.16). ☺

Theorem 12.2.11 (Pythagorean Theorem). *If the set* $\{e_k \mid k \in S\}$ *is orthonormal and the integer* $m, m+1, \ldots, n$ *are in* S, *then*

$$\|f\|^2 = \sum_{k=m}^{n} |\langle f \mid e_k \rangle|^2 + \left\| f - \sum_{k=m}^{n} \langle f \mid e_k \rangle e_k \right\|^2,$$

for all $f \in \mathscr{V}$.

Proof. This is part of the Projection Theorem, it can also be seen by setting $a_k = \langle f \mid e_k \rangle$ in (12.16) and rearranging the equality. ☺

Replacing the second term on the right-hand side of the equality in the Pythagorean Theorem by zero leads us to

$$\sum_{k=m}^{n} |\langle f \mid e_k \rangle|^2 \leq \|f\|^2 \tag{12.17}$$

for all $m \leq n$.

Everything above is true even if we only have a finite number of e_k. The next results require us to have an infinite number of e_k, we assume they are indexed by $k \in \mathbb{Z}$. The following inequality is named after Friedrich Wilhelm Bessel (Born: 22 July 1784, Minden to 17 March 1846, Königsberg).

Theorem 12.2.12 (Bessel's Inequality). *If the set* $\{e_k \mid k \in \mathbb{Z}\}$ *is orthonormal, then*

$$\sum_{k=-\infty}^{\infty} |\langle f \mid e_k \rangle|^2 \leq \|f\|^2,$$

for all $f \in \mathscr{V}$.

Proof. As a special case of (12.17) we have $\sum_{k=-N}^{N} |\langle f \mid e_k \rangle|^2 \leq \|f\|^2$ for all N, hence it follows from Monotone Convergence that $\lim_{N\to\infty} \sum_{k=-N}^{N} |\langle f \mid e_k \rangle|^2$ exists and is less than or equal to $\|f\|^2$. Consequently,

$$\sum_{k=-\infty}^{\infty} |\langle f \mid e_k \rangle|^2 = \lim_{N\to\infty} \sum_{k=-N}^{N} |\langle f \mid e_k \rangle|^2 \leq \|f\|^2,$$

as we needed to show. ☺

Remark 12.2.13. Parseval's identity states that for Fourier series we have equality in Bessel's inequality, this is Corollary 12.7.4.

Theorem 12.2.14 (Riemann–Lebesgue Lemma). *If the set $\{e_k \mid k \in \mathbb{Z}\}$ is orthonormal, then for any $f \in \mathscr{V}$, $\langle f \mid e_k \rangle \to 0$ as $|k| \to \infty$.*

Proof. Since the sum in Bessel's Inequality is convergent, the terms $|\langle f \mid e_k \rangle|^2 \to 0$ as $|k| \to \infty$, by the Test For Divergence. ☺

Henri Léon Lebesgue (28 June 1875, Beauvais to 26 July 1941, Paris) made other contributions to the theory of Fourier series, however his major contribution to mathematics was a new method of integration, now known as the Lebesgue integral. This integral is an essential part of modern analysis.

Some Properties of the Norm

We use Bessel's inequality to establish the Cauchy–Schwarz Inequality and the triangle inequality for $\|\cdot\|$.

Theorem 12.2.15 (Cauchy–Schwarz Inequality). *For all f and g in $\mathbb{V}$,*

$$|\langle f \mid g \rangle| \le \|f\| \, \|g\| \, .$$

Proof. If $\|g\| = 0$ both sides are zero. Suppose $\|g\| \neq 0$, and let $e_0 := g/\|g\|$. Then Bessel's inequality (12.17) with $m = n = 0$ (and $S = \{0\}$) gives

$$|\langle f \mid e_0 \rangle| \le \|f\| \, .$$

Using $e_0 := g/\|g\|$ this is

$$\left| \left\langle f \;\middle|\; \frac{g}{\|g\|} \right\rangle \right| \le \|f\| \, .$$

Multiplying both sides by $\|g\|$ yields the desired inequality. ☺

Theorem 12.2.16 (Triangle Inequality). *For all f and g in $\mathscr{V}$,*

$$\|f+g\| \le \|f\| + \|g\| \, .$$

Proof. This is a simple consequence of Cauchy–Schwarz.

$$\begin{aligned}
\|f+g\|^2 &= \langle f+g \mid f+g \rangle \\
&= \langle f \mid f \rangle + \langle f \mid g \rangle + \langle g \mid f \rangle + \langle g \mid g \rangle \\
&\le \langle f \mid f \rangle + |\langle f \mid g \rangle| + |\langle g \mid f \rangle| + \langle g \mid g \rangle \\
&\le \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \\
&= (\|f\| + \|g\|)^2
\end{aligned}$$

Taking square roots completes the proof. ☺

A map $f \in \mathscr{V} \to \|f\| \in \mathbb{R}$ is a *norm*, if for all f and g in $\mathscr{V}$ and all complex numbers a

$$\|af\| = |a|\,\|f\| \tag{12.18}$$
$$\|f+g\| \leq \|f\|+\|g\| \tag{12.19}$$
$$\|f\| \geq 0 \tag{12.20}$$
$$\|f\| = 0 \implies f = 0 \tag{12.21}$$

We established (12.19) above. Observe (12.18) follows from $\langle af \mid af\rangle = a\bar{a}$ $\langle f \mid f\rangle$. Setting $a=1$, gives (12.20). And also shows (12.21) is equivalent to (12.12). Hence, $\|(x_k)\| = \left(\sum_k |x_k|^2\right)^{1/2}$ is a norm on ℓ^2, $\|f\| = \left(\int_0^1 |f|^2\right)^{1/2}$ is a norm on the set of continuous functions and a norm on the set of Riemann integrable functions $\mathscr{R}$, expect on $\mathscr{R}$ there are nonzero functions f with $\|f\|=0$. For simplicity we will also refer to $\|f\|$ as a norm on $\mathscr{R}$. See Remark 12.2.4 for a way to eliminate functions of norm zero.

In terms of integrable functions f and g on the interval $[0,1]$ the Cauchy–Schwarz and triangle inequalities state

$$\left|\int_0^1 f\overline{g}\right| \leq \left(\int_0^1 |f|^2\right)^{1/2}\left(\int_0^1 |g|^2\right)^{1/2}$$
$$\left(\int_0^1 |f+g|^2\right)^{1/2} \leq \left(\int_0^1 |f|^2\right)^{1/2} + \left(\int_0^1 |g|^2\right)^{1/2}.$$

For sequences (a_k) and (b_k) in ℓ^2 these inequalities take the form

$$\left|\sum_{k=-\infty}^{\infty} a_k\overline{b_k}\right| \leq \left(\sum_{k=-\infty}^{\infty} |a_k|^2\right)^{1/2}\left(\sum_{k=-\infty}^{\infty} |b_k|^2\right)^{1/2}$$
$$\left(\sum_{k=-\infty}^{\infty} |a_k+b_k|^2\right)^{1/2} \leq \left(\sum_{k=-\infty}^{\infty} |a_k|^2\right)^{1/2} + \left(\sum_{k=-\infty}^{\infty} |b_k|^2\right)^{1/2}$$

Similar inequalities are established in the problems for Sect. 6.8.

12.3 Partial Sums

We establish some properties of the Dirichlet kernel and use it to write the partial sums $S_N f$ of the Fourier series of f as a convolution.

We now specialize to integrable functions. In this case the inner product is determined by

$$\langle f \mid g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$$

and the corresponding norm is

$$\|f\| = \int_0^1 |f(x)|^2 \, dx.$$

Recall also the Fourier coefficients are

$$\widehat{f}(k) = \langle f \mid e_k \rangle = \int_0^1 f(x) \, e^{-i2\pi kx} \, dx$$

and the partial sums of the Fourier series is

$$S_N f(x) = \sum_{k=-N}^{N} \widehat{f}(k) e_k(x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{-i2\pi kx}.$$

Before beginning our investigation of convergence of $S_N f$ we restate Bessel's Inequality and the Riemann–Lebesgue Lemma in terms of the Fourier coefficients $\widehat{f}(k)$.

Theorem 12.3.1 (Bessel's Inequality). *If the set $\{e_k \mid k \in \mathbb{Z}\}$ is orthonormal, then*

$$\sum_{k=-\infty}^{\infty} \left| \widehat{f}(k) \right|^2 \le \|f\|^2,$$

for all integrable f on $[0,1]$.

Theorem 12.3.2 (Riemann–Lebesgue Lemma). *If f is integrable on $[0,1]$, then*

$$\left| \widehat{f}(k) \right| \to 0 \text{ as } k \to \infty \text{ and as } k \to -\infty.$$

The Dirichlet Kernel

In the remainder of this section we rewrite the sum $S_N f$ as a convolution of f and a function D_N called the Dirichlet kernel.

An integrable function f defined on the interval $[0,1]$ can be extended to a periodic function on all of $\mathbb{R}$ by setting $f(x+n) = f(x)$ for all $0 \le x < 1$ and all $n \in \mathbb{Z}$. The so extended function has period one. Note this may change the value of f at $x = 1$, but it does not change any integrals we may want to calculate. For the remainder of this chapter we will assume that f has *period one*, that is $f(x+1) = f(x)$ for all real x, unless otherwise stated.

Exercise 12.3.3. For any real number a we have

$$\int_a^{a+1} f = \int_0^1 f.$$

Since both f and e_k have period one Exercise 12.3.3 implies

$$\widehat{f}(k) = \int_{-1/2}^{1/2} f(t)e^{-i2\pi kt}\,dt.$$

The *Dirichlet kernel* is

$$D_N(t) := \sum_{k=-N}^{N} e_k(t) = \sum_{k=-N}^{N} e^{i2\pi kt}. \tag{12.22}$$

This kernel appears naturally when we plug the definitions into $S_N f$:

$$\begin{aligned}
S_N f(x) &= \sum_{k=-N}^{N} \widehat{f}(k)e^{i2\pi kx} \\
&= \sum_{k=-N}^{N} \left(\int_{-1/2}^{1/2} f(t)e^{-i2\pi kt}\,dt \right) e^{i2\pi kx} \\
&= \int_{-1/2}^{1/2} f(t) \left(\sum_{k=-N}^{N} e^{i2\pi k(x-t)} \right) dt \\
&= \int_{-1/2}^{1/2} f(t)D_N(x-t)\,dt \\
&= -\int_{x+1/2}^{x-1/2} f(x-u)D_N(u)\,du \\
&= \int_{-1/2}^{1/2} f(x-t)D_N(t)\,dt,
\end{aligned}$$

where we, interchanged the *finite* sum and the integral, used the change of variables $u = x - t$, and applied Exercise 12.3.3. Hence, $S_N f = f \star D_N$ (Fig. 12.1).

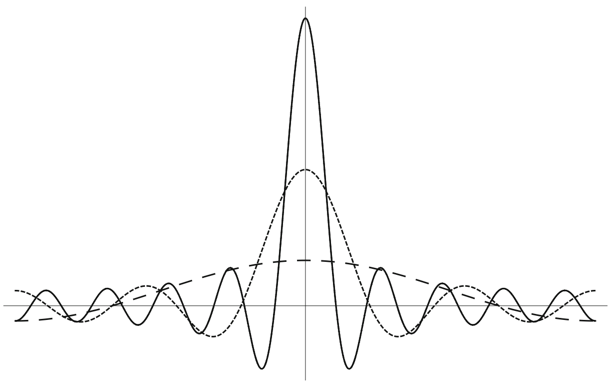

Fig. 12.1 A few samples D_1, D_4, and D_9 of the Dirichlet kernel

We have written $S_N f$ as a convolution of f and D_N. Unfortunately D_N is not positive, hence not an approximate identity. Consequently, we cannot apply the Approximate Identity Lemma. Nevertheless, the Dirichlet kernel is very useful when investigating pointwise convergence of Fourier series. Hence, we establish some properties of this kernel.

Exercise 12.3.4. Show that

$$\int_{-1/2}^{1/2} D_N = 1$$

for all $N \geq 1$.

Exercise 12.3.5. Show that

$$D_N(t) = \begin{cases} 2N+1 & \text{if } t = 0 \\ \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)} & \text{if } t \neq 0 \end{cases}.$$

Usually we simply write

$$D_N(t) = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)},$$

since the limit as $t \to 0$ equals $2N+1$. Using $D_N(t) = D_N(-t)$,

$$S_N f(x) = f \star D_N(x) = \int_{-1/2}^{1/2} f(x-t) D_N(t)\, dt = \int_{-1/2}^{1/2} f(x+t) D_N(t)\, dt,$$

where the last equality is an application of the change of variables $u = -t$.

12.4 Pointwise Convergence★

We establish Dini's criterion for pointwise convergence of a Fourier series. We use Dini's criterion to establish Dirichlet's Theorem on the pointwise convergence of Fourier series of piecewise smooth functions. We also show that if f has an integrable derivative, then the Fourier series converges uniformly.

Recall, if $\|f-g\| = 0$, then $S_N f = S_N g$ for all N. For example, if

$$g(x) := \begin{cases} f(x) & \text{if } x \neq \frac{1}{2} \\ f\left(\frac{1}{2}\right)+1 & \text{if } x = \frac{1}{2} \end{cases}$$

then $S_N f = S_N g$ for all N. In particular, if $(S_N f)\left(\frac{1}{2}\right) \to f\left(\frac{1}{2}\right)$, then $(S_N g)\left(\frac{1}{2}\right) \to f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) - 1$, So, to get pointwise convergence we need to impose some condition on f.

The following is a version of a Theorem due to Ulisse Dini (14 November 1845, Pisa to 28 October 1918, Pisa).

Theorem 12.4.1 (Dini's Criterion). *Let f be integrable on the closed interval $[0,1]$. Fix a point x_0 in $[0,1]$. Define a function g on the closed interval $[-1/2,1/2]$ by setting*

$$g(t) := \begin{cases} \frac{f(x_0+t)-f(x_0)}{\sin(\pi t)} & \text{it } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}. \tag{12.23}$$

If g is integrable on $[-1/2,1/2]$, then $S_N f(x_0) \to f(x_0)$ as $N \to \infty$.

Proof. The first equality below uses $S_N f = f \star D_N$ and $\int_{-1/2}^{1/2} D_N = 1$, the second equality the definition of g and $D_N(t) = \sin((2N+1)\pi t)/\sin(\pi t)$, and the third uses $\sin(y) = \left(e^{iy} - e^{-iy}\right)/2i$ and the definition of $\widehat{h}(k)$.

$$\begin{aligned} S_N f(x_0) - f(x_0) &= \int_{-1/2}^{1/2} (f(x_0+t) - f(x_0))\, D_N(t)\, dt \\ &= \int_{-1/2}^{1/2} g(t) \sin((2N+1)\pi t)\, dt \\ &= \frac{1}{2i} \int_{-1/2}^{1/2} g(t) \left(e^{i\pi(2N+1)t} - e^{-i\pi(2N+1)t}\right) dt \\ &= \frac{1}{2i} \int_{-1/2}^{1/2} g(t)\, e_{1/2}(t)\, e^{i2\pi Nt} - g(t)\, e_{-1/2}(t)\, e^{-i2\pi Nt}\, dt \\ &= \frac{1}{2i} \left(\widehat{ge_{1/2}}(-N) - \widehat{ge_{-1/2}}(N)\right) \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

The convergence is a consequence of the Riemann–Lebesgue Lemma. ☺

Exercise 12.4.2. Suppose f is integrable and g is determined by (12.23). If g is bounded, then g is integrable.

Corollary 12.4.3. *The Fourier series $\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{i2\pi k x_0}$ converges to $f(x_0)$ at any point where f is differentiable.*

Proof. If $f'(x_0)$ exists, then

$$\frac{f(x_0+t) - f(x_0)}{\sin(\pi t)} = \frac{f(x_0+t) - f(x_0)}{t} \frac{\pi t}{\sin(\pi t)} \frac{1}{\pi} \to f'(x_0) \cdot 1 \cdot \frac{1}{\pi}$$

as $t \to 0$. So g is bounded hence integrable by Exercise 12.4.2. Consequently, $S_N f(x_0) \to f(x_0)$, by Dini's criterion. ☺

In the following result we do not assume that $f(0) = f(1)$. It allows us to calculate the sum of the Fourier series of certain discontinuous functions. The following result is due to Johann Peter Gustav Lejeune Dirichlet (13 February 1805, Düren to 5 May 1859, Gättingen).

Theorem 12.4.4 (Dirichlet, 1829). *Let f be defined on the closed interval $[0,1]$. Suppose there is a partition*

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

such that f is differentiable on the open intervals $]x_{k-1}, x_k[$, the limits

$$f(x_k+) = \lim_{t \searrow x_k} f(t) \text{ and } f(x_k-) = \lim_{t \nearrow x_k} f(t)$$

exists with the understandings $f(0-) := f(1-)$ and $f(1+) = f(0+)$ and the limits

$$f'^{+}(x_k) = \lim_{h \searrow 0} \frac{f(x_k+h) - f(x_k)}{t}$$
$$f'^{-}(x_k) = \lim_{h \nearrow 0} \frac{f(x_k+h) - f(x_k)}{t}$$

all exists, with understandings similar to the ones above at the endpoints 0 and 1. Then

$$S_N f(x) \to \frac{1}{2}\left(f(x-) + f(x+)\right) \text{ as } N \to \infty$$

for all x in $[0,1]$.

Proof. This is a consequence of the Dini Criterion (Corollary 12.4.3), if $x \neq x_k$. When $x = x_k$, the proof is similar to the proof of Dini's criterion.

$$\begin{aligned} S_N f(x) &= \int_{-1/2}^{1/2} f(x+t) \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\, dt \\ &= \int_{-1/2}^{0} \cdots + \int_{0}^{1/2} \cdots \\ &= S_N^- f(x) + S_N^+ f(x). \end{aligned}$$

We will show $S_N^+ f(x) \to \frac{1}{2} f(x+)$. Similarly, $S_N^- f(x) \to \frac{1}{2} f(x-)$. Let

$$g(t) = \begin{cases} \frac{f(x+t) - f(x+)}{\sin(\pi t)} & \text{if } 0 < t \leq 1/2 \\ 0 & \text{if } -1/2 \leq t \leq 0 \end{cases}.$$

Then g is bounded, by the proof of Corollary 12.4.3. Hence

$$\begin{aligned} S_N^+ f(x) - \frac{1}{2} f(x+) &= \int_0^{1/2} \left(f(x+t) - f(x+)\right) \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\, dt \\ &= \int_{-1/2}^{1/2} g(t) \sin((2N+1)\pi t)\, dt \\ &\to 0 \text{ as N} \to \infty, \end{aligned}$$

The first equality used $\int_0^{1/2} D_N = 1/2$ and the convergence follows from the Riemann–Lebesgue Lemma as in the proof of Dini's Theorem. ☺

If we assume that the function in Dirichlet's Theorem is continuous we get uniform convergence of the Fourier series. This is a consequence of the following result. To prove this result we need the Cauchy–Schwarz inequality for infinite sums.

