International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: http://www.tandfonline.com/loi/tmes20

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To cite this article: Kin-Keung Poon (2012) Tour of a simple trigonometry problem, International Journal of Mathematical Education in Science and Technology, 43:4, 449-461, DOI: 10.1080/0020739X.2011.622798

To link to this article: http://dx.doi.org/10.1080/0020739X.2011.622798

Published online: 10 Oct 2011.

International Journal of Mathematical Education in Science and Technology, Vol. 43, No. 4, 15 June 2012, 449–461

Tour of a simple trigonometry problem

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(*Received 6 April 2011*)

This article focuses on a simple trigonometric problem that generates a strange phenomenon when different methods are applied to tackling it. A series of problem-solving activities are discussed, so that students can be alerted that the precision of diagrams is important when solving geometric problems. In addition, the problem-solving plan was implemented in a high school and the results indicated that students are relatively weak in problem-solving abilities but they understand and appreciate the thinking process in different stages and steps of the activities.

Keywords: problem solving; trigonometry; dynamics geometry

1. Introduction

Problem solving plays a crucial role in mathematics education. One of the aims of teaching through problem solving is to encourage students to refine and build their own processes over time as their experience allows them to discard some ideas and become aware of others [1]. In addition to developing their knowledge, students also acquire an understanding of when it is appropriate to use particular strategies. Emphasis is placed on making students responsible for their own learning rather than letting them feel that the methods they are using are the inventions of others. Considerable emphasis is placed on exploratory activities, observation and discovery, and trial and error. Students must develop their own ideas, test them, discard them if they are not consistent and try something else [2]. Students become more involved in problem solving by formulating and solving their own problems, or by rewriting problems in their own words to aid their understanding. Crucially, students are encouraged to discuss the processes they are using to improve their understanding, gain new insights into problems and communicate their ideas [3,4].

There are many problem-solving models, such as those of Garafalo and Lester [5], Mason, Burton, and Stacey [6], Schoenfeld [7] and Krulik and Rudnick [8]. Here, the model developed by Pólya [9] is applied. This model can be roughly divided into four stages. In the first stage, which involves understanding the problem, students must determine whether they comprehend what it is that they are to solve or find. When they understand what it is they are looking for or trying to solve, they can move on to the second stage, in which they choose and devise a plan to reach a solution. An example of applying Polya's model can be found in [10]. This article

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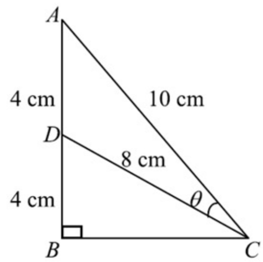
ISSN 0020–739X print/ISSN 1464–5211 online

http://dx.doi.org/10.1080/0020739X.2011.622798
http://www.tandfonline.com

focuses on a simple trigonometric problem that generates a strange phenomenon when different methods are applied to tackling it. A series of problem-solving activities are discussed, so that students can be alerted that the precision of diagrams is important when solving geometric problems.

2. Background of the problem

This is a simple trigonometry problem.



Find the value of θ.[1]

At first sight, this is an ordinary textbook problem. One of the ways to find the solution might be as follows.

Method 1:
Consider ΔABC, $\angle ACB = \sin^{-1}\left(\frac{4+4}{10}\right) \approx 53.13^\circ$, and consider ΔDBC, $\angle DCB = \sin^{-1}\frac{4}{8} \approx 30^\circ$. Hence, $\theta = \angle ACB - \angle DCB \approx 23.13^\circ$.

This seems to be the correct answer. However, if another method is used, another answer will be found.

Method 2:
Consider ΔABC, $\angle BAC = \cos^{-1}(\frac{4+4}{10}) \approx 36.87^\circ$, and ΔACD, applying the Sine Rule yields $\frac{4}{\sin\,\theta} = \frac{8}{\sin 36.87^\circ}$.

Hence, $\theta \approx 17.46^\circ$.