Theorem 12.4.5. *If f is continuous, $f(0) = f(1)$, and f' exists except at a finite number of points and is integrable, then $S_N f$ converges uniformly to f on $[0,1]$.*

Proof. By Bessel's inequality the series $\sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right|^2$ and $\sum_{k=-\infty}^{\infty} \left|\widehat{f'}(k)\right|^2$ are convergent. By integration by parts

$$\begin{aligned}\widehat{f'}(k) &= \int_0^1 f'(x)e^{2\pi i k x}\,dx \\ &= 2\pi i k \int_0^1 f(x)e^{2\pi i k x}\,dx \\ &= 2\pi i k \widehat{f}(k)\end{aligned}$$

the boundary terms in the integration by parts vanish, since $f(0) = f(1)$. Hence $\sum_{k=-\infty}^{\infty} k^2 \left|\widehat{f}(k)\right|^2$ is convergent, since $\sum_{k=-\infty}^{\infty} \left|\widehat{f'}(k)\right|^2$ is convergent. Ignoring the $k = 0$ terms, it follow from the Cauchy–Schwarz inequality that,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right| &= \sum_{k=-\infty}^{\infty} \left|\left(\tfrac{1}{k}\right)\left(k\widehat{f}(k)\right)\right| \\ &\le \left(\sum_{k=-\infty}^{\infty} \frac{1}{k^2}\right)^{1/2} \left(\sum_{k=-\infty}^{\infty} \left|k\widehat{f}(k)\right|^2\right)^{1/2}.\end{aligned}$$

Hence $\sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right|$ is convergent. So $S_N f$ converges uniformly by the Weierstrass M-test, in particular, $\lim S_N f$ is continuous. . By Dini's Criterion (Corollary 12.4.3) the limit is $f(x)$ at the points x where $f'(x)$ exists. Hence $\lim S_N f$ and f are two continuous functions that are equal except possibly at a finite number of point, thus there are equal everywhere. ☺

12.5 Cesàro Summability

Dirichlet proved a result, Theorem 12.4.4 is a version of this result, that implies that if f has a bounded derivative, then $S_N f$ converges pointwise to f. He also expressed the belief that, if f was integrable (and certainly, if f was continuous) then $S_N f$ would converge pointwise. During the next 40+ years many mathematicians, including Riemann, Weierstrass, and Julius Wilhelm Richard Dedekind (6 October 1831, Braunschweig to 12 February 1916, Braunschweig), agreed with Dirichlet's belief. It came as a surprise when a counter example was produced by Paul du Bois-Reymond in 1873 or 1876.

Theorem 12.5.1 (du Bois-Reymond). *There exists a continuous function whose Fourier series diverges to infinity at some point.*

As a consequence of du Bois-Reymond example, there is a continuous f and a point x_0 such that $S_N f(x_0) \not\to f(x_0)$. In response to this example a new question arose:

Suppose f is continuous. Can we recover f from its Fourier coefficients? Most mathematicians thought the answer was no, but to everyones surprise Lipót Fejér (or Leopold Fejér), (February 9, 1880, Pécs to October 15, 1959, Budapest), when 19 years old, showed that the answer is yes. In fact, he showed that

$$\frac{1}{N+1}\sum_{n=0}^{N} S_n f \to f \text{ uniformly}$$

in particular, pointwise.

In this section we will assume that $f : [0,1] \to \mathbb{C}$ is continuous and $f(0) = f(1)$. We extend f to a continuous function on $\mathbb{R}$ by setting $f(x+n) = f(x)$ for all $0 \leq x < 1$ and all $n \in \mathbb{Z}$.

Fejér made use of a method for summing possibly divergent series due to Ernesto Cesàro (12 March 1859, Naples to 12 September 1906, Torre Annunziata). The Cesàro sums are

$$\begin{aligned}\sigma_N f(x) = (\sigma_N f)(x) &:= \frac{1}{N+1}\sum_{n=0}^{N}(S_n f)(x)\\ &= \frac{1}{N+1}\sum_{n=0}^{N}\int_{-1/2}^{1/2} f(x-t)D_n(t)\,dt\\ &= \int_{-1/2}^{1/2} f(x-t)K_N(t)\,dt\\ &= f\star K_N(x),\end{aligned}$$

where

$$K_N(t) := \frac{1}{N+1}\sum_{n=0}^{N} D_n(t)$$

is called the *Fejér kernel*.

We will show the Fejér kernel is an approximate identity. So we can use (the proof of) the Approximate Identity Lemma to conclude $\sigma_N f$ converges uniformly to f.

To show that K_N is an approximate identity we must verify (*i*) it has integral one, (*ii*) is positive, and (*iii*) is concentrated near the origin. This is the next three results (Fig. 12.2).

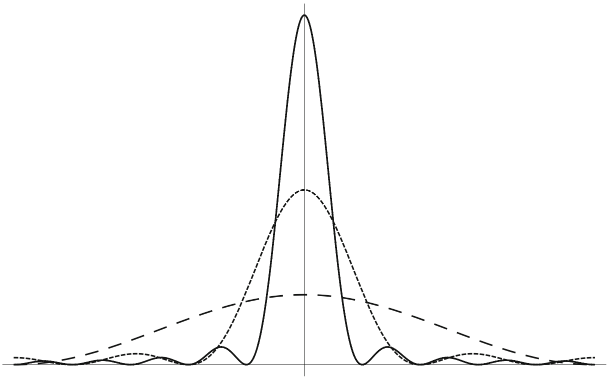

Fig. 12.2 A few samples K_1, K_4, and K_9 of the Fejér kernel

Exercise 12.5.2. For any $N \geq 1$,

$$\int_{-1/2}^{1/2} K_N = 1.$$

Since the Dirichlet kernel D_n is not positive, it is surprising that Fejér kernel K_N is positive.

Lemma 12.5.3. *For all* $N \geq 1$,

$$K_N(t) = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right)^2 \geq 0,$$

for all t.

Proof. One way to verify this formula is to use $2i\sin(x) = e^{ix} - e^{-ix}$ to convert it to a problems regarding exponentials and use that we know how to sum a geometric series.

Recall

$$D_k(t) = \frac{\sin(\pi(2k+1)t)}{\sin(\pi t)} = \frac{e^{i\pi(2k+1)t} - e^{-i\pi(2k+1)t}}{e^{i\pi t} - e^{-i\pi t}}.$$

Calculating, using the formula for the sum of a finite geometric series

$$\begin{aligned}
(N+1)\left(e^{i\pi t} - e^{-i\pi t}\right) K_N(t) &= \sum_{k=0}^{N} \left(e^{i\pi(2k+1)t} - e^{-i\pi(2k+1)t}\right) \\
&= e^{i\pi t} \sum_{k=0}^{N} \left(e^{i2\pi t}\right)^k - e^{-i\pi t} \sum_{k=0}^{N} \left(e^{-i2\pi t}\right)^k \\
&= e^{i\pi t} \frac{e^{i2\pi(N+1)t} - 1}{e^{i2\pi t} - 1} - e^{-i\pi t} \frac{1 - e^{-i2\pi(N+1)t}}{1 - e^{-i2\pi t}}
\end{aligned}$$

$$
\begin{aligned}
&= e^{i\pi t}\frac{e^{i\pi(N+1)t}}{e^{i\pi t}}\frac{e^{i\pi(N+1)t}-e^{-i\pi(N+1)t}}{e^{i\pi t}-e^{-i\pi t}} \\
&\quad - e^{-i\pi t}\frac{e^{-i\pi(N+1)t}}{e^{-i\pi t}}\frac{e^{i\pi(N+1)t}-e^{-i\pi(N+1)t}}{e^{i\pi t}-e^{-i\pi t}} \\
&= \left(e^{i\pi(N+1)t}-e^{-i\pi(N+1)t}\right)\frac{e^{i\pi(N+1)t}-e^{-i\pi(N+1)t}}{e^{i\pi t}-e^{-i\pi t}}.
\end{aligned}
$$

Dividing by $e^{i\pi t}-e^{-i\pi t}$, we get

$$
\begin{aligned}
(N+1)K_N(t) &= \left(\frac{e^{i\pi(N+1)t}-e^{-i\pi(N+1)t}}{e^{i\pi t}-e^{-i\pi t}}\right)^2 \\
&= \left(\frac{\sin\left(\pi\left(N+1\right)t\right)}{\sin\left(\pi t\right)}\right)^2
\end{aligned}
$$

as we needed to show.

Exercise 12.5.4. For any $0<\delta<1/2$

$$\int_{\delta\le|t|\le1/2} K_N \to 0 \text{ as } N\to\infty.$$

Where $\int_{\delta\le|t|\le1/2} K_N := \int_{-1/2}^{-\delta} K_N + \int_{\delta}^{1/2} K_N$.

Theorem 12.5.5 (Fejér). *If f is continuous and has period 1, then $\sigma_N f$ converges uniformly to f.*

Proof. Let $\varepsilon>0$ be given. Since$f$ is continuous and periodic, f is uniformly continuous, hence there exists $\delta>0$ such that for all x we have $|f(x-t)-f(x)|<\varepsilon/2$, when $|t|\le\delta$. Since f is continuous and periodic $M=\sup|f(x)|$ is finite. Pick N such that $\int_{\delta\le|t|\le1/2} K_n<\varepsilon/4M$ when $n\ge N$.

Exercise 12.5.6. Complete the proof of Fejér's Theorem.

Remark 12.5.7. Fejér's Theorem can be used to give a different proof of the Weierstrass approximation theorem.

12.6 Uniform Distribution of Sequences★

The connection between Fourier series and the uniform distribution of sequences in intervals is used to establish Weyl's Criterion for uniform distribution. A direct consequence of Weyl's Criterion is the uniform distribution of the fractional parts of the integer multiples of an irrational number in the unit interval.

Let (x_k) be a sequence of real numbers in the compact interval $[0,1]$. We say (x_k) is *uniformly distributed* in $[0,1]$, if for all $0 \le a < b \le 1$

$$\lim_{N\to\infty} \frac{\#\{k \in \mathbb{N} \mid k \le N \text{ and } a < x_k < b\}}{N} = b - a. \tag{12.24}$$

Remark 12.6.1. Intuitively, uniform distribution means that the x_k's are uniformly scattered over the interval. This is not just a property of the set $\{x_k \mid k \in \mathbb{N}\}$, the ordering imposed by the subscript k is also important. For example, suppose (y_k) is uniformly distributed in $[0,1]$. Let $a_k := \frac{1}{2}y_k$ and let $b_k := \frac{1}{2} + a_k$. Set $x_1 := \frac{1}{2}$, $x_2 := a_1$, $x_3 := a_2$, $x_4 := b_1$, $x_5 := a_3$, $x_6 := a_4$, $x_7 := b_2$, $x_8 := a_5, \ldots$. By construction,

$$\lim_{N\to\infty} \frac{\#\{k \in \mathbb{N} \mid k \le N \text{ and } 0 < x_k < \frac{1}{2}\}}{N} = \frac{2}{3}.$$

Hence, (x_k) is not uniformly distributed in $[0,1]$. On the other hand, if $z_1 := \frac{1}{2}$, $z_2 := a_1$, $z_3 := b_1$, $z_4 := a_2$, $z_5 := b_2$, $z_6 := a_3$, $z_7 := b_3$, $z_8 := a_4, \ldots$, then (z_k) is uniformly distributed in $[0,1]$. Of course, the sets $\{x_k \mid k \in \mathbb{N}\}$ and $\{z_k \mid k \in \mathbb{N}\}$ are equal.

For a set A the function

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$\mathbb{1}_A$ is called the *characteristic* function of A. We can write (12.24) as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{]a,b[}(x_k) = \int_0^1 \mathbb{1}_{]a,b[}, \tag{12.25}$$

for all $0 \le a < b \le 1$.

Exercise 12.6.2. If for some a, $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[a,a]}(x_k) > 0$, then (x_k) is not uniformly distributed. Here, $[a,a] := \{a\}$ is the degenerate interval containing only the point a.

Exercise 12.6.3. A sequence (x_k) in $[0,1]$ is uniformly distributed in $[0,1]$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_0^1 f \tag{12.26}$$

for all step functions f defined on $[0,1]$.

This leads directly to related characterizations of uniform distribution in terms of classes of functions we have encountered in other contexts.

Theorem 12.6.4. *Fix a sequence (x_k) in the closed interval $[0,1]$. Consider the equation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_0^1 f. \tag{12.27}$$

The following are equivalent:

(i) (12.27) *holds for all* $f = \mathbb{1}_{]a,b[}$ *with* $0 \le a < b \le 1$.
(ii) (12.27) *holds for all step functions* f *on* $[0,1]$.
(iii) (12.27) *holds for all integrable functions* f *on* $[0,1]$.
(iv) (12.27) *holds for all continuous functions* f *on* $[0,1]$.
(v) (12.27) *holds for all continuous functions* f *on* $[0,1]$ *with* $f(0) = f(1)$.
(vi) (12.27) *holds for all* $f = e_k$ *with* $k \in \mathbb{Z}$. *Recall,* $e_k(x) = e^{i2\pi kx}$.

By considering the real and complex parts separately we see that if (12.27) *is true for real valued* f *it is true for complex valued* f*. In particular, (iii) holds for real valued functions iff it holds for complex valued functions. Similarly, for (iv) and (v).*

Proof. We will show $(i) \implies (ii) \implies (iii) \implies (iv) \implies (v) \implies (vi)$, $(vi) \implies (v)$, and $(v) \implies (i)$.

$(i) \implies (ii)$. This is the nontrivial part of Exercise 12.6.3.

$(ii) \implies (iii)$. This is essentially the definition of the Riemann integral. As noted above, it is sufficient to consider real valued f. Suppose (x_k) in $[0,1]$ is uniformly distributed in $[0,1]$. Let f be some real values continuous function on $[0,1]$. Fix $\varepsilon > 0$. Pick lower and upper step functions s and S for f such that $\int_0^1 S - \int_0^1 s < \varepsilon$. Then $0 \le \int_0^1 f - \int_0^1 s < \varepsilon$ and $0 \le \int_0^1 S - \int_0^1 f < \varepsilon$, hence

$$\begin{aligned}
\int_0^1 f - \varepsilon \le \int_0^1 s &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N s(x_k) \\
&\le \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^N f(x_k) \le \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^N f(x_k) \\
&\le \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N S(x_k) = \int_0^1 S \\
&\le \int_0^1 f + \varepsilon.
\end{aligned}$$

Hence, the liminf equals the limsup so the limit exists and (12.27) holds.

$(iii) \implies (iv) \implies (v) \implies (vi)$ since the classes of functions are decreasing.

$(v) \implies (i)$. Let $0 \le a < b \le 1$. Fix $\varepsilon > 0$. Let f and F be continuous functions on $[0,1]$ such that $f \le \mathbb{1}_{]a,b[} \le F$, $f(0) = f(1)$, $F(0) = F(1)$, and $\int_0^1 F - \int_0^1 f < \varepsilon$. See Fig. 12.3 for the construction of f and F in the cases where $0 < a$ and $b < 1$. Using $\mathbb{1}_{]a,b[} \le F$, $\int_0^1 F - \int_0^1 f < \varepsilon$, (v), $f \le \mathbb{1}_{]a,b[}$, $\liminf \le \limsup$, $\mathbb{1}_{]a,b[} \le F$, (v), $\int_0^1 F - \int_0^1 f < \varepsilon$, and $f \le \mathbb{1}_{]a,b[}$ we have

$$\begin{aligned}
\int_0^1 \mathbb{1}_{]a,b[} - \varepsilon \le \int_0^1 F - \varepsilon \le \int_0^1 f &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N f(x_k) \\
&\le \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{]a,b[}(x_k) \le \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{]a,b[}(x_k)
\end{aligned}$$

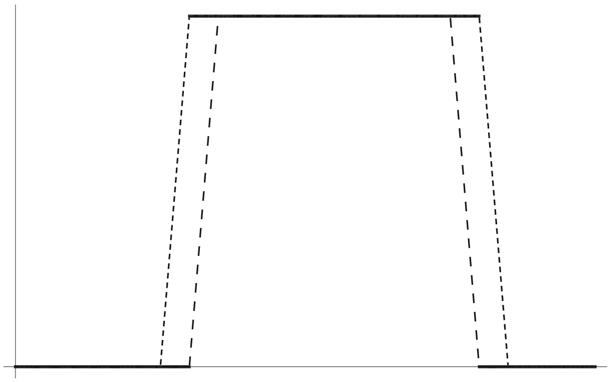

Fig. 12.3 The functions $f \le \mathbb{1}_{]a,b[} \le F$. Making the *dashed lines* sufficiently steep, forces $\int_0^1 F - \int_0^1 f < \varepsilon$

$$\le \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F(x_k) = \int_0^1 F$$
$$\le \int_0^1 f + \varepsilon \le \int_0^1 \mathbb{1}_{]a,b[} + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary (i) holds.

$(vi) \implies (v)$. Let f be a continuous function with $f(0) = f(1)$. Fix $\varepsilon > 0$. By Fejér's Theorem there exists

$$p(x) := \sum_{j=-m}^{m} a_j e^{i2\pi jx} = \sum_{j=-m}^{m} a_j e_j(x)$$

such that $|f(x) - p(x)| < \varepsilon/3$ for all $0 \le x \le 1$. We have

$$\begin{aligned} \left| \int_0^1 f - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right| \le & \left| \int_0^1 f - \int_0^1 p \right| \\ & + \left| \int_0^1 p - \frac{1}{N} \sum_{k=1}^{N} p(x_k) \right| \\ & + \left| \frac{1}{N} \sum_{k=1}^{N} p(x_k) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right|. \end{aligned}$$

The first and third terms are $\le \varepsilon/3$ for all N, since $|f(x) - p(x)| < \varepsilon/3$, for all x. We must show the middle term $\left| \int_0^1 p - \frac{1}{N} \sum_{k=1}^{N} p(x_k) \right|$ is $< \varepsilon/3$ for large N. Using the formula for p, that the sum $\sum_{j=-m}^{m}$ is a finite sum, and (vi) we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} p(x_k) = \lim_{N\to\infty} \sum_{j=-m}^{m} a_j \frac{1}{N} \sum_{k=1}^{N} e_j(x_k)$$

$$= \sum_{j=-m}^{m} a_j \left(\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} e_j(x_k) \right)$$
$$= \sum_{j=-m}^{m} a_j \left(\int_0^1 e_j \right)$$
$$= \int_0^1 p.$$

In particular, $\left| \int_0^1 p - \frac{1}{N} \sum_{k=1}^{N} p(x_k) \right|$ is $< \varepsilon/3$ for sufficiently large N. ☺

Remark 12.6.5. (iii) leads to the idea of Monte-Carlo integration.
The following criterion for uniform distribution was discovered by Hermann Klaus Hugo Weyl (9 November 1885, Elmshorn to 8 December 1955, Zurich).

Theorem 12.6.6 (Weyl's Criterion). *A sequence (x_k) of points in the closed interval $[0,1]$ is uniformly distributed in $[0,1]$ if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} e^{i2\pi j x_k} = 0 \text{ for all integers } j \neq 0. \tag{12.28}$$

Proof. By (vi) in Theorem 12.6.4 the sequence (x_k) is uniformly distributed in $[0,1]$, iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} e^{i2\pi j x_k} = \int_0^1 e^{i2\pi jx}\, dx,$$

for all integers j. Since $\int_0^1 e_0 = 1 = \frac{1}{N} \sum_{k=1}^{1} 1 = \sum_{k=1}^{N} e_0(x_k)$ and $\int_0^1 e_k(x)dx = \int_0^1 e^{i2\pi kx} dx = 0$ for all integers $k \neq 0$ we see that (vi) in Theorem 12.6.4 is equivalent to (12.28). ☺

Recall, $\{t\} := t - \lfloor t \rfloor$ denotes the fractional part of the real number t.

Corollary 12.6.7. *If α is irrational, then the sequence of fractional parts $(\{k\alpha\})$ is uniformly distributed in $[0,1]$.*

Proof. Let n be a nonzero integer, since α is irrational, $2n\alpha$ is not an integer. Hence $\sin(2\pi n\alpha) \neq 0$. Using $t \to e^{i2\pi nt}$ has period one, we compute

$$\left| \frac{1}{N} \sum_{k=1}^{N} e^{i2\pi n\{k\alpha\}} \right| = \left| \frac{1}{N} \sum_{k=1}^{N} e^{i2\pi nk\alpha} \right|$$
$$= \left| \frac{1}{N} \frac{e^{i2\pi n(N+1)\alpha} - e^{i2\pi n\alpha}}{e^{i2\pi n\alpha} - 1} \right|$$
$$\leq \frac{2}{N|\sin(2\pi n\alpha)|} \xrightarrow[N\to\infty]{} 0$$

The inequality used $\left| e^{i2\pi n(N+1)\alpha} - e^{i2\pi n\alpha} \right| \leq 2$ and $|y| \leq |x+iy|$ with $x = \cos(2\pi n\alpha) - 1$ and $y = \sin(2\pi n\alpha)$. ☺

12.7 Norm Convergence★

Norm convergence of Fourier series is established. Consequences of norm convergence are Parseval's Identity (equality in Bessel's inequality) and Plancherel's Formula. These identities allow us to evaluate the sums of certain series by calculating integrals.

Norm convergence is not convergence in a simple pointwise sense, but in the sense that the integrals

$$\int_0^1 |f(x) - (S_N f)(x)|^2 \underset{N\to\infty}{\longrightarrow} 0 \tag{12.29}$$

for all integrable f. Using the norm (12.8), i.e., $\|f\|_2 = \left(\int_0^1 |f|^2\right)^{1/2}$, we can write (12.29) as

$$\|f - S_N f\|_2 \underset{N\to\infty}{\longrightarrow} 0\,.$$

We have mostly omitted the subscript in $\|\cdot\|_2$ simple writing $\|\cdot\|$.

Remark 12.7.1. We have already establish some results that can be regarded as convergence with respect to a different norm. For a bounded function f on $[0,1]$ let

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in [0,1]\}\,.$$

It is easy to see that $\|af\|_\infty = |a|\,\|f\|_\infty$ and $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. Hence, $\|\cdot\|_\infty$ is a norm on the set of bounded functions. Let f_k, $k \in \mathbb{N}$, and f be bounded functions. It is not difficult to see that f_k converges uniformly to f on $[0,1]$ iff $\|f - f_k\|_\infty \to 0$ as $k \to \infty$. Since a continuous function on is bounded, in particular, $\|\cdot\|_\infty$ is a norm on the set of continuous functions on $[0,1]$. In particular, Theorem 12.4.5 and Theorem 12.5.5 can be interpreted as stating that

$$\|f - S_N f\|_\infty \underset{N\to\infty}{\longrightarrow} 0 \text{ and } \|f - \sigma_N f\|_\infty \underset{N\to\infty}{\longrightarrow} 0$$

for certain classes of functions f. Since we have no further use for $\|\cdot\|_\infty$, we will resume writing $\|\cdot\|$ in place of $\|\cdot\|_2$.

In this section we do not assume $f(0) = f(1)$. As a first step toward (12.29) we establish a relationship between the partial sums $S_N f = \sum_{k=-N}^{N} \widehat{f}(k)\, e_k$ of the Fourier series for f and f itself, we show if f and g are close, then so are the partial sums of their Fourier series, more precisely:

Theorem 12.7.2. *If f and g are integrable, then*

$$\|S_N f - S_N g\| \le \|f - g\|.$$

Proof. Since

$$\|S_N f\|^2 = \langle S_N f \mid S_N f\rangle$$

$$\begin{aligned}
&= \left\langle \sum_{j=-N}^{N} \widehat{f}(j)e_j \,\middle|\, \sum_{k=-N}^{N} \widehat{f}(k)e_k \right\rangle \\
&= \sum_{j=-N}^{N} \sum_{k=-N}^{N} \widehat{f}(j)\overline{\widehat{f}(k)}\langle e_j \mid e_k\rangle \\
&= \sum_{k=-N}^{N} \left|\widehat{f}(k)\right|^2 \\
&\le \|f\|^2,
\end{aligned}$$

the inequality is Bessel's inequality.

Using (12.10) we see

$$\begin{aligned}
S_N f - S_N g &= \sum_{k=-N}^{N} (\langle f \mid e_k\rangle - \langle g \mid e_k\rangle)e_k \\
&= \sum_{k=-N}^{N} \langle f - g \mid e_k\rangle e_k \\
&= S_N(f-g).
\end{aligned}$$

Hence, the previous calculation gives

$$\|S_N f - S_N g\| = \|S_N(f-g)\| \le \|f-g\|$$

as we needed to show. ☺

We use Theorem 12.7.2 to establish that, if f is integrable on $[0,1]$, then $\|f - S_N f\| \to 0$ as $N \to \infty$. The proof is in three steps. (i) Use Fejér's Theorem and the Projection Theorem to get convergence for continuous functions. (ii) A given step function can be approximated by a continuous functions, hence Theorem 12.7.2 and step (i) gives the result for step functions. (iii) Repeat step (ii), but this time approximate a given integrable function by a step function.

Theorem 12.7.3. *If f is integrable on* $[0,1]$, *then*

$$\|S_N f - f\| = \left(\int_0^1 |f - S_N f|^2\right)^{1/2} \to 0$$

as $N \to \infty$.

Proof. Since $S_N f = (S_N \operatorname{Re} f) + (S_N \operatorname{Im} f)$ and

$$\|f - S_N f\| = \|\operatorname{Re} f - (S_N \operatorname{Re} f)\| + \|\operatorname{Im} f - (S_N \operatorname{Im} f)\|$$

we will assume f is real valued.

Step (*i*): Suppose h is continuous and $h(0) = h(1)$. By Fejér's Theorem $\sigma_N h$ converges uniformly to h. The Projection Theorem, Theorem 12.2.10, states that

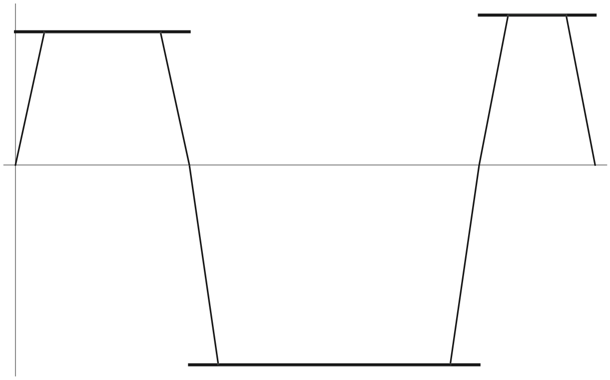

Fig. 12.4 Approximating a step function g with a continuous function h such that $\|g-h\|$ is arbitrarily small

$\left\|h-\sum_n^N a_k e_k\right\|$ is minimized by choosing $a_k = \widehat{h}(k)$. In particular,

$$\|h - S_N h\| \leq \|h - \sigma_N h\|.$$

But

$$\|h-\sigma_N h\|^2 = \int_0^1 |h-\sigma_N h|^2 \to 0$$

because $|h-\sigma_N h|$ converges uniformly to 0.

Step (*ii*): Suppose g is a step function on $[0,1]$. The idea is to approximate g by a continuous h and then use Theorem 12.7.2 and Step (i). Let M satisfy $|g(x)| \leq M$ for all x. Let $\varepsilon > 0$. Construct a continuous function h by narrowing each step in g slightly and connecting the ends to the $x-$axis. By similarly modifying the right end of the last step we can arrange $h(1) = h(0) = 0$. See Fig. 12.4. Note that $|h(x)| \leq |g(x)| \leq M$ for all x. Hence, by making the sum of the lengths of the intervals where we modify the step function less than $\varepsilon^2/4M^2$ we obtain

$$\begin{aligned}
\|g-h\|^2 &= \int_0^1 |g-h|^2 = \int_0^1 |g-h|\,|g-h| \\
&\leq 2M \int_0^1 |g-h| \\
&\leq 2M\,(2M)\left(\varepsilon^2/4M^2\right) = \varepsilon^2.
\end{aligned}$$

The last inequality used that $|g-h| \leq 2M$ and that the sum of the lengths of the intervals where $g \neq h$ is less than $\varepsilon^2/4M^2$. Hence $\|g-h\| \leq \varepsilon$. Using Step (i) we conclude $\|h-S_N h\| \leq \varepsilon$ for N sufficiently large. Since $\|g-h\| \leq \varepsilon$ also $\|S_N g - S_N h\| \leq \varepsilon$, by Theorem 12.7.2. So

$$\begin{aligned}
\|g-S_N g\| &\leq \|g-h\| + \|h-S_N h\| + \|S_N h - S_N g\| \\
&\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\end{aligned}$$

for N sufficiently large. Since $\varepsilon > 0$ is arbitrary, this means $\|g - S_N g\| \to 0$ as $N \to \infty$.

Step (*iii*): Suppose f is integrable on $[0,1]$. This is similar to the previous part. We approximate f by a step function g and use Theorem 12.7.2 and the second part of this proof. Let M be such that $|f(x)| \le M$ for all x. Let g be an upper step function for f such that $\int_0^1 g - \int_0^1 f \le \varepsilon^2/2M$. Then

$$\begin{aligned}\|g - f\|^2 &= \int_0^1 |g - f|^2 = \int_0^1 |g - f|\,|g - f| \\ &\le 2M \int_0^1 |g - f| = 2M \int_0^1 g - f \\ &\le 2M(\varepsilon^2/2M) = \varepsilon^2.\end{aligned}$$

Thus $\|f - g\| \le \varepsilon$, by Theorem 12.7.2 also $\|S_N g - S_N f\| \le \varepsilon$. By Step (*ii*) we have $\|g - S_N g\| \le \varepsilon$ for N sufficiently large. Hence

$$\begin{aligned}\|f - S_N f\| &\le \|f - g\| + \|g - S_N g\| + \|S_N g - S_N f\| \\ &\le \varepsilon + \varepsilon + \varepsilon = 3\varepsilon\end{aligned}$$

for N sufficiently large. ☺

As a consequence of this theorem and the Pythagorean Theorem we have equality in Bessel's inequality, this is due to Marc-Antoine Parseval (27 April 1755, Rosières-aux-Salines to 16 August 1836, Paris).

Corollary 12.7.4 (Parseval's Identity). *If f is integrable on $[0,1]$, then*

$$\int_0^1 |f|^2 = \sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right|^2.$$

Hence, $\|f\| = \left\|\left(\widehat{f}(k)\right)\right\|$, where the norm on the left-hand side is in the sense of the integral and the norm on the right-hand side in the norm in the sequence space ℓ^2.

Exercise 12.7.5. Prove Parseval's Identity.

Example 12.7.6. It follows from Parseval's Indentity applied to $f(x) = e_\alpha(x) = e^{i2\pi\alpha x}$ that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)} \tag{12.30}$$

for real non-integer α. Setting $\alpha = 1/2$ gives

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \text{ and } \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8.}.$$

Proof. With $f = e_\alpha$, the left-hand side of Parseval is $\|e_\alpha\|^2 = \int_0^1 |e_\alpha|^2 = \int_0^1 1 = 1$. To calculate the right-hand side we begin by calculating $\widehat{e_\alpha}$. Fix an integer k, then

$$\begin{aligned}\widehat{e_\alpha}(k) &= \int_0^1 e^{i2\pi\alpha x} e^{-i2\pi kx} dx = \int_0^1 e^{i2\pi(\alpha-k)x} dx \\ &= \frac{1}{i2\pi(\alpha-k)}\left(e^{-i2\pi(\alpha-k)} - 1\right) \\ &= \frac{1}{i2\pi(\alpha-k)}\left(e^{-i2\pi\alpha} - 1\right)\end{aligned}$$

Hence,

$$|e_\alpha(k)|^2 = \frac{1-\cos(2\pi\alpha)}{2\pi^2(\alpha-k)^2} = \frac{\sin^2(\pi\alpha)}{\pi^2(\alpha-k)^2},$$

since $\left|e^{-i2\pi(\alpha-k)} - 1\right|^2 = (\cos(2\pi(\alpha-k)) - 1)^2 + (\sin(2\pi(\alpha-k)))^2$. Using Parseval we get

$$1 = \sum_{k=-\infty}^{\infty} \frac{\sin^2(\pi\alpha)}{\pi^2(\alpha-k)^2},$$

rearranging gives (12.30).

Setting $\alpha = 1/2$ in (12.30) gives

$$2\sum_{k=1}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} = \sum_{k=-\infty}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} = \pi^2.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} = \frac{\pi^2}{8.}.$$

Let $a := \sum_{k=1}^{\infty} \frac{1}{k^2}$. Then

$$\begin{aligned}a - \frac{\pi^2}{8.} &= \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \\ &= \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= \frac{a}{4}.\end{aligned}$$

Solving for a, we get $a = \pi^2/6$. ☺

Applying the Polarization Identity

$$4\langle f \mid g\rangle = \|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2$$

to both sides of Parseval's Identity leads to:

Theorem 12.7.7 (Plancherel's Formula). *If f and g are integrable on $[0,1]$, then*

$$\int_0^1 f\overline{g} = \sum_{-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}.$$

This is a special case of a result due to Michel Plancherel (16 January 1885, Bussy to 4 March 1967, Zurich).

Exercise 12.7.8. Prove the Polarization Identity.

Exercise 12.7.9. Prove Plancherel's Formula.

Problems

Problems for Sect. 12.1

1. Let $f(x) := x$ on the interval $[0,1]$. Calculate $\widehat{f}(k)$.
2. Same as the previous problem, but for

$$f(x) := \begin{cases} 1 & \text{when } 0 < x < 1/2 \\ 0 & \text{when } 1/2 < x < 1 \end{cases}.$$

Problems for Sect. 12.2

1. State the triangle inequality and the Cauchy–Schwarz inequality in the case of $\mathbb{C}^d$.
2. If $x_k > 0$ for all k and $\sum_{k=1}^{\infty} x_k$ is convergent, then $\sum_{k=1}^{\infty} x_k^2$ is convergent.

Problems for Sect. 12.3

1. Show that $S_N(af+bg) = aS_N f + bS_N g$ for any integrable functions f, g and any constants a, b.
2. Calculate $D_N\left(\frac{1}{2}\right)$ and $D_N\left(\frac{1}{4}\right)$.

Problems for Sect. 12.4

1. Suppose f is integrable and g is determined by (12.23). If f has a jump discontinuity at x_0, then g is not integrable.

2. **(Riemann Localization)** If f is integrable on $[0,1]$ and $f(t)=0$ for all t in (a,b), then $S_N f(t)\to 0$ for all $t\in(a,b)$.
3. If two continuous functions are equal except possibly at one point in an interval, then they are equal everywhere on that interval.
4. Let $f(x) := \begin{cases} -1 & \text{for } -\frac{1}{2}\le x<0 \\ 1 & \text{for } 0\le x<\frac{1}{2} \end{cases}$.

   a. Find $S_N f(x)$.
   b. Why does $S_N f$ not converge uniformly to f.
   c. For which $-\frac{1}{2}\le x<\frac{1}{2}$ does $S_N f(x)$ converge to $f(x)$?

Problems for Sect. 12.5

1. If $a_k\to a$ as $n\to\infty$, then $\frac{1}{N}\sum_{k=1}^{N} a_k\to a$ as $N\to\infty$.
2. If $a_k=(-1)^k$, then (a_k) is divergent and the sequence $\left(\frac{1}{N}\sum_{k=1}^{N} a_k\right)$ is convergent.
3. Find $a_{N,k}$ such that $\sigma_N f(x)=\sum_{k=-N}^{N} a_{N,k}e^{i2\pi kx}$.
4. Show, if f is continuous and periodic, then f is uniformly continuous on $\mathbb{R}$.
5. Show, if f is continuous and periodic, then f is bounded.

Problems for Sect. 12.6

1. Prove $(\{\sin(k)\})$ is dense in $[0,1]$, but not uniformly distributed in $[0,1]$.
2. $\left(\left\{\sqrt{2}^k\right\}\right)$ is not uniformly distributed in $[0,1]$. [*Hint*: Since $\left\{\sqrt{2}^k\right\}=0$ when k is even, this follows from the definition of uniform distribution.]
3. If $\alpha=\frac{1+\sqrt{5}}{2}$, then $\left(\left\{\alpha^k\right\}\right)$ is not uniformly distributed. [*Hint*: If $\beta:=\frac{1-\sqrt{5}}{2}$ and $f_k:=\alpha^k+\beta^k$, then $f_0=2$, $f_1=1$, and $f_{k+2}=f_{k+1}+f_k$. In particular, f_k is an integer ≥ 2 for all $k\ge 2$. Hence, $\alpha^{2k+1}=f_{2k+1}+|\beta|^{2k+1}$ implies $\left\{\alpha^{2k+1}\right\}=|\beta|^{2k+1}\to 0$ as $k\to\infty$. And $\alpha^{2k}=f_{2k}-|\beta|^{2k}$ implies $\left\{\alpha^{2k}\right\}=1-|\beta|^{2k}\to 1$ as $k\to\infty$. Consequently, $\lim_{N\to\infty}\frac{\#\{k\in\mathbb{N}\mid k\le N \text{ and } \frac{1}{3}<\left\{\alpha^k\right\}<\frac{2}{3}\}}{N}=0$.]
4. Replacing the closed intervals in the definition of uniformly distributed by open intervals gives an equivalent concept.
5. $\left(\left\{\sqrt{k}\right\}\right)$ is uniformly distributed in $[0,1]$.