If the problem is considered more deeply, the non-uniqueness of BC is also found, i.e. in ΔABC, $BC = \sqrt{AC^2 - AB^2} = 6\,\text{cm}$, but in ΔDBC, $BC = \sqrt{DC^2 - DB^2} = 4\sqrt{3}\,\text{cm}$, which is not the same as in ΔABC. This does not make sense at all, and the problem should be reconsidered through a series of problem-solving activities.

3. Problem-solving activities

Step 1: Analyse and understand the problem
We may suspect that the problem is difficult because inconsistent conditions are given in the diagram. To analyse the problem, the information should first be listed as follows:

Condition 1: $AD = 4\,\text{cm}$.
Condition 2: $DB = 4\,\text{cm}$.
Condition 3: ADB is a straight line.
Condition 4: $AC = 10\,\text{cm}$.
Condition 5: $DC = 8\,\text{cm}$.

Condition 6: $\angle DBC = 90^\circ$.

If we assume Conditions 1, 4 and 5, that is AC, AD and DC are fixed, then ΔADC can be uniquely determined (Congruence Triangle SSS). That is, we can find $\angle ACD$ right here. If we further assume Conditions 2 and 3, we can determine that $\angle DBC$ is unique (Congruence Triangle SAS). Thus, Condition 6 seems to be inconsistent with others and hence we can remove it and make the figure consistent. The rephrased problem is as follows.

New problem

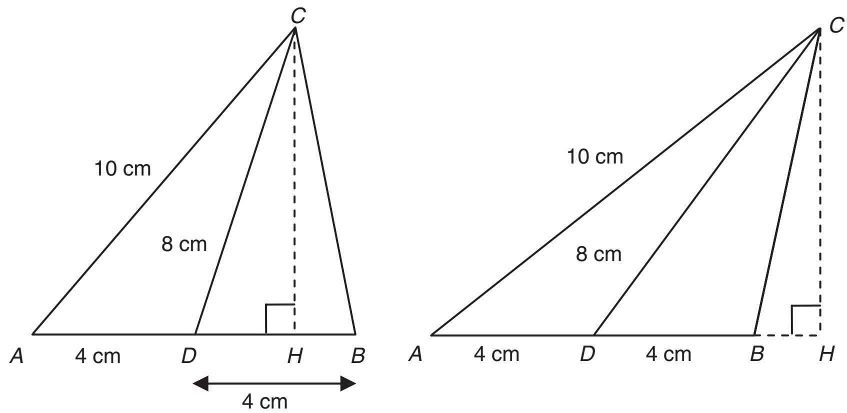
Assume that all the lengths are correct, that is $AC = 10\,\text{cm}$, $DC = 8\,\text{cm}$ *and* $AD = DB = 4\,\text{cm}$, *and that* ADB *is a straight line and* $\angle DBC$ *is not necessarily equal to* 90°. *Draw the figure exactly and find* $\angle DBC$ *and* θ.

The new problem is more difficult than the conventional exercise as it requires figuring out the exact shape of the diagram before doing calculations and thus requires students to have a higher proficiency in mathematics.

Step 2: Devising a plan

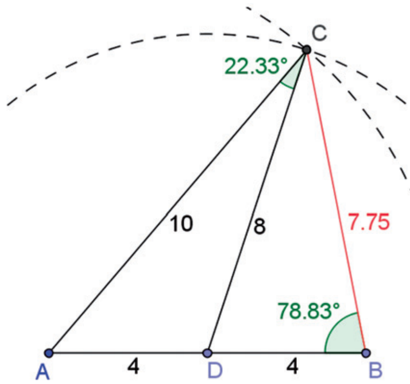
The second step in solving this problem is to devise a plan. Students need to find the connections between the given information and the unknowns. Pólya [9] mentioned that an accurate figure can help students to solve the problem. In addition, drawing an exact figure can help students to explore more geometrical properties and more easily investigate other meaningful relationships by themselves. In fact, figures have an auxiliary role for high achievers. The important thing is that the problem solver should understand the logical relations of the quantities in a figure [9]. Given the popularity of computers, we can use dynamic geometry software (DGS) to replace the work of drawing by hand.