6. Let $\gamma_k := \left\{\sqrt{k}\right\}$. By uniform distribution there is a subsequence $\left(\gamma_{n_k}\right)$ of the sequence (γ_k) such that $\gamma_{n_k} \to 1$. Construct such a subsequence.
7. Is $\gamma_k := \left\{\sqrt[3]{k}\right\}$ uniformly distributed?
8. For which $x \in \mathbb{R}$ is $\gamma_k := \left\{\sqrt[x]{k}\right\}$ uniformly distributed?

Problems for Sect. 12.7

1. Let $f(x) := x$ on $[0,1]$. Calculate both sides of Parseval's Identity.
2. Same as the previous problem, but with f replaced by
$$g(x) := \begin{cases} 0 & \text{when } -\frac{1}{2} < x < 0 \\ 1 & \text{when } 0 < x < \frac{1}{2} \end{cases}.$$
3. Calculate both sides of Plancherel's Formula, if $f(x) := x$ on $[0,1]$ and
$$g(x) := \begin{cases} 0 & \text{when } -\frac{1}{2} < x < 0 \\ 1 & \text{when } 0 < x < \frac{1}{2} \end{cases}.$$
4. Let a be a complex number and let f, g be bounded functions on $[0,1]$. Prove that $\|af\|_\infty = |a|\,\|f\|_\infty$ and $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.
5. Let f_k, $k \in \mathbb{N}$, and f be bounded functions. Prove f_k converges uniformly to f on $[0,1]$ iff $\|f - f_k\|_\infty \to 0$ as $k \to \infty$.
6. Let f be integrable.
   a. Why is $\widehat{f}(k) = 1$ for $k \ge 1$ not possible?
   b. Why is $\widehat{f}(k) = \frac{1}{\sqrt{k}}$ for $k \ge 1$ not possible?

Solutions and Hints for the Exercises

Exercise 12.2.2. $\int_0^1 |f| = 0$ implies $\int_0^1 |f\overline{g}| = 0$, since g is bounded. But $\int_0^1 |f| > 0$ implies $\int_0^1 |f|^2 > 0$. One way to see this is to use that, if $\sum_{k=1}^n m_k \chi_{]x_{k-1},x_k[}$ is a lower step function for $|f|$, then $\sum_{k=1}^n m_k^2 \chi_{]x_{k-1},x_k[}$ is a lower step function for $|f|^2$.

Exercise 12.3.3 Since f has period one we may assume $0 \le a < 1$. Since f has period one, it follows by substitution that $\int_1^{a+1} f = \int_0^a f$.

Exercise 12.3.5 Evaluate the geometric series used to define D_N.

Exercise 12.4.2 Let $\varepsilon > 0$ be given. Let M be an upper bound for $|g|$. Pick $0 < \delta < 1/2$ such that $4M\delta < \varepsilon/2$. Then g is integrable on the interval $I_+ := [\delta, 1/2]$ because it is a product of two integrable functions this interval. Hence we can find upper and lower step functions s_+, S_+ for g restricted to I_+ such that $\int_\delta^{1/2} S_+ - \int_\delta^{1/2} s_+ < \varepsilon/4$. Similarly, we can find upper and lower step functions s_-, S_- for g restricted to $I_- := [-1/2, -\delta]$ such that $\int_\delta^{1/2} S_- - \int_\delta^{1/2} s_- < \varepsilon/4$. Let $S_0(t) = M$ and $s_0(t) = -M$ for t in the interval $I_0 := [-\delta, \delta]$. Combining S_-, S_0, and S_+ we get an upper step function S for g on $[-1/2, 1/2]$. Similarly, combining s_-, s_0, and s_+ we get a lower step function s for g on $[-1/2, 1/2]$. By construction $\int_{-1/2}^{1/2} S - \int_{-1/2}^{1/2} s < \varepsilon$.

Exercise 12.5.2 Since $\int_{-1/2}^{1/2} D_n = 1$ for all $n \geq 1$, this follows from the definition of K_N.

Exercise 12.5.4 Since $t \to \sin(\pi t)$ is increasing on the interval $[0, 1]$, we have

$$K_N(t) \leq \frac{1}{N+1}\left(\frac{1}{\sin(\pi t)}\right)^2 \leq \frac{1}{N+1}\left(\frac{1}{\sin(\pi\delta)}\right)^2$$

for $\delta \leq |t| \leq 1/2$.

Exercise 12.5.6 Imitating the proof of the Approximate Identity Lemma.

Exercise 12.6.3 (12.26) implies (12.25) because any characteristic function is a step function.

Since any step function is a linear combination of characteristic functions the converse follows from theorems about linear combinations of limits.

Exercise 12.7.5 A consequence of the Pythagorean Theorem and Theorem 12.7.3.

Exercise 12.7.8 Expand the right-hand side of the Polarization Identity using $\|f\|^2 = \langle f \mid f \rangle$ and equation (12.10).

Exercise 12.7.9 Apply the Polarization Identity and Parseval's Identity.

Chapter 13
Topology

This chapter contains a brief introduction to point set topology. The main aim is to extend some of the important results about continuous functions on a compact intervals to continuous functions on a larger class of sets, the compact sets.

Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} = \mathbb{C} = \mathbb{R}^2$. In either case

$$|xy| = |x||y| \text{ for all } x, y \in \mathbb{K}$$

and the triangle inequality

$$|a+b| \le |a| + |b| \text{ for all } a, b \in \mathbb{K}$$

holds. That is $|\cdot|$ is a norm on $\mathbb{K}$. By the triangle inequality $|(x-y)+(y-z)| \le |x-y| + |y-z|$, equivalently,

$$\boxed{|x-z| \le |x-y| + |y-z| \text{ for all } x, y, z \in \mathbb{K}.}$$

13.1 Open Sets

A subset D of $\mathbb{K}$ is *open*, if any point in D is the center of a ball contained in D, that is

$$\forall x \in D, \exists r > 0, B_r(x) \subseteq D. \tag{13.1}$$

Recall, $B_r(x) = \{y \in \mathbb{K} \mid |y-x| < r\}$ and if $\mathbb{K} = \mathbb{R}$, then $B_r(x) =]x-r, x+r[$ is an open interval. The use of the term "open" is consistent with our previous use of this terminology in the sense that:

Example 13.1.1. An open ball is an open set.

Proof. (This is Exercise E.2.2.) An open ball is a set of the form $D := B_s(y)$ for some $s > 0$ and some $y \in \mathbb{K}$. Let $x \in B_s(y)$, then $|x-y) < r$, hence $r := s - |x-y| > 0$. We

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will show $B_r(x) \subseteq B_s(y)$. Let $z \in B_r(x)$. Then

$$\begin{aligned} |z-y| &\leq |z-x| + |x-y| \\ &< r + |x-y| \\ &= s \end{aligned}$$

hence $z \in B_s(y)$. Thus $B_s(y)$ is an open set. ☺

Theorem 13.1.2. *The empty set is an open set, the whole space $\mathbb{K}$ is an open set, any union of open sets is an open set, and any finite intersection of open sets is an open set.*

Proof. The first two claims are trivial. If A_i is a collection of open sets and $x \in \bigcup_i A_i$, then for some i_0 we have $x \in A_{i_0}$. Since A_{i_0} is open, there is a ball B with center x, such that $B \subseteq A_{i_0}$. But then

$$B \subseteq A_{i_0} \subseteq \bigcup_i A_i,$$

hence $\bigcup_i A_i$ is an open set.

Finally, if A_i, $i = 1, 2, \ldots, n$ are open sets and $x \in \bigcap_{i=1}^n A_i$, then $x \in A_i$ for all $i = 1, 2, \ldots, n$. Since each A_i is open, there are $r_i > 0$, such that $B_{r_i}(x) \subseteq A_i$. Let $r := \min\{r_1, r_2, \ldots r_n\}$, then $r = r_{k_0} \leq r_i$ for some k_0 and all $i = 1, 2, \ldots, n$. Hence

$$B_r(x) = \bigcap_{i=1}^n B_{r_i}(x) \subseteq \bigcap_{i=1}^n A_i.$$

Consequently, $B_r(x)$ is the required ball. ☺

Interior

The *interior* of A is

$$\overset{\circ}{A} := \{x \in A \mid \exists r > 0, B_r(x) \subseteq A\}.$$

Clearly, $\overset{\circ}{A} \subseteq A$. The reverse inclusion is the definition of an open set.

Proposition 13.1.3. *If A is open, then $\overset{\circ}{A} = A$.*

Proof. If A is open and $x \in A$, then $B_r(x) \subseteq A$ for some $r > 0$, by the definition of an open set. Hence $x \in \overset{\circ}{A}$. ☺

Exercise 13.1.4. The interior of an close ball $\overline{B}_r(x)$ is the corresponding open ball $B_r(r)$.

Exercise 13.1.5. The interior of a set is an open set. In symbols, for any subset A of $\mathbb{K}$, $\overset{\circ}{A}$ is an open set.

Open Sets and Continuity

Example 13.1.6. The open interval $]a,b[$ is an open subset of $\mathbb{R}$. The open interval $]a,b[$ is not an open subset of $\mathbb{C}$, because the set $]a,b[\times\{0\} = \{(x,0) \mid a < x < b\}$ is not an open subset of $\mathbb{R}^2$.

Example 13.1.7. If U is an open subset of $\mathbb{R}$, then $U\times]-1,1[$ is open in $\mathbb{R}^2$. Consequently, $O := \{x+iy \mid x \in U, y \in]-1,1[\}$ is open in $\mathbb{C}$ and $U = \mathbb{R}\cap O$.

Definition 13.1.8. Let D be a subset of $\mathbb{K}$. A subset A of D is *open in* D, if $A = D\cap O$ for some open subset O of $\mathbb{K}$, i.e., for some set O satisfying (13.1).

This definition makes the following theorem possible. Recall, $f : D \to \mathbb{C}$ is continuous at $a \in D$, means

$$\forall\varepsilon > 0, \exists\delta > 0, f(D\cap B_\delta(x)) \subseteq B_\varepsilon(f(a)),$$

and that f is continuous on D, if f is continuous at each point a in D.

Theorem 13.1.9. *Let D be a subset of $\mathbb{K}$ and let $f : D \to \mathbb{C}$. Then f is continuous on D iff the pre-image $f^{-1}(U)$ is open in D, for any open subset U of $\mathbb{C}$.*

Proof. Suppose f is continuous on D. Let U be an open subset of $\mathbb{C}$. Let $a \in f^{-1}(U)$, then $f(a)$ is in U. Since U is open, there is a $\varepsilon > 0$, such that $B_\varepsilon(f(a)) \subseteq U$. Since f is continuous there is a $\delta = \delta(a) > 0$, such that $f(D\cap B_\delta(a)) \subseteq B_\varepsilon(f(a))$. Consequently, $D\cap B_{\delta(a)}(a) \subseteq f^{-1}(B_\varepsilon(f(a))) \subseteq f^{-1}(U)$. Taking the union over $a \in f^{-1}(U)$ leads to, $\bigcup_{a\in f^{-1}(U)} \left(D\cap B_{\delta(a)}(a)\right) \subseteq f^{-1}(U)$. The reverse inclusion follows from $a \in D\cap B_{\delta(a)}(a)$. Thus

$$f^{-1}(U) = \bigcup_{a\in f^{-1}(U)} \left(D\cap B_{\delta(a)}(a)\right) = D\cap \bigcup_{a\in f^{-1}(U)} B_{\delta(a)}(a).$$

Since $\bigcup_{a\in f^{-1}(U)} B_{\delta(a)}(a)$ is a union of open sets, it is an open set. Thus $f^{-1}(U)$ is open in D.

Conversely, suppose the pre-image of any open set in $\mathbb{C}$ is open in D. Let $x \in D$. We must show f is continuous at x. Let $\varepsilon > 0$. Then $B_\varepsilon(f(x))$ is an open subset of $\mathbb{C}$, hence $f^{-1}(B_\varepsilon(f(x)))$ is open in D. Let V be an open set in $\mathbb{K}$ such that $f^{-1}(B_\varepsilon(f(x))) = D\cap V$. Since $x \in f^{-1}(B_\varepsilon(f(x))) \subseteq V$ and V is open, there is a $\delta > 0$, such that $B_\delta(x) \subseteq V$. Hence,

$$f(D\cap B_\delta(x)) \subseteq f(D\cap V) = f\left(f^{-1}(B_\varepsilon(f(x)))\right) \subseteq B_\varepsilon(f(x)).$$

Thus f is continuous at x. ☺

13.2 Closed Sets

Recall, $a \in \mathbb{C}$ is an accumulation point of $D \subseteq \mathbb{C}$, if

$$\forall r > 0, \exists x \in D, 0 < |x-a| < r$$

that is, if

$$\forall r > 0, (D \cap B_r(a)) \setminus \{a\} \neq \emptyset.$$

Let D' be the set of accumulation points of D. We say D is a *closed* set, if D contain all its accumulation points, that is, if $D' \subseteq D$.

Example 13.2.1. Some simple examples of sets of accumulation points are :

1. If $D = B_1(0)$, then $D' = \overline{B}_1(0)$.
2. If $D = \mathbb{N}$, then $D' = \emptyset$.
3. If $D = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, then $D' = \{0\}$.

The following result establishes a correspondence between the open and the closed subsets of $\mathbb{K}$.

Theorem 13.2.2. *D is closed iff $\mathbb{C} \setminus D$ is open.*

Proof. We must show $D' \subseteq D$ iff $\mathbb{C} \setminus D$ is open. Suppose $\mathbb{C} \setminus D$ is open. Let $a \in \mathbb{C} \setminus D$. We must show a is not an accumulation point of D. Since $\mathbb{C} \setminus D$ is open and $a \in \mathbb{C} \setminus D$ there is an $r > 0$, such that $B_r(a) \subseteq \mathbb{C} \setminus D$. For such an $r > 0$, $D \cap B_r(a) = \emptyset$. Consequently, a is not an accumulation point of D.

Conversely, suppose $D' \subseteq D$. Let $a \in \mathbb{C} \setminus D$. We must show $B_r(a) \subseteq \mathbb{C} \setminus D$ for some $r > 0$. Since D contain all its accumulation points, the point a is not an accumulation point of D. Consequently, for some $r > 0$,

$$(D \cap B_r(a)) \setminus \{a\} = \emptyset.$$

For such an r, $D \cap B_r(a) = \emptyset$, since $a \notin D$. Thus $B_r(a) \subseteq \mathbb{C} \setminus D$. ☺

Theorem 13.2.3. *The whole space $\mathbb{K}$ is a closed set, the empty set $\emptyset$ is a closed set, any intersection of closed sets is a closed set, and any finite union of closed sets is a closed set.*

Proof. Take complements in the corresponding theorem for open sets. ☺

Definition 13.2.4. A subset A of D is *closed in D*, if there is a closed subset K of $\mathbb{K}$, such that $A = D \cap K$.

Exercise 13.2.5. Let A be a subset of D. Then A is closed in D iff $D \setminus A$ is open in D.

Exercise 13.2.6. Let $f : D \to \mathbb{C}$. Then f is continuous on D iff $f^{-1}(K)$ is closed in D for any closed subset K of $\mathbb{C}$.

Closure

The *closure* of a set is the union of the set and its accumulation points. Hence if D is a set, then the closure of D is the set $\overline{D} := D \cup D'$.

Exercise 13.2.7. The closure of an open ball $B_r(x)$ is the corresponding closed ball $\overline{B}_r(x)$. So the notation used for the closed ball agrees with the notation used for the closure of the open ball.

Exercise 13.2.8. If K is closed, then $\overline{K} = K$.

Let D be a subset of $\mathbb{C}$. A point a is a *contact point* of D, if $\forall r > 0, D \cap B_r(a) \neq \emptyset$.

Exercise 13.2.9. Let D be a subset of $\mathbb{C}$. The closure of D is the set of contact points of D.

Theorem 13.2.10. *The closure of a set is a closed set.*

Proof. Let a be a contact point of $\overline{D}$. We must show $a \in \overline{D}$, that is, we must show a is a contact point of D. Let $r > 0$. Since $r/2 > 0$, and a is a contact point of $\overline{D}$, there is a $b \in \overline{D}$, such that $|a-b| < r/2$. Since $r/2 > 0$ and b is a contact point of D, there is a $c \in D$, such that $|b-c| < r/2$. Hence,

$$|a-c| \leq |a-b| + |b-c| < r.$$

Consequently, a is a contact point of D, as we needed to show. ☺

Lemma 13.2.11. *Let A be a closed set. The distance $D_A(x)$ from A to x is zero iff $x \in A$.*

Proof. If $x \in A$, then $D_A(x) = \inf\{|x-a| \mid a \in A\} \leq |x-x| = 0$. Conversely, if $D_A(x) = 0$, then $\inf\{|x-a| \mid a \in A\} = 0$, so x is an contact point of A. Hence, $x \in A$, since A is closed. ☺

Proposition 13.2.12. *If A is a closed and $[a,b] \cap A = \emptyset$, then there is a $r > 0$, such that $D_A(x) \geq r$ for all $x \in [a,b]$.*

Proof. By Example 5.4.4, the function $D_A : [a,b] \to \mathbb{R}$ is continuous. By the Extreme Value Theorem it has a smallest value $D_A(x_{\min})$. By the lemma $D_A(x) > 0$ for all $x \in [a,b]$. So $r := D_A(x_{\min}) > 0$, since $x_{\min} \notin A$ and A is closed. ☺

13.3 Compact Sets

We established some useful results, for example, the Extreme Value Theorem, for continuous functions defined on compact (i.e, closed and bounded) intervals. In this section, we extend these results to all closed and bounded subsets of $\mathbb{K}$.

Covering Compactness

The notion of covering compactness is useful because it allows for simple proofs of important properties of continuous functions.

Definition 13.3.1. Let F be a subset of $\mathbb{K}$.

- A collection of sets $(A_b)_{b\in B}$, is a *cover* of F, if $F \subseteq \bigcup_{b\in B} A_b$.
- A cover $(A_b)_{b\in B}$ is an *open cover*, if each A_b is an open set.
- If $(A_b)_{b\in B}$ is a cover of F, $C \subseteq B$, and $(A_b)_{b\in C}$ is a cover of F, then $(A_b)_{b\in C}$, is a *subcover* of $(A_b)_{b\in B}$.
- The subcover $(A_b)_{b\in C}$ is a *finite subcover*, if C is finite.

Definition 13.3.2. A subset K of $\mathbb{C}$ is *covering compact*, if any open cover of a K has a finite subcover.

It is customary to say compact in place of covering compact. We say covering compact to distinguish the covering definition from some of the alternative definitions of compactness, see Corollary 13.3.21.

Example 13.3.3. $\mathbb{R}$ is not covering compact.

Proof. $(]-n,n[)_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}$. We will show this cover does not have a finite subcover. If $C \subset \mathbb{N}$ is finite, then C has a largest member $\max(C)$. Hence, $\bigcup_{n\in C}]-n,n[=]-\max(C),\max(C)[$. In particular, $\max(C) \notin \bigcup_{n\in C}]-n,n[$, hence $\bigcup_{n\in C}]-n,n[$ is not a cover of $\mathbb{R}$. Consequently, $\mathbb{R}$ is not compact. ☺

A subset K of $\mathbb{C}$ is *bounded*, if there is an M, such that $|k| \le M$, for all $k \in K$.

Exercise 13.3.4. If K is a covering compact subset of $\mathbb{C}$, then K is bounded.

Example 13.3.5. $]0,1]$ is not covering compact.

Proof. $\left(\left]\frac{1}{n},2\right[\right)_{n\in\mathbb{N}}$ is an open cover of $]0,1]$. We will show this cover does not have a finite subcover. If $C \subset \mathbb{N}$ is finite, then C has a largest member $\max(C)$. Hence $\bigcup_{n\in C} \left]\frac{1}{n},2\right[= \left]\frac{1}{\max(C)},2\right[$. So $\frac{1}{\max(C)} \notin \bigcup_{n\in C} \left]\frac{1}{n},2\right[$ ☺

Exercise 13.3.6. If K is covering compact subset of $\mathbb{C}$, then K is a closed subset of $\mathbb{C}$.

Proposition 13.3.7. *A closed subset of a covering compact set is covering compact.*

Proof. Let K be covering compact. Suppose F is a closed subset of K. Let $(U_\alpha)_{\alpha\in A}$ be an open cover of F. Since F is closed, $\widetilde{U} := \mathbb{R}^2 \setminus F$ is open. Hence, the collection $\widetilde{U}$, U_α, $\alpha \in A$ is an open cover of K. Since K is covering compact, there is a finite set B, such that $\widetilde{U}$, $(U_\alpha)_{\alpha\in B}$ is a subcover of K. The collection $(U_\alpha)_{\alpha\in B}$ is the sought for finite subcover of F. ☺

Compact Sets and Continuity

Covering compactness is a topological property in the sense that any continuous image of a covering compact set is covering compact. Since covering compact sets are bounded the following is a generalization of Theorem 5.3.3.

Theorem 13.3.8. *If K is covering compact and $f : K \to \mathbb{C}$ is continuous, then $f(K)$ is covering compact.*

Proof. Let $(U_\alpha)_{\alpha\in A}$ be an open cover of $f(K)$. Since f is continuous each $f^{-1}(U_\alpha)$ is open. Hence,

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(U_\alpha)$$

shows $\left(f^{-1}(U_\alpha)\right)_{\alpha\in A}$ is an open cover of K. Since K is covering compact, there is a finite subcover $\left(f^{-1}(U_\alpha)\right)_{\alpha\in B}$ of K. Since

$$f(K) \subseteq f\left(\bigcup_{\alpha\in B} f^{-1}(U_\alpha)\right) = \bigcup_{\alpha\in B} f\left(f^{-1}(U_\alpha)\right) \subseteq \bigcup_{\alpha\in B} U_\alpha$$

$(U_\alpha)_{\alpha\in B}$ is a finite subcover of $f(K)$. Thus $f(K)$ is compact. ☺

The following is a generalization of Theorem 5.3.5.

Corollary 13.3.9 (Extreme Value Theorem). *If K is covering compact and $f : K \to \mathbb{R}$ is continuous, then there are $x_{\min}$ and $x_{\max}$ in K, such that*

$$f(x_{\min}) \le f(x) \le f(x_{\max}) \textit{ for all } x \in K.$$

Proof. By the theorem $f(K)$ is a covering compact subset of $\mathbb{R}$. Since any covering compact set is closed and bounded, $f(K)$ is a closed and bounded subset of $\mathbb{R}$. Since $f(K)$ is bounded, $\inf(f(K))$ and $\sup(f(K))$ are real numbers. Since $f(K)$ is closed, $\inf(f(K)), \sup(f(K))$ are in $f(K)$, consequently,

$$\inf(f(K)) = f(x_{\min}) \text{ and } \inf(f(K)) = f(x_{\max})$$

for some $x_{\min}, x_{\max} \in K$. ☺

The following is a generalization of Corollary 5.1.10.

Corollary 13.3.10. *Suppose A, B are covering compact sets and $f : A \to B$ is a continuous bijection, then the inverse function $f^{-1} : B \to A$ is continuous.*

Proof. Let F be a closed subset of A. We must show $\left(f^{-1}\right)^{-1}(F)$ is closed. But F is compact, since F is a closed subset of a compact set. Hence, continuity of f implies $\left(f^{-1}\right)^{-1}(F) = f(F)$ is compact, hence closed. ☺

The following is a generalization of Theorem 5.4.6.

Theorem 13.3.11. *If K is covering compact and $f : K \to \mathbb{C}$ is continuous, then f is uniformly continuous on K.*

Proof. Let $\varepsilon > 0$ be given. Since f is continuous on K, and $\varepsilon/2 > 0$

$$\forall x \in K, \exists \delta_x > 0, \forall y \in K, f\left(B_{\delta(x)}(x)\right) \subseteq B_{\varepsilon/2}(f(x)). \tag{13.2}$$

Now $\left(B_{\delta(x)/2}(x)\right)_{x \in K}$ is an open cover of K. Since K is covering compact, there is a finite subset $A = \{a_1, a_2, \ldots, a_n\}$ of K such that $\left(B_{\delta(a_i)/2}(a_i)\right)_{i=1}^n$ is a cover of K. Let

$$\delta := \min\left\{\frac{\delta(a_1)}{2}, \frac{\delta(a_2)}{2}, \ldots, \frac{\delta(a_n)}{2}\right\}.$$

It remains to show that δ works. Suppose $x, y \in K$ and $|x - y| < \delta$. Since the open balls $\left(B_{\delta(a_i)/2}(a_i)\right)_{i=1}^n$ form a cover of K, there is an i, such that $x \in B_{\delta(a_i)/2}(a_i)$. By the triangle inequality

$$|y - a_i| \leq |y - x| + |x - a_i| < \delta + \frac{\delta(a_i)}{2} \leq \delta(a_i).$$

Since both x and y are in $B_{\delta(a_i)}(a_i)$, we have

$$|f(x) - f(y)| \leq |f(x) - f(a_i)| + |f(a_i) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

by Eq. (13.2). Thus f is uniformly continuous. ☺

Heine–Borel Theorem

Since covering compactness allows us to prove important theorems as illustrated above, we would like to have a simple description of this class of sets. Providing a simple characterization of covering compact sets is the purpose of this subsection.

Lemma 13.3.12. *A closed and bounded interval is covering compact.*

Proof. Let $(U_\alpha)_{\alpha \in A}$ be some open cover of $[a, b]$. Let

$$S := \{t \in [a, b] \mid [a, t] \text{ has a finite subcover}\}.$$

Since a is in $[a, b]$, a is in some U_α, so $a \in S$. Hence, $S \neq \emptyset$. Let $c := \sup(S)$. Since b is an upper bound for S, $c \leq b$. Suppose $c < b$. Let U_{α_0} contain c. Let $r > 0$ be such that $B_r(c) \subseteq U_{\alpha_0}$. Let $(U_\alpha)_{\alpha \in D}$ be a finite subcover of $\left[a, c - \frac{r}{2}\right]$. Then $(U_\alpha)_{\alpha \in D \cup \{\alpha_0\}}$ is a finite subcover of $\left[a, c + \frac{r}{2}\right]$. Contradicting the definition of c. Consequently, $c = b$. The argument also shows $c \in S$. Hence, $[a, b]$ has a finite subcover. ☺

The same argument shows that $[a, b] \times \{y\}$ is covering compact.

Lemma 13.3.13. *Any closed rectangle $[a,b] \times [c,d]$ is covering compact.*

Proof. To simplify the notation we will show $[0,1]^2$ is covering compact. Let $(U_\alpha)_{\alpha \in A}$ be some open cover of $[0,1]^2$. Let

$$S := \{t \in [0,1] \mid [0,1] \times [0,t] \text{ has a finite subcover}\}.$$

Then $(U_\alpha)_{\alpha \in A}$ is an open cover of $[0,1] \times \{0\}$. By the previous lemma, there is a finite subcover $(U_\alpha)_{\alpha \in B}$ of $[0,1] \times \{0\}$. Hence, $0 \in S$, so $S \neq \emptyset$. Let $c := \sup(S)$. Since 1 is an upper bound for S, $c \leq 1$. Suppose $c < 1$. By the previous lemma, there is a finite subcover $(U_\alpha)_{\alpha \in C}$ of $[0,1] \times \{c\}$.

Let $\Phi := \mathbb{R}^2 \setminus \bigcup_{\alpha \in C} U_\alpha$. Recall, $D_\Phi(y) := \inf\{|x-y| \mid x \in \Phi\}$ is the distance from the point y to the set Φ. The function $f : [0,1] \to \mathbb{R}$ determined by $f(t) := D_\Phi((t,c))$ is continuous, by Example 5.4.4. Hence, f has a minimal value r by the Extreme Value Theorem. Since Φ is closed and $\Phi \cap ([0,1] \times \{c\}) = \emptyset$, the minimal value satisfies $r > 0$. If $x \in \Phi$ and $y \in [0,1] \times \{c\}$, then $|x-y| \geq D_\Phi(y) \geq r$. In particular, $[0,1] \times]c-r, c+r[\subseteq \bigcup_{\alpha \in C} U_\alpha$.

Let $(U_\alpha)_{\alpha \in D}$ be a finite subcover of $[0,1] \times \left[0, c - \frac{r}{2}\right]$. Then $(U_\alpha)_{\alpha \in D \cup C}$ is a finite subcover of $[0,1] \times \left[0, c + \frac{r}{2}\right]$. Contradicting the definition of c. Consequently, $c = 1$. The argument also shows $c \in S$. Hence, $[0,1] \times [0,1]$ has a finite subcover (Fig. 13.1). ☺

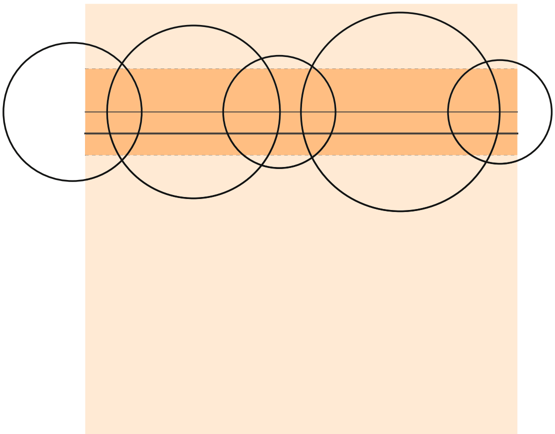

Fig. 13.1 Illustrating proof of Lemma 13.3.13, by showing the square $[0,1]^2$, the (*thin*) line $L_c := [0,1] \times \{c\}$, the finite subcover U_α, $\alpha \in A$ of L_c (in the case where the open sets U_α are balls) the inclusion $[0,1] \times]c-r, c+r[\subseteq \bigcup_{\alpha \in C} U_\alpha$, and the (*thick*) line $L_{c-\frac{r}{2}} = [0,1] \times \left\{c - \frac{r}{2}\right\}$

We can now establish a useful characterization of covering compactness named after Félix Édouard Justin Émile Borel (7 January 1871, Saint-Affrique to 3 February 1956, Paris) and Heinrich Eduard Heine (16 March 1821, Berlin to 21 October 1881, Halle).

Theorem 13.3.14 (Heine–Borel). *A subset of $\mathbb{C}$ is covering compact iff it is closed and bounded.*

Proof. We have already seen that, if K is covering compact, then K is closed and bounded. So suppose K is closed and bounded. Since K is bounded, there is a square $[a,b]^2$ containing K. Since K is a closed subset of a covering compact set, K is covering compact. ☺

Sequential Compactness

A subset K of $\mathbb{C}$ is *sequentially compact*, if any sequence of points in K has a subsequence converging to a point *in* K.

Example 13.3.15. The set of reals $\mathbb{R}$ is not sequentially compact. For example, the sequence $a_n := n$ does not have a convergent subsequence. In fact, since $a_n \to \infty$, so does any subsequence.

Exercise 13.3.16. Any sequentially compact set is bounded.

Example 13.3.17. The set $]0,1]$ is not sequentially compact. For example, the sequence $a_n := \frac{1}{n}$ does not have a subsequence converging to a point in $]0,1]$, because the sequence converges to 0, any subsequence converges to 0. And $0 \notin]0,1]$.

Exercise 13.3.18. Any sequentially compact set contains all its accumulation points, hence is closed.

Exercise 13.3.19. Let (x_n) be a sequence of points in A. If $x_n \to x_0$, then x_0 is in the closure of A.

Recall, we proved in Sect. 9.1 that any bounded sequence of complex numbers has a convergent subsequence. We called this sequential compactness, or the Bolzano–Weierstrass Theorem.

Theorem 13.3.20 (Bolzano–Weierstrass). *Let K be a subset of $\mathbb{C}$. The set K is sequentially compact iff it is closed and bounded.*

Proof. We saw above that a sequentially compact set is closed and bounded. So suppose K is closed and bounded. Let (x_n, y_n) be a sequence of points in K. Since K is bounded, the sequence (x_n, y_n) is bounded. Hence, it has a convergent subsequence (x_{i_n}, y_{i_n}) by the result from Sect. 9.1 mentioned above. Let (x_0, y_0) be the limit of this subsequence. Since K is closed, (x_0, y_0) is in K. Thus K is sequentially compact. ☺

Corollary 13.3.21. *A subset of the complex plane is covering compact iff it is sequentially compact iff it is closed and bounded.*

In particular, an interval is a compact set iff it is closed and bounded. This explains the terminology "compact interval".

13.4 Connected Sets

We will discuss pathwise connected sets. This allows us to show a space filling curve cannot be one-toone. We will not discuss the related, but different notion of connectedness, of topologically connected sets (Fig. 13.2) .

A subset $D \subseteq \mathbb{C}$ is *pathwise connected*, if given any two points $a, b \in D$ there is a continuous function $\phi : [0,1] \to D$, such that $\phi(0) = a$ and $\phi(1) = b$. The function ϕ is a *path* in D connecting a and b.

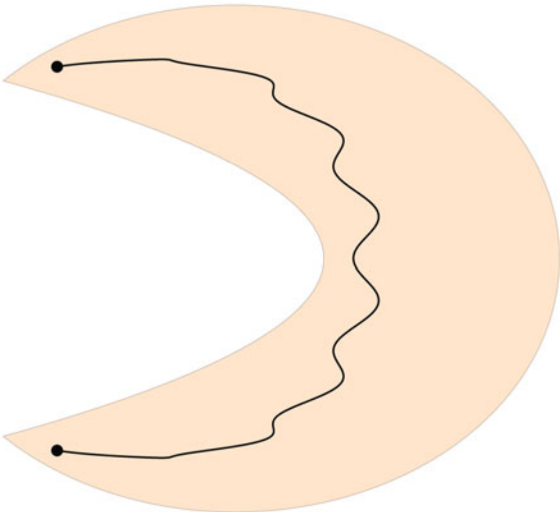

Fig. 13.2 A region D and two points in the region connected by a path in the region

Exercise 13.4.1. Let D be a subset of $\mathbb{R}$. The set D is pathwise connected iff D is an interval.

Exercise 13.4.2. The square $[0,1]^2$ is pathwise connected.

Theorem 13.4.3 (Netto's Theorem). *No function $f : [0,1] \to [0,1]^2$ is 1-1, onto, and continuous.*

Proof. Suppose $f : [0,1] \to [0,1]^2$ is 1-1, onto and continuous. Since f is continuous and $[0,1]$ is compact, the inverse function $g = f^{-1}$ is continuous. Let $x_0 = f(1/2)$. Then $g([0,1]^2 \setminus \{x_0\})$ is pathwise connected, since g is continuous and $[0,1]^2 \setminus \{x_0\}$ is pathwise connected. But $g([0,1]^2 \setminus \{x_0\}) = [0,1/2) \cup (1/2,1]$ is not pathwise connected. ☺

Problems

Problems for Sect. 13.1

Infinite intersections of open sets need not be open sets:

1. Suppose $\mathbb{K}=\mathbb{R}$. Prove that $\bigcap_{n=1}^{\infty}\left]0,1+\frac{1}{n}\right[$ is an intersection of open sets that is not an open set.

2. Construct a sequence A_n of open sets in $\mathbb{R}^2$, such that $\bigcap_{n=1}^{\infty}A_n$ is not an open subset of $\mathbb{R}^2$.

3. The triangle $\{(x,y)\in\mathbb{R}^2 \mid x>0, y>0, x+y<1\}$ is an open set in $\mathbb{R}^2$.

4. Show that $\bigcap_{n=1}^{\infty}B_{1+\frac{1}{n}}(x)=\overline{B}_1(x)$.

5. The set $\mathbb{R}\setminus\left\{\frac{1}{n}\mid n\in\mathbb{N}\right\}$ is not an open subset of $\mathbb{R}$, because any interval of the form $B_r(0)=]-r,r[$ will contain all $\frac{1}{n}$ with $n>\frac{1}{r}$.

6. Show the set $\mathbb{R}\setminus\left(\left\{\frac{1}{n}\mid n\in\mathbb{N}\right\}\cup\{0\}\right)$ is an open subset of $\mathbb{R}$.

7. If C is the Cantor set, then $[0,1]\setminus C$ is an open subset of $\mathbb{R}$.

8. If C is the Cantor set, then $\overset{\circ}{C}=\emptyset$.

9. If A and B are open subsets of the real line $\mathbb{R}$, then $A\times B$ is an open subset of the plane $\mathbb{R}^2$.

Problems for Sect. 13.2

1. If A and B are closed subsets of the real line $\mathbb{R}$, then $A\times B$ is an closed subset of the plane $\mathbb{R}^2$.

2. The Cantor set is a closed subset of $\mathbb{R}$.

3. Find a sequence of closed sets K_n, such that $\bigcup_{n=1}^{\infty}K_n$ is not a closed set.

4. Find a sequence of closed sets K_n, such that $\bigcup_{n=1}^{\infty}K_n$ is an open set.

5. Establish the following:
(i) $\overline{Q}=\mathbb{R}$
(ii) $\overline{\mathbb{Q}\times\mathbb{Q}}=\mathbb{R}\times\mathbb{R}$.

6. The closure of the open ball $B_r(x)$ is the closed ball $\overline{B}_r(x)$.

7. If C is the Cantor set, then $\overline{\mathbb{R}\setminus C}=\mathbb{R}$.

8. For any sets A,B we have $\overline{A\cup B}=\overline{A}\cup\overline{B}$.

9. For any sets A,B we have $\overline{A\cap B}\subseteq\overline{A}\cap\overline{B}$.

10. If $A := \mathbb{Q}$ and $B := \mathbb{R} \setminus \mathbb{Q}$, then $\overline{A \cap B} = \emptyset$ and $\overline{A} \cap \overline{B} = \mathbb{R}$.

The *boundary* of the set D is the set $\partial D := \overline{D} \setminus \mathring{D}$.

11. $\partial[a,b] = \{a,b\}$.