Students who have not yet learned the sine or cosine theorems should solve the problem by dividing the triangle into right-angled triangles: that is, they should add an auxiliary straight line CH such that $CH \perp ADB$, and then the figure is divided into several right-angled triangles. In this way, students can solve the problem by setting up suitable equations. However, the position of CH can affect those equations.



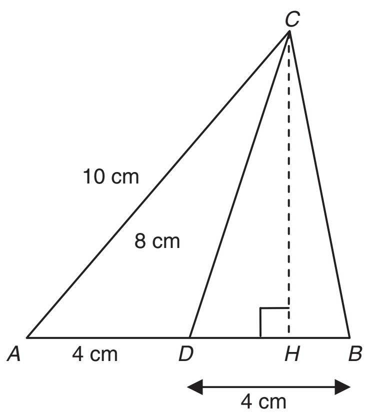
Without the correct figures, we cannot determine the two cases shown above.

To help themselves come up with a plan, students need to draw a scale diagram on a worksheet. This will help them to determine the relations of the unknowns and the given lengths. Using DGS is a good way of doing this. The following figure was drawn in GeoGebra.[2]



The correct figure, drawn and measured in GeoGebra.

The scale diagram will aid students in that they will be able to clearly see the position of the auxiliary straight line.



Expected strategy: Adding the auxiliary line $CH \perp ADB$.

In this problem, students are expected to figure out that three right-angled triangles, that is ΔACH, ΔCDH and ΔCHB, can be used to set up the equations. Thus, when they are devising a plan they should be able to figure out the following equations.

For ΔACH:

$$CA^2 = AH^2 + CH^2 = (AD + DH)^2 + CH^2, \quad \text{i.e. } 100 = (4 + DH)^2 + CH^2 \tag{1}$$

For ΔCDH:

$$CD^2 = DH^2 + CH^2, \quad \text{i.e. } 64 = DH^2 + CH^2, \tag{2}$$

For ΔCBH:

$$CB^2 = CH^2 + HB^2, \quad \text{i.e. } CB^2 = CH^2 + (4 - DH)^2 \tag{3}$$

After setting up the equations, students should realize that they need to solve them with three unknowns. Hence, the next step is to make a plan to solve the equations. The teacher should remind students that they can solve the problem of two simultaneous equations with two unknowns, and then ask them which two equations should be chosen. Students should figure out they need to choose (1) and (2), so that they can solve CH and DH.

Students who have learned the sine or cosine theorems may take a different approach to the problem, and the diagram will help them in that it indicates the given information and the angles they need to find. They may find that the required $\angle ACD$ is the angle of the triangle with three given lengths. Thus, they can apply the cosine theorem directly. In fact, the diagram will help them by showing the logical relations between $\angle ACD$ and AC, AD and CD. For the remaining part, they can find out $\angle DBC$ by first finding out $\angle CDB$ and applying the sine theorem.

Step 3: Carry out the plan

The third step is to carry out the plan, for which there are two methods.

Method 1:

After choosing suitable equations, the solution is clear. Students will need their algebraic skills to solve the equations:

In ΔCDH,

$$64 = DH^2 + CH^2. \tag{4}$$

In ΔACH,

$$100 = (4 + DH)^2 + CH^2 \tag{5}$$

$$DH = \frac{5}{2}\text{cm}.$$

Therefore,

$$CH = \sqrt{64 - DH^2} = \frac{\sqrt{231}}{2}\text{cm}.$$

Now students have information on the length; they can calculate the angles easily by applying trigonometric ratios on suitable right-angled triangles:

In ΔACH,

$$\angle CAD = \sin^{-1}\frac{CH}{CA} = \sin^{-1}\frac{\sqrt{231}}{20} = 49.5^\circ.$$

In ΔCDH,

$$\angle CDB = \sin^{-1}\frac{CH}{CD} = \sin^{-1}\frac{\sqrt{231}}{16} = 71.8^\circ,$$
$$\angle ACD = \angle CDB - \angle CAD = \underline{\underline{22.3^\circ}}.$$