12. $\partial B_r(a) = \{x \mid |x-a| = r\}$.

13. If C is the Cantor set, then $\partial C = C$.

14. If C is the Cantor set, then $\partial(\mathbb{R} \setminus C) = C$.

Problems for Sect. 13.3

1. Find an open cover of $[0,1] \cap \mathbb{Q}$ that does not have a finite subcover.

2. Give an example of an increasing sequence (a_n) such that the set $\{a_n \mid n \in \mathbb{N}\}$ is compact.

3. If $a_n < a_{n+1}$ for all n, then the set $\{a_n \mid n \in \mathbb{N}\}$ is not compact.

4. Let (z_n) be a convergent sequence with limit $\widetilde{z}$. The set $\{z_n \mid n \in \mathbb{N}\}$ is compact iff $z_n = \widetilde{z}$ for some n.

5. Use the definition of covering compactness to show that if A and B are covering compact, then the union of A and B is also covering compact.

Problems for Sect. 13.4

1. Suppose $f : A \to B$ is continuous. If D is a pathwise connected subset of A, then $f(D)$ is a pathwise connected subset of B.

2. Suppose $f : A \to \mathbb{R}$ is continuous and A is pathwise connected. Let a, b be points in A, such that $f(a) < f(b)$. If $f(a) < k < f(b)$, show there is a c in A, such that $f(c) = k$.

3. Let I be an interval and suppose $f : I \to \mathbb{R}$ is differentiable. Let

$$D := \left\{ f'(x) \mid x \in I \right\}$$

and

$$C := \left\{ \frac{f(b) - f(a)}{b - a} \mid a, b \in I, a < b \right\}.$$

(a) Prove $C \subseteq D \subseteq \overline{C}$.

(b) Prove C is an interval. [*Hint*: One way is to prove C is pathwise connected. To this end, note for x,y in I, the function $\varphi_{x,y}(t) := (1-t)x+ty$ determines a path in I beginning at $\varphi_{x,y}(0) = x$ and ending at $\varphi_{x,y}(1) = y$. Show

$$\psi(t) := \frac{f\left(\varphi_{b,d}(t)\right) - f\left(\varphi_{a,c}(t)\right)}{\varphi_{b,d}(t) - \varphi_{a,c}(t)}, 0 \le t \le 1$$

is a path in C connecting $\frac{f(b)-f(a)}{b-a}$ to $\frac{f(d)-f(c)}{d-c}$ for all $a < b$ and all $c < d$ in I.]

Solutions and Hints for the Exercises

Exercise 13.1.4. By Example 13.1.1 $B_r(x)$ is in the interior of the closed ball $\overline{B_r}(x)$. If $|x-y| = r$ and $s > 0$, then $B_s(y)$ is not a subset of $B_r(x)$.

Exercise 13.1.5. Let $x \in \overset{\circ}{A}$. Pick $r > 0$ such that $B_r(x) \subseteq A$. Let $y \in B_r(x)$. For some $s > 0$, $B_s(y) \subseteq B_r(x)$. Hence, $y \in \overset{\circ}{A}$.

Exercise 13.2.5. If A is closed in D, then $A = D \cap K$ for some closed set K. Hence, $D \setminus A = D \setminus (D \cap K) = D \cap (\mathbb{C} \setminus K)$. The converse is similar.

Exercise 13.2.6. By Exercise 13.2.5 $f^{-1}(K)$ is closed in D iff $\mathbb{C} \setminus f^{-1}(K) = f^{-1}(\mathbb{C} \setminus K)$ is open in D.

Exercise 13.2.7. If $|x-y| = r$ then $B_s(y) \cap B_r(x) \neq \emptyset$. Hence, y is an accumulation point of $B_r(x)$.

Exercise 13.2.8. If K is closed, then $K' \subseteq K$. Hence $K \cup K' = K$.

Exercise 13.2.9. Let $\widehat{K}$ be the set of contact points of K. Any point in K is a contact point of K and any accumulation point of K is a contact point of K, hence $\overline{K} \subseteq \widehat{K}$. Conversely, if $x \notin K$ is a contact point of K, then x is an accumulation point of K. Consequently, $\widehat{K} \subseteq \overline{K}$.

Exercise 13.3.4. If K is not bounded, then $(B_1(k))_{k \in K}$ is an open cover without a finite subcover.

Exercise 13.3.6. If K is not closed, then K has an accumulation point a that is not in K. Let A_0 be the complement of the closed ball $\overline{B_{1/2}}(a)$ and let $A_n = B_{1/n}(a)$ for $n \in \mathbb{N}$. Then $(A_n)_{n \in \mathbb{N}_0}$ is an open cover of K without a finite subcover.

Exercise 13.3.16. Let $x_1 \in K$, $x_2 \in K$ with $|x_2| > 1 + |x_1|$, $x_3 \in K$ with $|x_3| > 1 + |x_2|$, and so on. Then $|x_k| \to \infty$. Hence (x_k) is not convergent.

Exercise 13.3.18. Suppose K is sequentially compact and a is an accumulation point of K. For each n, let $x_n \in B_{1/n}(a) \cap (K \setminus \{a\})$. Then $x_n \to a$. Hence, any subsequence of (x_n) converges to a. Thus $a \in K$.

Exercise 13.3.19. By definition of convergence, x_0 is a contact point of A.

Exercise 13.4.1. $\Longrightarrow$ is a consequence of the Intermediate Value Theorem and the Interval Theorem. $\Longleftarrow$ Suppose D is an interval and $a < b$ are in D. Then $f(t) := a + t(b-a)$ is the required path.

Exercise 13.4.2. If (a,b) and (α,β) are in $[0,1]^2$, then $f(t) := (a,b) + t(\alpha - a, \beta - b)$ is the required path.

Appendix A
Logic and Set Theory

This chapter serves to introduce notation and notions from logic and set theory. Many of the details are left as exercises for the reader. Hopefully, the reader is already familiar with most of this material. We will use this material in the body of the text, usually without calling attention to the details.

A.1 Logic

In the following P, Q, R are *statements*, that is, sentences that either are true or false.

A.1.1 The Connective "or"

$P \vee Q$ means at least one of P and Q is true. We say *P or Q*.

A.1.2 The Connective "and"

$P \wedge Q$ means both P and Q are true. We say *P and Q*.

A.1.3 Implication

$P \implies Q$ means "if P is true, then Q is true" or "given P is true, we can conclude Q is true". Often this is abbreviated "if P, then Q". We say *P implies Q*. Clearly, the only way $P \implies Q$ can be false, is when P is true and Q is false.

Implication is the fundamental form of a mathematical claim.

S. Pedersen, *From Calculus to Analysis*, DOI 10.1007/978-3-319-13641-7

A.1.4 Bi-implication

$P \iff Q$ means $P \implies Q$ and $Q \implies P$. When $P \iff Q$ we say P and Q are *equivalent* and will sometimes write P *if and only if* Q, or more briefly P *iff* Q.

A.1.5 Quantifiers

Most "interesting" statements depend on one or more variables, e.g., $P(x), P(x,y), \ldots$ They may be true for some values of the variables and false for others, usually this state of affairs is dealt with using quantifiers. There are two quantifiers, the universal quantifier $\forall$ and the existential quantifier $\exists$.

A.1.5.1 All

$\forall$ is shorthand for statements like "for all" and "for each".

$$\forall x, P(x) \text{ means: for all } x, P(x) \text{ is true}$$

A.1.5.2 Some

$\exists$ is shorthand for statements like "there is", "for some", and "there exists".

$$\exists x, P(x) \text{ means: } P(x) \text{ is true for at least one } x$$

A typical statement in this book is of the form: "f is continuous at x", which in the notation introduced above is written as

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y, |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

A.1.6 Negation

$\neg P$ is shorthand for P is false. This may take the form of crossing out a binary operator, for example, $x \neq y$ means $\neg(x = y)$.

Some useful formulas involving negation are:

$$\begin{aligned}
[\neg(P \vee Q)] &\iff [(\neg P) \wedge (\neg Q)] \\
[\neg(\forall x, P(x))] &\iff [\exists x, \neg P(x)] \\
[P \implies Q] &\iff [(\neg Q) \implies (\neg P)] \\
[\neg(P \implies Q)] &\iff [(\neg P) \wedge Q]
\end{aligned}$$

$$[P \implies Q] \iff [P \vee (\neg Q)]$$

The third formula is the foundation for "proof by contraposition" and the fourth for "proof by contradiction". In a proof by contraction we assume that P is false and Q is true and then deduces some contradiction, i.e, we show $[(\neg P) \wedge Q] \implies [R \wedge (\neg R)]$, where R is some statement. Since R is false, the only way $[(\neg P) \wedge Q] \implies [R \wedge (\neg R)]$ can hold is if $[(\neg P) \wedge Q]$ is false, hence by the fourth formula $P \implies Q$ is true.

A.2 Sets of Numbers

The symbol "$a := b$" is used to indicate that b has already been constructed and that we introduce the notation a to equal b.

Set notation	Name
$\mathbb{N} := \{1, 2, 3, \ldots\}$	Natural numbers
$\mathbb{N}_0 := \{0, 1, 2, \ldots\}$	Natural numbers
$\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	Integers
$\mathbb{Q} := \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N} \right\}$	Rational numbers
$\mathbb{R} := \{\text{all infinite decimals}\}$	Real numbers
$\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$	Complex numbers

A *closed interval* is a set of the form

$$\begin{aligned}
[a,b] &:= \{x \in \mathbb{R} \mid a \leq x \leq b\}, \\
[a,\infty[&:= \{x \in \mathbb{R} \mid a \leq x\} = \{x \in \mathbb{R} \mid a \leq x < \infty\}, \\
]-\infty,b] &:= \{x \in \mathbb{R} \mid x \leq b\} = \{x \in \mathbb{R} \mid -\infty < x \leq b\}, \text{ or} \\
]-\infty,\infty[&:= \{x \in \mathbb{R} \mid -\infty < x < \infty\} = \mathbb{R}.
\end{aligned}$$

Where $a \leq b$ are real numbers. Intervals of the form $[a,b]$ are important enough to have their own terminology, such an interval is called a compact interval. An *open interval* is a set of the form

$$\begin{aligned}
]a,b[&:= \{x \in \mathbb{R} \mid a < x < b\}, \\
]a,\infty[&:= \{x \in \mathbb{R} \mid a < x\} = \{x \in \mathbb{R} \mid a < x < \infty\}, \\
]-\infty,b[&:= \{x \in \mathbb{R} \mid x < b\} = \{x \in \mathbb{R} \mid -\infty < x < b\}, \text{ or} \\
]-\infty,\infty[&:= \{x \in \mathbb{R} \mid -\infty < x < \infty\} = \mathbb{R}.
\end{aligned}$$

Where $a < b$ are real numbers. A *half-open interval* is a set of the form

$$\begin{aligned}
]a,b] &:= \{x \in \mathbb{R} \mid a < x \leq b\} \text{ or} \\
[a,b[&:= \{x \in \mathbb{R} \mid a \leq x < b\}.
\end{aligned}$$

Where $a < b$ are real numbers. An *interval* is a subset of $\mathbb{R}$ that is either an open, a closed, or a half-open interval.

A.3 Set Theory

A set is a collection of object, usually called *elements*. We will write $x \in A$, if the object x *is in* the set A. and $x \notin A$, if the object x *is not in* the set A. The set without any elements is the *empty set* $\emptyset$, mostly we do our best to ignore this set.

A.3.1 Subset

$A \subseteq B$ means $x \in A \implies x \in B$, that is any element in A is an element of B. $A = B$ means $A \subseteq B$ and $B \subseteq A$, that is A and B have the same elements. $A \subset B$, means $A \subseteq B$ and $A \neq B$. Finally, $A \supseteq B$ means $B \subseteq A$.

A.3.2 Union

$A \cup B$ is the *union* of A and B.

$$A \cup B := \{x \mid x \in A \vee x \in B\}$$

A.3.3 Intersection

$A \cap B$ is the *intersection* of A and B.

$$A \cap B := \{x \mid x \in A \vee x \in B\}$$

A.3.4 Set Difference

$A \setminus B$ is the *difference* of A and B.

$$A \setminus B := \{a \in A \mid a \notin B\}.$$

$A \setminus B$ is also called the *complement* of B in A.

A.3.5 General Unions and Intersections

Let I be a set and suppose for each $i \in I$, A_i is some set.

$$\bigcup_{i\in I} A_i := \{x \mid \exists i \in I, x \in A_i\} \text{ and } \bigcap_{i\in I} A_i := \{x \mid \forall i \in I, x \in A_i\}.$$

Some frequently occurring special cases have their own notation, for example,

$$\bigcup_{n=1}^{\infty} A_n := \bigcup_{n\in\mathbb{N}} A_n \text{ and } \bigcap_{n=1}^{\infty} A_n := \bigcap_{n\in\mathbb{N}} A_n.$$

By De Morgan's Laws

$$B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} (B \setminus A_i) \text{ and } B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I} (B \setminus A_i).$$

A.4 Functions

The notation $f : A \to B$ means that f is a function defined on the set A with values in the set B, that is a unique element of B is assigned to each element of A by f. If b is the element of B assigned to $a \in A$ we write $f(a) = b$, and we say that b is the image of a under f.

A.4.1

If $f : A \to B$, we say f is *onto* or *surjective,* if each element of B is assigned to some element of A, that is

$$\forall b \in B, \exists a \in A, f(a) = b.$$

A.4.2

If $f : A \to B$, we say f is *one-to-one* or $1-1$, or *injective*, if each element in B is assigned to at most one element of A, that is

$$\forall a_1, a_2 \in A, f(a_1) = f(a_2) \implies a_1 = a_2.$$

A.4.3

If $f : A \to B$, we say f is a *one-to-one correspondence* or *bijective* if f is both onto and $1-1$. If f is bijective that any $b \in B$ is assigned to exactly one $a \in A$ by f, given b the a that is assigned to b by f is denoted by $f^{-1}(b) := a$, this determines a function $f^{-1} : B \to A$. f^{-1} is called the *inverse* function of f.

A.4.4 Set Functions

A.4.4.1 Image

If $C \subseteq A$, then

$$f(C) := \{f(c) \mid c \in C\}$$

is called the *image* of C under f. The *range* of f is $f(A)$. If $f(A) = B$, then f is onto. If f is one-to-one, we can consider the inverse function $f^{-1} : f(A) \to A$.

A.4.4.2 Pre-image

If $D \subseteq B$, then

$$f^{-1}(D) := \{a \in A \mid f(a) \in D\}$$

is called the *pre-image* of D under f. Note, this does not require f^{-1} to be a function. If for each $b \in B$, the set $f^{-1}(\{b\})$ contains at most one element, then f is one-to-one.

The use of the f^{-1} for inverse function and for pre-image are distinguished by the argument in the first case being an element and in the second case a subset.

The text assumes the reader knows the following results.

Exercise A.4.1. Prove

$$f^{-1}\left(\bigcap_i B_i\right) = \bigcap_i f^{-1}(B_i)$$

for any sets A, B, any $f : A \to B$, and any subsets B_i of B.

Exercise A.4.2. Prove

$$f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i)$$

for any sets A, B, any $f : A \to B$, and any subsets B_i of B.

Exercise A.4.3. Let $f : A \to B$. Prove

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

for any $A_1, A_2 \subseteq A$.

Exercise A.4.4. Let $f : A \to B$. Prove

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

for any subsets A_1, A_2 of A.

Exercise A.4.5. Let $f : A \to B$. Prove f is one-to-one iff

$$f(A_1 \cap A_2) \supseteq f(A_1) \cap f(A_2)$$

for any subsets A_1, A_2 of A?

Exercise A.4.6. If $f : A \to B$ and $C \subseteq A$, then $C \subseteq f^{-1}(f(C))$.

Exercise A.4.7. Let $f : A \to B$. Prove f is $1-1$ iff

$$\forall C \subseteq A, f^{-1}(f(C)) \subseteq C.$$

Exercise A.4.8. If $f : A \to B$ and $D \subseteq B$, then $f\left(f^{-1}(D)\right) \subseteq D$.

Exercise A.4.9. Let $f : A \to B$. Prove f is onto iff

$$D \subseteq f\left(f^{-1}(D)\right)$$

for all $D \subseteq B$.

Appendix B
The Principle of Induction

We assume the reader is familiar with proofs by induction. As applications of induction we establish the Fundamental Theorem of Arithmetic, that there are infinitely many primes, and the Binomial Theorem.

B.1 Formulations of Induction

Induction is a fundamental property of the set of natural numbers. There are three standard formulations. Let $A \subseteq \mathbb{N}$.

- **Weak induction**: If (i) $1 \in A$ and (ii) $\forall n \in \mathbb{N}, n \in A \implies n+1 \in A$, then $A = \mathbb{N}$.
- **Strong induction**: If $\forall n \in \mathbb{N}_0, \{1,2,\ldots,n\} \subseteq A \implies n+1 \in A$, then $A = \mathbb{N}$. [If $n = 0$, then $[1,2,\ldots,n\} = \emptyset$.]
- **Well ordering**: If $A \neq \emptyset$, then A has a smallest element.

It is well known that these formulations are in some sense equivalent.

The following provides examples of proofs by induction.

A number $p \in \mathbb{N}$ is *prime*, if $p > 1$ and for all $k, m \in \mathbb{N}$, $p = km \implies k = 1$ or $m = 1$.

Theorem B.1.1 (Fundamental Theorem of Arithmetic). *Any positive integer is a product of primes. That is given any $n \in \mathbb{N}$ there exists $m_j \in \mathbb{N}_0$, such that*

$$n = p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots$$

where $p_1 < p_2 < \cdots$ is the primes.

Proof. Suppose the claim is false. Let n be the smallest positive integer that is not a product of primes. The product is $= 1$ if all $m_j = 0$ and any prime is obtained by setting one $m_j = 1$ and all other $m_j = 0$. So n is not 1 and n is not a prime. Hence, $n = n_1 n_2$, where $n_1 > 1$ and $n_2 > 1$. If both n_1 and n_2 are products of primes, then n is a product of primes, so at least one of n_1 and n_2 is not a products of primes. Since,

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$n_1 < n$ and $n_2 < n$ this contradicts that n is the *smallest* positive integer that is not a product of primes. ☺

Exercise B.1.2. The exponents m_j are unique.

As an application of the Fundamental Theorem of Arithmetic let me prove:

Theorem B.1.3. *The are infinitely many primes.*

Proof. The number $p = 2$ is prime, so the set of primes is not empty. Let $q_{1,}q_2, \ldots, q_n$ be some finite list of primes. Let $a := q_1 q_2 \cdots q_n$ be the product of the primes $q_1, q_2, \ldots, q_n$. Let $b := a+1$. By the Fundamental Theorem of Arithmetic, b is a product of primes. Let $\widetilde{p}$ be one of these primes. Then $b = \widetilde{p}m$ for some integer $m \geq 1$. Suppose $\widetilde{p} = q_k$ for some $k \in \{1, 2, \ldots, n\}$. Then

$$1 = b - a = \widetilde{p}\left(m - \frac{a}{q_k}\right).$$

Since $\widetilde{p} > 1$ and $m - \frac{a}{q_k}$ is an integer, this is a contradiction. Hence $\widetilde{p} \neq q_k$ for all $k = 1, 2, \ldots, n$. Thus, no finite list contains all the primes. ☺

This proof is essentially due to Euclid. Euclid worked in Alexandria during the reign of Ptolemy I (323–283 BC).

B.2 Binomial Theorem

The *factorial* function is defined inductively by the basis clause: $0! = 1$ and the inductive clause for $n \in \mathbb{N}_0$ let $(n+1)! = (n!)(n+1)$. Hence, $n! = 1 \cdot 2 \cdots (n-1) \cdot n$. For $n, k \in \mathbb{N}_0$ with $n \geq k$ let

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

These numbers are called the *binomial coefficients*.

Exercise B.2.1. Let $n \geq k \in \mathbb{N}$ with $n \geq k$. Prove

$$\binom{n}{k-1}\binom{n}{k} = \binom{n+1}{k}.$$

[*Hint*: This is a simple consequence of the inductive clause: $(j+1)! = (j!)(j+1)$ in the definition of factorials and algebra.]

Theorem B.2.2 (Binomial Theorem). *For any complex numbers x, y and any $n \in \mathbb{N}_0$ we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Exercise B.2.3. Use induction to prove the Binomial Theorem. [*Hint*: The previous exercise is one step in this proof.]

Exercise B.2.4. Why are the binomial coefficient integers?

Appendix C
The Field Axioms

C.1 Statement of the Axioms

A *field* is a set $\mathbb{F}$ together with two functions $a : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$and $m : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, usually called addition and multiplication and written as

$$x + y = a(x, y)$$
$$xy = x \cdot y = m(x, y)$$

satisfying the following axioms:

Axioms for addition:

$\forall x, y, z \in \mathbb{F}, x + (y + z) = (x + y) + z$	associativity
$\forall x, y \in \mathbb{F}, x + y = y + x$	commutativity
$\exists 0 \in \mathbb{F}, \forall x \in \mathbb{F}, x + 0 = x$	existence of identity
$\forall x \in \mathbb{F}, \exists -x \in \mathbb{F}, x + (-x) = 0$	existence of inverse

Remark C.1.1. An ordered pair $(\mathbb{F}, +)$ consisting of a set $\mathbb{F}$ and a function $(x, y) \to x + y$ mapping $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying the axioms for addition is called a *commutative group*. Such a pair is also called an *abelian group* after Niels Henrik Abel (5 August 1802 Finnoy to 6 April 1829 Froland).

Axioms for multiplication:

$\forall x, y, z \in \mathbb{F}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$	associativity
$\forall x, y \in \mathbb{F}, x \cdot y = y \cdot x$	commutativity
$\exists 1 \in \mathbb{F}, \forall x \in \mathbb{F}, x \cdot 1 = x$	existence of identity
$\forall x \in \mathbb{F} \setminus \{0\}, \exists x^{-1} \in \mathbb{F}, x \cdot \left(x^{-1}\right) = 1$	existence of inverse

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Axioms for the interaction of addition and multiplication:

$$1 \neq 0 \qquad \text{nontriviality}$$

$$\forall x,y,z \in \mathbb{F}, x \cdot (y+z) = (x \cdot y) + (x \cdot z) \qquad \text{distributivity}$$

The set of rational numbers, the set of real numbers, and the set of complex numbers are examples of fields when equipped with the usual notions of addition and multiplication. But there are other examples of fields. For example, if p is a prime, then

$$\mathbb{Z}_p := \{0, 1, \ldots, p-1\}$$

equipped with addition and multiplication modulo p is a field. We will not prove this. For small values of p it can be verified by checking all cases. For $p = 3, 5$ the addition and multiplication tables for $\mathbb{Z}_p$ are given in Tables C.1 and C.2.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Table C.1 The addition table and the multiplication table in $\mathbb{Z}_3$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Table C.2 The addition table and the multiplication table in $\mathbb{Z}_5$

C.2 Some Consequences of the Axioms

It is usual to define subtraction as $x - y := x + (-y)$ and division as $\frac{x}{y} := x\left(y^{-1}\right)$. The point of the field axioms is that arithmetic works in any field. To verify this requires some work. How one can do this is indicated below.

Theorem C.2.1 (Uniqueness of Identity). (a) *If $x + y = x$ and $x + z = x$ for all x, then $y = z$.* (b) *If $x \cdot y = x$ and $x \cdot z = x$ for all x, then $y = z$.*

Proof. For (a):

$$z = z + y = y + z = y.$$

Where we used $x+y=x$ with $x=z$, commutativity of addition, and $x+z=x$ with $x=y$.

Case (b) is similar. ☺

Theorem C.2.2 (Uniqueness of Inverse). *(*a*) If $x+y=0$ and $x+z=0$, then $y=z$. (*b*) If for some $x\neq 0$, $x\cdot y=x$ and $x\cdot z=x$, then $y=z$.*

Proof. For (a):

$$y=0+y=(x+z)+y=x+(z+y)=(z+y)+x=z+(y+x)=z+0=z.$$

Case (b) is similar. ☺

Theorem C.2.3. *For all x we have $x\cdot 0=0$.*

Proof. As for the previous theorems, this is a simple calculation:

$$x\cdot 0=x\cdot(0+0)=(x\cdot 0)+(x\cdot 0).$$

Adding $-(x\cdot 0)$ to both sides and simplifying gives $0=x\cdot 0$. ☺

Theorem C.2.4. *The equality $(-x)\,y=-(xy)$ holds for all x,y.*

Proof. Again a simple calculation suffices:

$$x\cdot y+(-x)\cdot y=(x+(-x))\cdot y=0\cdot y=0.$$

Since $x\cdot y$ has a unique additive inverse, the proof is complete. ☺

Exercise C.2.5. Show $-(x+y)=(-x)+(-y)$ for all x,y.

Exercise C.2.6. Show $-0=0$ and $1^{-1}=1$.

Exercise C.2.7. (a) For any x, $-(-x)=x$. (b) If $x\neq 0$, then $\left(x^{-1}\right)^{-1}=x$.

Exercise C.2.8. If $x\cdot y=0$, then $x=0$ or $y=0$.

Corollary C.2.9. *If $x\neq 0$ and $y\neq 0$, then $x\cdot y\neq 0$. In particular, $(\mathbb{F}\setminus\{0_{\mathbb{F}}\},\cdot)$ is an abelian group.*

Recall, $\frac{x}{y}:=x\cdot\left(y^{-1}\right)$, if $y\neq 0$. Setting $x=1$ gives $\frac{1}{y}=y^{-1}$.

Exercise C.2.10. Show $\frac{x}{1}=x$, for all x.

Exercise C.2.11. Show $\frac{1}{x}\cdot\frac{1}{y}=\frac{1}{x\cdot y}$ for all $x,y\neq 0$.

Exercise C.2.12. Show $\frac{xy}{xz}=\frac{y}{z}$ for all x,y,z such that $xz\neq 0$.

Exercise C.2.13. Show $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$ for all $x,y\neq 0$.

Below, we will write $0_{\mathbb{F}}$ for the additive identity in $\mathbb{F}$ and $1_{\mathbb{F}}$ for the multiplicative identity in $\mathbb{F}$ to distinguish them from the natural numbers 0 and 1.

For $x \in \mathbb{F}$ inductively define a map $n \to x^n$ mapping $\mathbb{N}_0 \to \mathbb{F}$ by setting by $x^0 = 1_{\mathbb{F}}$ and $x^{n+1} = x^n \cdot x$.

Exercise C.2.14. Prove that $x^m \cdot x^n = x^{m+n}$ for all $m, n \in \mathbb{N}_0$.

Define a map $n \to n_{\mathbb{F}}$ mapping $\mathbb{N}_0 \to \mathbb{F}$ inductively by

$$\begin{aligned} 0 &\to 0_{\mathbb{F}} \\ n+1 &\to n_{\mathbb{F}} + 1_{\mathbb{F}}. \end{aligned}$$

For example, $3_{\mathbb{F}} = 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}}$.

Exercise C.2.15 (Binomial Theorem). If $a, b \in \mathbb{F}$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}_{\mathbb{F}} a^k b^{n-k}$$

for any $n \in \mathbb{N}_0$. Here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the usual binomial coefficient, in particular, $\binom{n}{k}$ is in $\mathbb{N}$.

If $a_k \in \mathbb{F}$ for $k = 0, 1, \ldots, n$ we can consider the polynomial (called a polynomial over $\mathbb{F}$)

$$p(x) := \sum_{k=0}^{n} a_k x^k.$$

The collection of these polynomials is usually denoted by $\mathbb{F}[x]$.

Exercise C.2.16. Let $p \in \mathbb{F}[x]$ and $r \in \mathbb{F}$. Show $p(r) = 0_{\mathbb{F}}$ iff there is a $q \in \mathbb{F}[x]$ such that $p(x) = (x-r)\, q(x)$. [*Hint*: See the proofs of Lemma 1.4.12 and Lemma 1.4.13.]

One cannot construct square roots using a finite number of arithmetic operations. Hence, we cannot expect every member of a field to have a square root. For example, $\sqrt{2}$ is not rational, hence 2 does not have a square root in the field of rational numbers.

Example C.2.17. In $\mathbb{Z}_3$ the squares are $0^2 = 0$, $1^2 = 1$, and $2^2 = 1$. Hence, both 1 and $2 = -1$ are square roots of 1 and 2 does not have a square root.

In $\mathbb{Z}_5$ the squares are $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 4$, and $4^2 = 1$. Hence, both 1 and $4 = -1$ are square roots of 1, both 2 and $3 = -2$ are square roots of 4, and neither 2 nor 3 have a square root.

In $\mathbb{Z}_7$ the squares are $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 2$, $4^2 = 2$, $5^2 = 4$, and $6^2 = 1$. Hence, both 1 and $6 = -1$ are square roots of 1, both 2 and $5 = -2$ are square roots of 4, both 3 and $4 = -3$ are square roots of 2, and none of 3, 5, and 6 have a square root.

Hence, $\sqrt{2}$ exists in $\mathbb{Z}_p$ for some primes p and not for other primes p.

Appendix D
Working with Inequalities

The field axioms allow us to manipulate equalities in the usual ways, for example, to "complete the square". Some notable exceptions are, they do not allow us to extract roots, or calculate values of the transcendental functions sin, cos, log, etc. We do not use these operations/functions until we have established their existence. For our development of roots see Sect. 3.5, for logarithms and exponentials see Chap.8, and for trigonometric functions see Sect. 11.2.

Inequalities form the basis for most proofs in analysis. Thus, it is important to know how to work correctly with inequalities. In this chapter, we will develop the basic properties of inequalities based on two axioms: (*i*) trichotomy and (*ii*) positive closure.

D.1 Inequalities

The universe in this section is a set satisfying the field axioms. In particular, this could be the set of all real numbers or the set of all rational numbers. We will also assume that there is an order, that is a notion of positivity, on this set satisfying the axioms of trichotomy and positive closure. Hence we are studying ordered fields. Not all fields admit an order that turn them into an ordered field. We will see below that the complex numbers do not admit such an order and neither does the finite fields $\mathbb{Z}_p$.

Axioms forInequality:

Trichotomy For any a exactly one of $0 < a$, $0 < -a$, or $0 = a$ is true.
Positive Closure If $0 < a$ and $0 < b$, then $0 < a + b$ and $0 < ab$.

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Trichotomy provided a way to compare any number to zero. The following definition extends this to a comparison between any two numbers.

Definition D.1.1. Given numbers a and b

1. $a < b$ means $0 < b - a$
2. $a \leq b$ means $a < b$ or $a = b$
3. $b > a$ means $a < b$
4. $b \geq a$ means $a \leq b$

Every property of inequality must be proven using the above axioms for inequality and this definition.

Theorem D.1.2. $a < 0$ *iff* $0 < -a$

Proof. If $a < 0$, then $0 < 0 - a$. But $0 - a = -a$, so $0 < -a$. Conversely, suppose $0 < -a$. Using $-a = 0 - a$, this means $0 < 0 - a$. Hence $a < 0$. ☺

Exercise D.1.3. Prove $0 < 1$ and $-1 < 0$.

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of natural numbers, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ be the set of natural numbers including zero, and let $\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$ be the set of all integers.

Theorem D.1.4. *If* $n \in \mathbb{N}$, *then* $0 < n$.

Proof. Since $2 = 1 + 1$ and $0 < 1$, then $0 < 2$ by positive closure. Since $3 = 2 + 1$ and $0 < 1$ and $0 < 2$, then $0 < 3$ by positive closure. It follows by induction and positive closure that for any $n \in \mathbb{N}$, $0 < n$. In fact, suppose $0 < n$. Since $0 < 1$ and $0 < n$, then $0 < n + 1$ by positive closure. ☺

Exercise D.1.5 (Extended Trichotomy). For any a and b exactly one of

$$a < b, b < a, \text{ or } a = b$$

is true.

Exercise D.1.6. $a < b$ iff $-b < -a$.

Exercise D.1.7 (Transitivity). If $a < b$ and $b < c$, then $a < c$.

It is customary to interpret $a < b$ to mean that a is to the left of b, or equivalently, b is to the right of a. Transitivity makes this reasonable, see Fig. D.1.

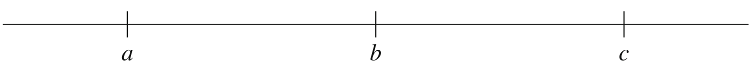

Fig. D.1 If a is to the left of b and b is to the *left* of c, then a is to the *left* of c

Definition D.1.8. $a < b < c$ means that $a < b$ and $b < c$.

So transitivity can be written as $a < b < c \implies a < c$.

Inequalities Involving Addition

Exercise D.1.9. For any a, b, and c. If $a < b$, then $a + c < b + c$.

Exercise D.1.10. If $a < b$ and $c < d$, then $a + c < b + d$.

Inequalities Involving Multiplication

Exercise D.1.11. If $0 < a$ and $n \in \mathbb{N}$, then $0 < a^n$.

Exercise D.1.12. If $a < b$ and $0 < c$, then $ac < bc$.

A converse of Exercise D.1.12 is:

Exercise D.1.13. If $a < b$ and $ac < bc$, then $0 < c$.

Exercise D.1.14. If $a \neq 0$, then $a > 0$ iff $\frac{1}{a} > 0$.

Exercise D.1.15. For any $a \neq 0$, $0 < a^2$.

Since the imaginary unit i, satisfies $i^2 = -1 < 0$, this shows it is not possible to define $0 < a$ on $\mathbb{C}$, in such a way that both trichotomy and positive closure holds.

Exercise D.1.16. $0 < a < b \Rightarrow a^2 < b^2$.

Exercise D.1.17. $a < b \not\Rightarrow a^2 < b^2$.

Exercise D.1.18. If $0 < a$, $0 < b$, and $a^2 < b^2$, then $a < b$.

The field axioms do not imply the existence of square roots. In fact, the field axioms are satisfied by the set of all rational numbers, but some roots are not rational, see Sect. 3.5. Hence statements involving, for example, $a^{1/n}$, make the implicit assumption that $a^{1/n}$ does exist.

If $0 < x$, then $\sqrt{x}$ denotes a number > 0, such that $\left(\sqrt{x}\right)^2 = x$. Similarly, if n is a positive integer, then $a^{1/n}$ denotes a number such that $\left(a^{1/n}\right)^n = a$. If $a > 0$ we assume $a^{1/n} > 0$ and if n is odd and $a < 0$, then $a^{1/n} < 0$.

Exercise D.1.19. If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

Exercise D.1.20. If $n \in \mathbb{N}$, and $0 < a < b$, then $a^n < b^n$.

Exercise D.1.21. If $0 < a < b$, then $a^{1/n} < b^{1/n}$.

If $n > 0$ is an integer, then it follows from the Binomial Theorem and Exercise D.1.10 that

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \cdots + x^n \geq 1 + nx + 0 + 0 + \cdots + 0 = 1 + nx$$

when $x \geq 0$. This argument does not work when $x < 0$. However, when $-1 < x$, the inequality can be established by induction:

Exercise D.1.22 (Bernoulli's Inequality). If $n \in \mathbb{N}$, then

$$1 + nx \leq (1+x)^n,$$

for all $-1 < x$.

Bernoulli's inequality is named after Jacob Bernoulli (6 January 1655 Basel to 16 August 1705 Basel).

D.2 Absolute Value

Consider a number system with a relation $<$ satisfying the Axioms for Inequality.

Definition D.2.1. The absolute value $|a|$ of a number a is determined by

$$|a| := \begin{cases} a & \text{if } 0 \leq a \\ -a & \text{if } a < 0 \end{cases}$$

Exercise D.2.2. Prove that $-|a| \leq a \leq |a|$ for all a.

Exercise D.2.3. Prove that $0 \leq |a|$ for all a.

The following group of exercises explore how the absolute value interact with multiplication.

Exercise D.2.4. Prove that $|a|^2 = a^2$ for all a.

Exercise D.2.5. Prove that $|-a| = |a|$ for all a.

Exercise D.2.6. Prove that $|ab| = |a|\,|b|$ for all a and b.

Exercise D.2.7. Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for all a and b with $b \neq 0$.

A very useful way to understand inequalities involving absolute value is:

Exercise D.2.8. For all a and b, prove that $|a| \leq b \iff -b \leq a \leq b$.

The two exercises below relate absolute value and addition.

Exercise D.2.9 (Triangle Inequality). Prove that $|a+b| \leq |a| + |b|$ for all a and b.

There are no triangles in the set of real numbers, so the name "Triangle Inequality" may seem strange. See Sect. E for the reason this inequality is called the triangle inequality.

Exercise D.2.10 (Reverse Triangle Inequality). Prove that $||a| - |b|| \leq |a-b|$ for all a and b.

Problems

Problems for Sect. D.1

1. If $a < b$, then there is a number c, such that $a < c < b$. [In other words, given a number a, there is no number b, that is the "next" number.]

2. If $a < 0$ and $0 < b$, then $ab < 0$.

3. If $a < b$ and $c < 0$, then $ac > bc$.

Problems for Sect. D.2

1. If x, y are points in the closed interval $[a, b]$, then $|x - y| \leq b - a$.

Solutions and Hints for the Exercises

Exercise D.1.3. If $0 < -1$, then $0 < (-1)(-1)$ by positive closure. But $0 < -1$ and $0 < 1$ contradicts trichotomy.

Exercise D.1.5. Apply trichotomy to $c := b - a$.

Exercise D.1.6. You cannot multiply by -1, because we only know *positive* closure.

Exercise D.1.7. Use positive closure.

Exercise D.1.22. This follows from a proof by induction using $0 \leq x^2$.

Exercise D.2.9. Prove $-(|a| + |b|) \leq a + b \leq |a| + |b|$.