Therefore,

$$\angle DBC = \tan^{-1}\frac{CH}{HB} = \tan^{-1}\frac{\sqrt{231}}{3} = \underline{\underline{78.8^\circ}}.$$

Method 2:
Apply the cosine theorem in ΔACD,

$$\angle ACD = \cos^{-1}\frac{CA^2 + CD^2 - AD^2}{2CA \times CD} = \cos^{-1}\frac{37}{40} = \underline{\underline{22.3^\circ}},$$
$$\angle CDA = \cos^{-1}\frac{AD^2 + CD^2 - CA^2}{2AD \times CD} = \cos^{-1}\left(-\frac{5}{16}\right) = 108.2^\circ,$$
$$\angle CDB = 180^\circ - \angle CDA = 71.8^\circ.$$

Apply the cosine and sine theorems in ΔCDB,

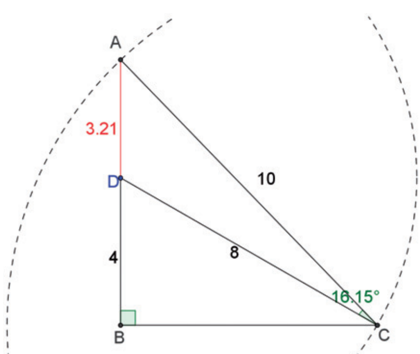
$$CB = \sqrt{CD^2 + DB^2 - 2CD \times DB \cos \angle CDB}$$
$$= 2\sqrt{15}\,\text{cm},$$
$$\frac{8}{\sin \angle CBD} = \frac{CB}{\sin \angle CDB},$$
$$\angle CBD = \underline{\underline{78.8^\circ}}.$$

The plan should be carried out in a fashion similar to the steps previously outlined. Students may encounter problems, so the teacher should help them to check the feasibility of their plan, and then give them hints to carry out the plan or to devise a more feasible plan.

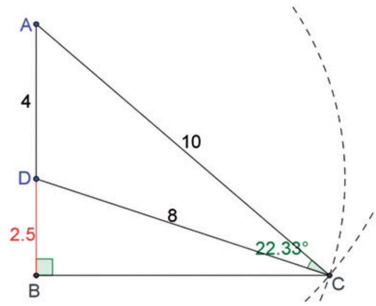
Step 4: Review the problem
The final step in solving the problem is to look back. We can first examine the solution. The scale diagram that students have drawn by DGS will help them to know if they have made the correct calculations.

The teacher should further discuss with students how to correct the figure in other ways. One way of doing this would be to ask students to choose five conditions out of six and construct the correct figure by DGS and then find out the corresponding value of $\angle ACD$. The following are examples of the corrected diagram when one condition is removed.

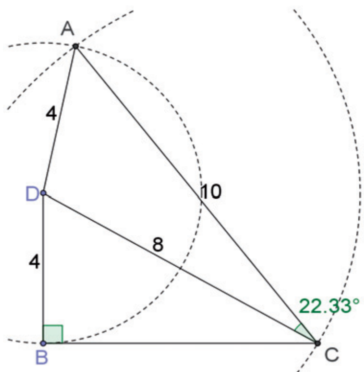
Case 1: Remove condition 1; the figure can be determined uniquely and $\angle ACD$ is approximately equal to 16.15°.



Case 2: Remove condition 2; the figure can be determined uniquely and $\angle ACD$ is approximately equal to 22.33°.



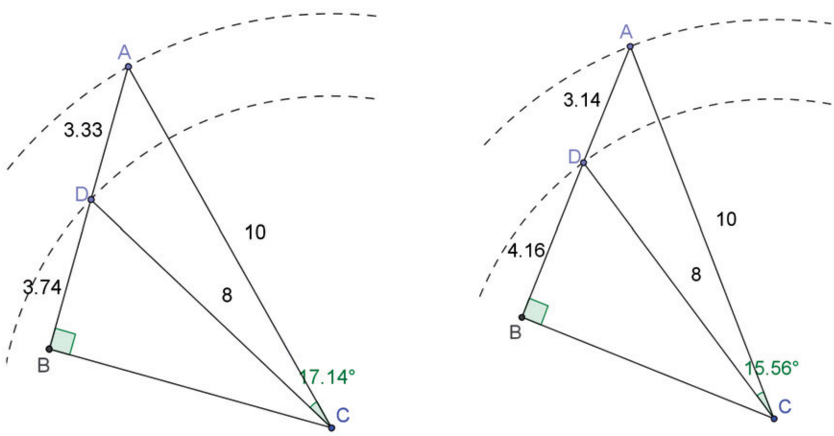
Case 3: Remove condition 3; the figure can be determined uniquely and $\angle ACD$ is approximately equal to 22.33°.



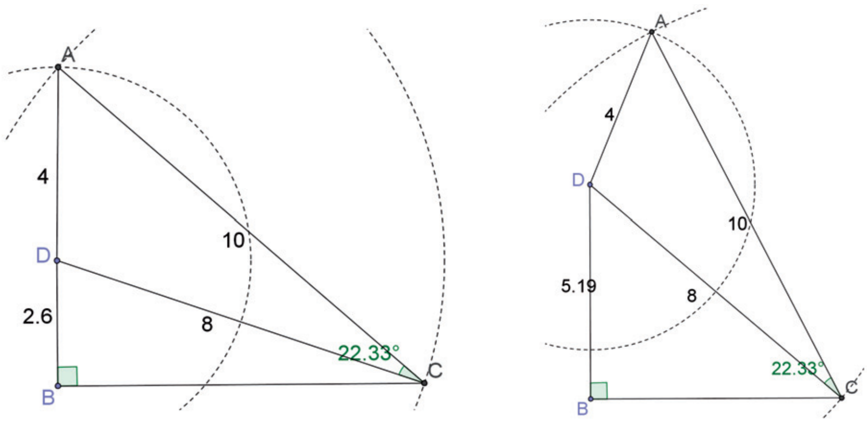
It may be interesting to construct figures for other cases.

The teacher should encourage higher achievers to discuss the uniqueness of the figures if only four conditions are chosen out of the six's. Students may find that this is not sufficient to ensure the uniqueness of the figure.

Example 1: Remove Conditions 1 and 2; *A* and *D* can be freely chosen, so *B* cannot be uniquely determined.



Example 2: Remove Conditions 2 and 3; D can be freely chosen so A cannot be uniquely determined.



4. Data collection

We implemented this problem-solving activity in a sixth form high school class (equivalent with Grades 11 and 12 in the US system) in Hong Kong. The class size was 30 students and they had basic knowledge in trigonometry and had already learnt sine and cosine laws. In the first stage, they were required to solve the original simple trigonometry problem with inconsistent information. During the activity, no guidance is provided.

Without guidance, nearly all students did not notice the error. They just simple found the angle θ. Here is an example:

Student 1: Did not notice the error.

$$4^2 = 8^2+10^2-2\cdot10\cdot8\cos\theta$$

$$\cos\theta = \frac{4^2-8^2-10^2}{-2\cdot10\cdot8}$$

$$\cancel{\cos\theta} = \cos^{-1}(0.925)$$

$$\theta = 22.3^\circ$$

However, in the second stage, the teacher asked the students to use different methods to find the angle θ in order to draw their attention to the inconsistent answers. The following are the key questions that students were asked:

- Does it make a difference if you use different methods?
- Which answer is correct?
- Are there any errors in the original problem?

With guidance, students noticed there were some problems. However, not all the students were able to identify the errors. Fourteen students only noticed that there were some problems with different methods but did not know why the answers were different; 10 students noticed and knew the information of the figure was not consistent; three students noticed and even pointed out the errors.

Below are some students' works:

Student 2: Worked the problem with different methods, but did not know why the answers were different.