Exercise D.2.10. Prove $|a| - |b| \leq |a - b|$ and $|b| - |a| \leq |a - b|$.

Appendix E
Complex Numbers

The set of *complex numbers* $\mathbb{C}$ is the set of ordered pairs of real numbers

$$\mathbb{C} := \mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}.$$

Abbreviating $a = (a,0)$ identifies $\mathbb{R}$ with the subset $\mathbb{R} \times \{0\}$ of $\mathbb{C}$. In particular, $1 = (1,0)$. The imaginary unit is $i := (0,1)$. With this notation we have

$$(a,b) = a\,(1,0) + b\,(0,1) = a + ib.$$

The first equality being a standard property of $\mathbb{R}^2$. With this notation, in particular, $0 = 0 + i0$.

The *real part* of a complex number is $\operatorname{Re}(a+ib) := a$ and the *imaginary part* is $\operatorname{Im}(a+ib) := b$. Hence for a complex number z, $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$.

Arithmetic in $\mathbb{C}$ is determined by

$$(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2) \quad \text{and}$$
$$(a_1 + ib_1)(a_2 + ib_2) := (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).$$

In particular,

$$i^2 = i \cdot i = (0,1)\,(0,1) = (0-1) + (0+0)\,i = -1.$$

Division of complex numbers is determined by

$$\frac{a_1 + ib_1}{a_2 + ib_2} := \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2},$$

when $a_2 + ib_2 \neq 0$. If $b_1 = b_2 = 0$ this agrees with arithmetic in $\mathbb{R}$.

Exercise E.0.1. If $z = a + ib \neq 0$, then $z/z = 1$.

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E.1 Basic Properties of Complex Numbers

Manipulations with complex numbers is often most efficiently done without considering the real and imaginary parts.

Exercise E.1.1. If z, z_1, and z_2 are complex numbers, then

1. $z_1 + z_2 = z_2 + z_1$
2. $z_1 z_2 = z_2 z_1$
3. $z(z_1 + z_2) = zz_1 + zz_2$

The *complex conjugate* of $a+ib$ is $\overline{a+ib} := a - ib$.

Exercise E.1.2. If z and w are complex numbers, then

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z}\,\overline{w}$
3. $\overline{\overline{z}} = z$

The *length* or *modulus* of a complex number $a+ib$ is

$$|a+ib| := \sqrt{a^2+b^2}.$$

This agrees with the way one normally defines the length $|(a,b)| = \sqrt{a^2+b^2}$ in $\mathbb{R}^2$. Also, $|a+i0| = \sqrt{a^2}$ is the absolute value of a. Hence, results proven about the modulus of a complex number are also true for the absolute value of a real number.

Exercise E.1.3. Let z and w be complex numbers, then

1. $z\overline{z} = |z|^2$
2. $|\overline{z}| = |z|$
3. $|zw| = |z|\,|w|$
4. $\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$, if $w \neq 0$

Proposition E.1.4. *For any complex number z, $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.*

Proof. If $z = a+ib$, then $|a| = \sqrt{a^2+0} \leq \sqrt{a^2+b^2}$. Similarly, $|b| \leq \sqrt{a^2+b^2}$. ☺

Exercise E.1.5. $\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ and $\mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w)$ for any complex numbers z and w.

Exercise E.1.6. $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = i2\,\mathrm{Im}(z)$ for any complex number z.

Using some of the properties listed above we now establish how the modulus interact with addition.

Theorem E.1.7 (Triangle Inequality). *The modulus $|\cdot|$ satisfies the* triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

for all z_1 and z_2 in $\mathbb{C}$.

Proof. Calculating we have

$$\begin{aligned} z_1\overline{z_2}+\overline{z_1}z_2 &= z_1\overline{z_2}+\overline{z_1\overline{z_2}} \\ &= 2\,\mathrm{Re}(z_1\overline{z_2}). \end{aligned}$$

Hence

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\ &= z_1\overline{z_1}+z_2\overline{z_2}+z_1\overline{z_2}+\overline{z_1}z_2 \\ &= |z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\overline{z_2}) \\ &\le |z_1|^2+|z_2|^2+2|z_1\overline{z_2}| \\ &= |z_1|^2+|z_2|^2+2|z_1|\,|z_2| \\ &= (|z_1|+|z_2|)^2 \end{aligned}$$

for any complex numbers z_1 and z_2. Taking the square root gives the desired inequality. ☺

For any three points a,b,c in the complex plane

$$a-c=(a-b)+(b-c)$$

so the triangle inequality with $z_1 := a-b$ and $z_2 := b-c$ says

$$\boxed{|a-c| \le |a-b|+|b-c|.}$$

The triangle inequality, written in this way, is the most important inequality in analysis. We will use it hundreds of times in this text. We will often use it when we want to estimate $|a-c|$. The "trick" is to "guess" b so that we can estimate $|a-b|$ and $|b-c|$.

Geometrically it says that the length of any side of the triangle is $\le$ the sum of the lengths of the other two sides. See Fig. E.1. This is the reason it is called

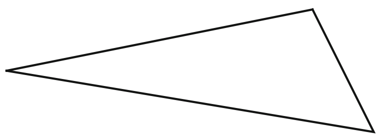

Fig. E.1 Triangle Inequality. Label the vertices a,b,c in any manner

the triangle inequality. Thinking of the triangle inequality this way may helps us understand, pictorially, when we may need to us the triangle inequality.

Restating the triangle inequality using $z_1 = a_1 + ia_2$ and $z_2 = b_1 + ib_2$ gives

$$\sqrt{(a_1+b_1)^2+(a_2+b_2)^2} \le \sqrt{a_1^2+a_2^2}+\sqrt{b_1^2+b_2^2}$$

for all real numbers a_1, a_2, b_1, and b_2. This is the triangle inequality in $\mathbb{R}^2$, since $|(a,b)| = \sqrt{a^2+b^2}$ is the distance function, i.e., norm, in $\mathbb{R}^2$.

Exercise E.1.8. Show that the reverse triangle inequality

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

holds for z_1, z_2 in $\mathbb{C}$.

E.2 Balls

The *open* ball with center c and radius $r > 0$ is the set of points whose distance from c is smaller than r. A *closed* ball is defined similarly, with the distance being at most r. The sphere with center c and radius r is the set of points whose distance from c equals r.

Balls in $\mathbb{C}$

Let $c \in \mathbb{C}$ and let $r > 0$. The set

$$B_r(c) := \{z \in \mathbb{C} \mid |z-c| < r\}$$

is an *open ball* with *center* c and *radius* r. The set

$$\overline{B}_r(c) := \{z \in \mathbb{C} \mid |z-c| \leq r\}$$

is a *closed ball* with center c and radius r. The set

$$S_r(c) := \{z \in \mathbb{C} \mid |z-c| = r\}$$

is a sphere. Clearly, the sphere is the difference between the closed and the open ball: $\overline{B}_r(c) \setminus B_r(c) = S_r(c)$. Geometrically, balls in $\mathbb{C}$ are disks and spheres in $\mathbb{C}$ are circles.

Balls in $\mathbb{R}$

Let $c \in \mathbb{R}$ and let $r > 0$. The set

$$B_r(c) := \{z \in \mathbb{R} \mid |z-c| < r\}$$

is an *open ball* with *center* c and *radius* r. The set

$$\overline{B}_r(c) := \{z \in \mathbb{R} \mid |z-c| \leq r\}$$

is a *closed ball* with center c and radius r. The set

$$S_r(c) := \{z \in \mathbb{R} \mid |z - c| = r\}$$

is a sphere. Geometrically, balls in $\mathbb{R}$ are intervals and spheres are the endpoints of intervals.

Exercise E.2.1. Suppose we are in $\mathbb{R}$. If $c \in \mathbb{R}$, then $B_r(c) =]c - r, c + r[$.

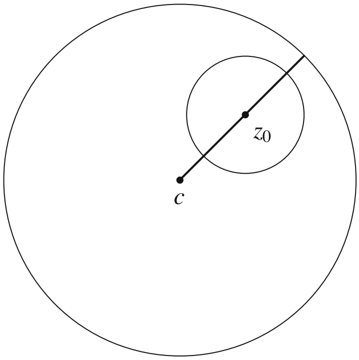

Fig. E.2 Illustration for Exercise E.2.2. The large circle has radius r and the small circle has radius s

Exercise E.2.2. If $z_0 \in B_r(c)$, then there exists $s > 0$, such that $B_s(z_0) \subseteq B_r(c)$. Give a proof based on the triangle inequality that works both in $\mathbb{R}$ and in $\mathbb{C}$. See, Fig. E.2.

Problems

Problems for Chap. E

1. Verify the triangle inequality in $\mathbb{R}^2$ without using complex numbers.
2. If z is a complex number, then

$$|1 + nz| \leq (1 + |z|)^n$$

for all $n \in \mathbb{N}$.

3. Clearly, $B_r(c) \subset \overline{B}_r(c)$. Prove $\overline{B}_{r/2}(c) \subset B_r(c)$.

A *linear order* on a set A is a relation on A satisfying extended trichotomy and transitivity. You can think of a linear order on A as a way of alphabetizing the set A.

On the set of complex number $\mathbb{C}$ define $a+ib < c+id$ to mean either

$$a < c \text{ or both}$$
$$a = c \text{ and } b < d.$$

4. Draw the region $\{1+i2 < x+iy \mid x+iy \in \mathbb{C}\}$.

5. Show that $a+ib < c+id$ is a linear order on $\mathbb{C}$.

The next two problems investigate positive closure of this linear order on $\mathbb{C}$.

6. Show that if $0 < a+ib$ and $0 < c+id$, then $0 < (a+ib)+(c+id)$.

7. Give an example of numbers $0 < a+ib$ and $0 < c+id$, such that $(a+ib)(c+id) < 0$.

Solutions and Hints for the Exercises

Exercise E.1.8. Look at the corresponding result in Sect. D.2.

Exercise E.2.2. Drawing a picture, see Fig. E.2, shows that any $0 < s \leq r - |c - z_0|$ should work. Suppose z satisfies $|z - z_0| < s$, then an application of the triangle inequality shows that $|z - c| < r$.

Appendix F
Greek Alphabet

Some of the greek letters are written in more than one form. For the upper case letters the variations are small. In some cases one of the forms looks like a symbol used in another context, notably $\in$ and ε.

Upper case	Lower case	Name
A	α	alpha
B	β	beta
$\mathit{\Gamma}, \Gamma$	γ	gamma
$\mathit{\Delta}, \Delta$	δ	delta
E	ε, ε	epsilon
Z	ζ	zeta
H	η	eta
$\mathit{\Theta}$	θ, ϑ	theta
I	ι	iota
K	$\kappa, \varkappa$	kappa
$\mathit{\Lambda}, \Lambda$	λ	lambda
M	μ	mu
N	ν	nu
$\mathit{\Xi}$	ξ	ksi
O	o	omicron
$\mathit{\Pi}, \Pi$	π, ϖ	pi
P	ρ, ρ	rho
$\mathit{\Sigma}, \Sigma$	σ, ς	sigma
T	τ	tau
$\mathit{\Upsilon}, \Upsilon$	υ	upsilon
$\mathit{\Phi}, \Phi$	ϕ, φ	phi
X	χ	chi
$\mathit{\Psi}, \Psi$	ψ	psi
$\mathit{\Omega}, \Omega$	ω	omega

Table F.1 Greek alphabet

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Appendix G
Credits

The author has taught, parts of, the material included in this text using several different textbooks. For example, he has taught from (various editions of) Bartle and Sherbert (1999), Burn (2000), Belding (2008), Lay (2004), and Strichartz (2000). The present text is clearly influenced by the works cited above. The list below contains the cases where the author consciously adapted material from other sources to fit into this text.

- Arranging the theory of inequalities as a sequence of inequalities is adapted from Burn (2000).
- A table similar to Table 1.1 can be found in Boester (2010), the use of the word "tolerance" is also taken from Boester (2010).
- The proof of Steinhaus' Three Distance Conjecture, Theorem 1.8.1, is from Slater (1967).
- The proof of Theorem 3.4.1 is from Dunham (1990), where it is credited to Volterra.
- The proof of Theorem 3.5.4 is a modification of the proof in Ferreño (2009).
- The notion "increasing at a point" is from Strichartz (2000).
- The proof of Theorem 6.5.3 is adapted from Körner (1989), where it is credited to Liouville.
- The proof that e is transcendental is adapted from Gelfond (1960).
- The problems for Sect. 8.3 showing that nonzero rational powers of e are irrational are adapted from Aigner and Ziegler (2004).
- The proof of Theorem 9.2.12 is an interpretation of (part of) Remark 2 in Aksoy and Martelli (2002)
- The proof of Theorem 9.4.1 is essentially from Scheinerman and Schep (2009).
- The construction of the space filling curve in Sect. 10.3 is adapted from Sagan (1994).
- The proof of Theorem 10.5.1 is adapted from Körner (1989), who credits it to Weierstrass.
- The construction of π in Sect. 11.2 is adapted from Bartle and Sherbert (1999).

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- The proof in Sect. 11.4 is adapted from a paper Weierstrass presented to the Königliche Akademie der Wissenschaften on 18 July 1872, Weierstrass (1895).
- The proof of Theorem 11.5.2 is adapted from Zhou and Markov (2010).
- The problems for Sect. 11.5 outline Niven's proof that π is irrational, see Niven (1947).
- Problem 2 for Sect. 13.4 is adapted from the proof of Darboux's Intermediate Value Theorem in Nadler (2010).
- The proof that the number of primes is infinite is adapted from Proposition 20, Book IX of Euclid's Elements.

Appendix H
Names

The list below contains refences to (some of) the pages where a mathematician is mentioned by name. The page number in bold refers to the page containing a small amount of biographical information.

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Nomenclature

$\mathbb{1}_A$	characteristic function of the set A, page 24
(x_n)	sequence $x_1, x_2, \ldots$, page 28
(a_{i_n})	subsequence of (a_n), page 182
$\lceil x \rceil$	smalles integer n such that $x \le n$, page 29
$\lfloor t \rfloor$	the largest integer $\le t$, page 30
$\lvert x \rvert$	absolute value of the real number x, page 322
$\lvert z \rvert$	modulus of the complex number z, page 326
$\{x\}$	fractional part of the real number x, page 32
$a_n \to a$	(a_n) converges to a, page 182
$f_n \rightrightarrows f$	f_n converges uniformly to f, page 191
$f_n \to f$	f_n converges pointwise to f, page 191
a^x	exponential function base a, page 164
$B'_r(c)$	punctured ball radius r and center c, page 7
$\binom{n}{k}$	binomial coefficients, page 312
$B_r(c)$	open ball with radius r and center c, page 328
$\overline{B}_r(c)$	closed ball with radius r and center c, page 328
$\mathbb{C}$	set of all complex numbers, page 325
$\mathscr{C}^1(D)$	f' exists and is continuous on D, page 112
C	Cantor set, page 66
$\mathscr{C}^\infty$	the derivative $f^{(n)}$ exist for all n, page 161
$\mathscr{C}^n(D)$	f is n times differentiable and the n^{th} derivative $f^{(n)}$ is continuous on D, page 107
$\cos(x)$	the cosine function, page 236
$d_0.d_1d_2\cdots$	infinite decimal (base 10), page 3
$1.67\overline{2345}$	repeating decimal, page 4
$d_0 \underset{b}{.} d_1d_2\cdots$	infinite decimal (base b), page 65
$D_A(x)$	distance from the point x to the set D, page 86
e	base of the natural exponential, page 164
e^x	exponential function, page 164
$\exp(x)$	exponential function, page 163

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$f * g$ convolution of f and g, page 198
$f : A \to B$ f is a function defined on A with values in B, page 9
$\widehat{f}(k)$ Fourier coefficients, page 255
$f^-(x)$ negative part of f, page 142
$f^+(x)$ positive part of f, page 142
$f'(a)$ derivative of the function f at the point a, page 93
$f\big|_E$ restriction of f to the set E, page 23
f_x partial derivative of f with respect to x, page 196
γ Euler constant, page 162
i imaginary unit, page 325
$\mathrm{Im}(z)$ imaginary part of z, page 325
$\inf(A)$ greatest lower bound of A, page 56
$\int_a^b f$ integral of f, page 133
$[a,b]$ closed interval, page 57
]a,b[open interval, page 57
$\lim_{x\to a} f(x)$ limit of $f(x)$ as $x \to a$, page 10
$f(x) \underset{x\to a}{\longrightarrow} L$ limit of $f(x)$ as $x \to a$, page 10
$\underline{\lim} \inf a_n$ limit inferior of a_n, page 190
$\overline{\lim} a_n$ limit superior of a_n, page 189
$\limsup a_n$ limit superior of a_n, page 189
$\underline{\lim} a_n$ limit inferior of a_n, page 190
$\log(x)$ logarithm, page 161
$\max(A)$ largest number in A, if any, page 53
$\min(A)$ smallest number in A, if any, page 56
$\mathbb{N}$ set of all natural numbers $\{1,2,3,\ldots\}$, page 320
$\mathbb{N}_0$ the set $\{0,1,2,\ldots\}$, page 320
$o(\phi(x))$ little oh notation, page 121
$\partial_x f$ partial derivative of f with respect to x, page 196
π pi, page 238
$\mathbb{Q}$ set of all rational numbers, page 4
$\mathbb{R}^2$ set of all ordered pairs of real numbers, page 325
$\mathrm{Re}(z)$ real part of z, page 325
$\sigma(x)$ pseudo-sine function, page 14
$\sin(x)$ the sine function, page 236
$S_r(c)$ sphere with radius r and center c, page 328
$\sum_{k=1}^n x_k$ the finite sum $x_1 + x_2 + \cdots + x_n$, page 30
$\sum_{k=0}^\infty x_k$ infinite series, page 31
$\sum s$ sum of a step function, page 132
$\sup(A)$ least upper bound of A, page 53
$x_{\max}$ point where a function assumes its largest value, page 84
$x_{\min}$ point where a function assumes its smallest value, page 84
$\mathbb{Z}$ set of all integers $\{\ldots,-2,-1,0,1,2,\ldots\}$, page 320
$\bar{z}$ complex conjugate of z, page 326
f is $1-1$ f is one-to-one, page 307
$f(C)$ image of the set C under f, page 308

$f^{-1}(D)$ pre-image of the set D under f, page 308
$\exists x$ for some x, page 304
$\forall x$ for all x, page 304
$\neg P$ negation of P, page 304
$P \iff Q$ P is true if and only if Q is true, page 304
$P \implies Q$ if P is true, then Q is true, page 303
$\bigcap_{i\in I} A_i$ intersection of A_i, $i \in I$, page 307
$\bigcup_{i\in I} A_i$ union of A_i, $i \in I$, page 307
$A \cap B$ intersection of A and B, page 306
$A \cup B$ union of A and B, page 306
$A \setminus B$ set difference of A and B, page 306
$A \subset B$ A is a proper subset of B, page 306
$A \subseteq B$ A is a subset of B, page 306

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