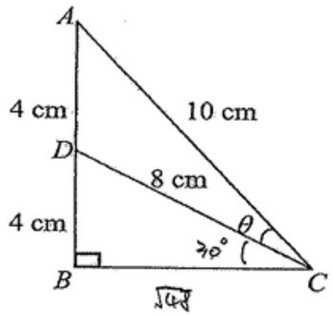


Figure 1

Find the value of θ.

$$4^2 = 8^2 + 10^2 - 2(8)(10)\cos\theta$$

$$\theta = 22.3^\circ$$

$$8^2 = 4^2 + 10^2 - 2(4)(10)\cos\angle CAD$$

$$\angle CAD = 49.5^\circ$$

$$\theta = 10.5^\circ$$

$$\sin(\angle DCB + \theta) = \frac{8}{10}$$

$$\theta = 23.1^\circ$$

$$\cos(\angle DCB + \theta) = \frac{BC}{10}$$

$$\theta = 16.1^\circ$$

Student 3: Noticed and knew the information of the figure was not consistent.

Consider $\triangle ADC$

$$\cos\theta = \frac{8^2+10^2-4^2}{2(8)(10)} \quad \text{(By cosine law)}$$

$$= 0.925$$

$$\theta = 22.3^\circ \quad \text{(cor to 3 sig fig)}$$

case 1:

Consider $\triangle DBC$

$$DC^2 = CB^2 + BC^2$$

$$8^2 = 4^2 + BC^2$$

$$BC^2 = 48$$

$$BC = \sqrt{48}$$

case 2

Consider $\triangle ABC$

$$AC^2 = AB^2 + BC^2$$

$$10^2 = 8^2 + BC^2$$

$$BC^2 = 36$$

$$BC = \sqrt{36}$$

case 1 $\neq$ case 2

the figure is impossible or the numbers were rounded

Student 4: Noticed and even identified the errors.

This is a simple trigonometry problem.

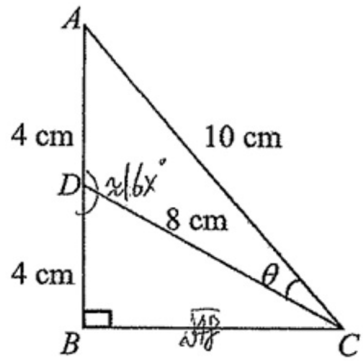


Figure 1

Find the value of θ.

$$\cos\theta = \frac{8^2+10^2-4^2}{2\times 8\times 10}$$

$$\theta = 22.3^\circ$$

AB is not straight line.

$\theta = \angle ACD$

After discussion of the original problem, the teacher started the problem-solving activities as mentioned in the previous section. The whole process went smoothly and students understood and appreciated the thinking process in different stages and steps of the activities.

The whole problem-solving activity was evaluated by a standard instrument developed by the Education Bureau in Hong Kong:

http://www.edb.gov.hk/FileManager/EN/Content_3644/chapter4_form_1-eng.pdf

5. Conclusion

Problem solving allows students to experience a range of emotions associated with the different stages of the solution process. Mathematicians who successfully solve problems say that the experience of having done so contributes to an appreciation of the 'power and beauty of mathematics' [2, p. 77], or the 'joy of banging your head against a mathematical wall, and then discovering that there might be ways of either going around or over that wall' [11, p. 43].

To enhance the students' problem-solving skills is one of the key objectives in learning and teaching mathematics in Hong Kong [12,13]. However, in real life teaching, students tend to focus on textbook exercises and examination drills. They seldom participate in problem solving in mathematics. The results found that students are relatively weak in problem-solving abilities, but they understand and appreciate the thinking process in different stages and steps of the activities. Students gain a distinct advantage from participating in problem-solving activities: they can build up their mathematical or logical thinking, their ability to express ideas. Our activity allows students to understand what kind of thinking strategies they can have, and to apply their mathematical knowledge when dealing with a mathematical problem.

Notes

1. Thanks to Mr Yiu-Ping Tsang who worked out some of the calculations in Steps 2 and 3 in the problem-solving process.
2. GeoGebra is a freeware that can be downloaded from www.geogebra.org.

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